An Approximation Algorithm for the $\rho$-Hub Median Problem

André Luis Vignatti
Camile Frazão Bordini

DINF - Federal University of Paraná (UFPR), Curitiba-PR, Brazil

September 13, 2017
1 The $p$-Hub Median Problem

2 Linear Program

3 Rounding Algorithm
The $p$-Hub Median Problem Linear Program Rounding Algorithm

$p$-Hub Median Problem (pHM)

- Metric pHM

**Instance**
- a set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- a set of demands $D \subseteq V \times V$
- a cost function $\rho : V \times V \rightarrow \mathbb{R}^+$
- an integer $p > 0$
The $p$-Hub Median Problem (pHM)

- **Metric pHM**

**Instance**
- A set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- A set of demands $D \subseteq V \times V$
- A cost function $\rho : V \times V \rightarrow \mathbb{R}^+$
- An integer $p > 0$
The $p$-Hub Median Problem

**Linear Program**

**Rounding Algorithm**

### $p$-Hub Median Problem (pHM)

- **Metric pHM**

---

**Instance**

- A set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- A set of demands $D \subseteq V \times V$
- A cost function $\rho : V \times V \to \mathbb{R}^+$
- An integer $p > 0$
The $p$-Hub Median Problem

Linear Program

Rounding Algorithm

$p$-Hub Median Problem (pHM)

- a set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- a set of demands $D \subseteq V \times V$
- a cost function $\rho : V \times V \rightarrow \mathbb{R}^+$
- an integer $p > 0$
The $p$-Hub Median Problem (pHM)

- Metric pHM

**Instance**
- a set of clients $V = \{v_1, v_2, \ldots, v_n\}$
- a set of demands $D \subseteq V \times V$
- a cost function $\rho : V \times V \rightarrow \mathbb{R}^+$
- an integer $p > 0$

**Objective**
Select $T \subseteq V$ of terminals, where $|T| \leq p$, and assign each demand to a terminal, in order to minimize the total cost between demands and terminals.
## $p$-Hub Median Problem (pHM)

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$-Median is a particular case of pHM.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem [Jain et. al., 2002]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$-Median has a approximation factor $\geq 1 + \frac{2}{e}$ if NP $\not\in$ DTIME($n^{O(log \ log n)}$).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Corollary</th>
</tr>
</thead>
<tbody>
<tr>
<td>pHM has a approximation factor $\geq 1 + \frac{2}{e}$ if NP $\not\in$ DTIME($n^{O(log \ log n)}$).</td>
</tr>
</tbody>
</table>
The $p$-Hub Median Problem (pHM)

Theorem

A $(4\alpha)$-approximation algorithm that opens at most \( \left( \frac{2\alpha}{2\alpha-1} \right) p \) terminals, where $\alpha > 1$ is a trade off parameter.
First, we preprocess the input, defining a new cost function \( \hat{\rho} \) such that \( \hat{\rho}(d, i) = \rho(u, i) + \rho(i, v), \forall d = (u, v) \in D, i \in V. \)

The integer program (IP) formulation for pHM:

\[
\begin{align*}
\text{minimize} & \quad \sum_{d \in D} \sum_{i \in V} x_{di} \hat{\rho}(d, i) \\
\text{subject to} & \quad \sum_{i \in V} y_i \leq p \\
\text{(IP)} & \quad \sum_{i \in V} x_{di} = 1, \quad \forall d \in D \\
& \quad x_{di} \leq y_i, \quad \forall d \in D, i \in V \\
& \quad x_{di} \in \{0, 1\}, \quad \forall d \in D, i \in V \\
& \quad y_i \in \{0, 1\}, \quad \forall i \in V
\end{align*}
\]

Our algorithm uses the LP relaxation of IP 1, where \( x_{di} \geq 0 \) and \( y_i \geq 0, \forall d \in D, i \in V. \)
Rounding Algorithm

To each demand $d=(u, v) \in D$:
- "Mean Distance": $C_d = \sum_{i \in V} x_{di} \hat{\rho}(d, i)$

For each $u \in V$, let:
- $B(u, \alpha C_d)$
For each \( d = (u, v) \in D \), let:

- **Neighborhood**: \( I_d = \{ u' \in B(u, \alpha C_d) \cap B(v, \alpha C_d) \} \)
For each $d \in D$, let:

- **Extended neighborhood:**
  \[
  \overline{V}_d = \{(u', v') \in V^2 : (u', v') \in D \text{ and } I_d \cap l_{(u', v')} \neq \emptyset\}
  \]
Rounding Algorithm

**Algorithm 1:** Rounding Algorithm.

1. Solve the LP and use it to compute the $C_d$ values
2. $T := \{\}$
3. $\overline{D} := D$
4. while $\overline{D} \neq \emptyset$ do
5.     Choose $d = (u, v) \in \overline{D}$ with the lowest value of $C_d$
6.     $T := T \cup \{u\}$
7.     for $(u', v') \in \overline{D}$ do
8.         if $(u' \in \overline{V_d})$ and $(v' \in \overline{V_d})$ then
9.             $\overline{D} := \overline{D} \setminus (u', v')$
10.            $\overline{D} := \overline{D} \setminus (u, v)$
11.     return $T$
Conclusions and Future Works

- We got a \((4\alpha)\)-approximation algorithm, with \(\alpha > 1\), that opens at most \((\frac{2\alpha}{2\alpha - 1})p\) terminals.
- We used the rounding technique of linear programs.
- As future works: opening the right number of terminals and improve the approximation factor.
An optimal bifactor approximation algorithm for the metric uncapacitated facility location problem.

Integer programming formulations of discrete hub location problems.

Hub location problems: A review of models, classification, solution techniques, and applications.

A threshold of ln n for approximating set cover.

Greedy strikes back: improved facility location algorithms.

Heuristics for the fixed cost median problem.

A new greedy approach for facility location problems.


Improved approximation algorithms for metric facility location problems.

The location of interacting hub facilities.