Facet-inducing inequalities and a cut-and-branch algorithm for the bandwidth coloring polytope based on orientation model

Bruno Dias, **Rosiane de Freitas**, Javier Marenco, Nelson Maculan

UFAM/UFRJ, Brazil and UNGS/UBA, Argentina

IX Latin and American Algorithms, Graphs and Optimization Symposium

September 15th, 2017

Marseille - France
**Definition 1: Bandwidth Coloring Problem (BCP).**

Let $G = (V, E)$ be an undirected graph and $d : E \rightarrow \mathbb{Z}_+$. A feasible coloring of $G$ and $d$ for the BCP is an assignment of colors $c : V \rightarrow \mathbb{Z}_+$ such that for each $(i, j) \in E$, the condition $|c(i) - c(j)| \geq d(i, j)$ is true. The span, defined as $\max_{i \in V} c(i)$, must be the minimum possible.

\[
\max_{i \in V} c(i) = \max\{1, 3, 4, 5\} \rightarrow \text{span} = 5.
\]
Important application: channel assignment in mobile wireless networks.

- Network consists of a number of transmitters, each responsible for calls in its area.
- Channels must respect interference constraints.
- Spectrum usage must be minimized.
BCP is a particular case of T-coloring, which asks for a coloring $c : V \rightarrow \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.

- $T_{i,j}$: forbidden sets [Hale, 1980].
BCP is a particular case of T-coloring, which asks for a coloring $c : V \rightarrow \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.

- $T_{i,j}$: forbidden sets [Hale, 1980].
- BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$. 
BCP is a particular case of **T-coloring**, which asks for a coloring $c : V \rightarrow \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.

- $T_{i,j}$: forbidden sets [Hale, 1980].
- BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$.

If $d_{i,j} = 1$ for every $(i, j) \in E$, then the BCP is equivalent to the classic $k$-coloring problem.
BCP is a particular case of T-coloring, which asks for a coloring $c : V \rightarrow \mathbb{Z}_+$ such that $|c(i) - c(j)| \notin T_{i,j}$ for every $(i, j) \in E$.

- $T_{i,j}$: forbidden sets [Hale, 1980].
- BCP is equivalent to T-coloring with $T_{i,j} = \{0, 1, \ldots, d_{i,j} - 1\}$ for all $(i, j) \in E$.

If $d_{i,j} = 1$ for every $(i, j) \in E$, then the BCP is equivalent to the classic $k$-coloring problem.

- In this case, the span $\max_{i \in V} c(i)$ is equivalent to the number of used colors.
Previous works on the BCP include:

- Tabu search [Dorne and Hao, 1998, Lai and Lu, 2013]
- Local search [Lau and Tsang, 1998, Galinier and Hertz, 2006]
- Heuristic framework [Phan and Skiena, 2002]
- Integer programming [Mak, 2007]
- Evolutionary algorithms [Malaguti et al., 2008]
- Constraint programming [Prestwich, 2008]
- GRASP [Marti et al., 2010]
Previous works on the BCP include:
- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
Previous works on the BCP include:

- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
Introduction

Previous works on the BCP include:
- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
- heuristic framework [Phan and Skiena, 2002];
Previous works on the BCP include:

- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
- heuristic framework [Phan and Skiena, 2002];
- integer programming [Mak, 2007];
Previous works on the BCP include:

- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
- heuristic framework [Phan and Skiena, 2002];
- integer programming [Mak, 2007];
- evolutionary algorithms [Malaguti et al., 2008];
Previous works on the BCP include:

- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
- heuristic framework [Phan and Skiena, 2002];
- integer programming [Mak, 2007];
- evolutionary algorithms [Malaguti et al., 2008];
- constraint programming [Prestwich, 2008];
Previous works on the BCP include:

- tabu search [Dorne and Hao, 1998, Lai and Lu, 2013];
- local search [Lau and Tsang, 1998, Galinier and Hertz, 2006];
- heuristic framework [Phan and Skiena, 2002];
- integer programming [Mak, 2007];
- evolutionary algorithms [Malaguti et al., 2008];
- constraint programming [Prestwich, 2008];
- GRASP [Marti et al., 2010].
Some of our results:
Some of our results:
- simulated annealing [Dias, Freitas and Maculan, 2013],
Our research

Some of our results:
- simulated annealing [Dias, Freitas and Maculan, 2013],
- constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
Some of our results:

- simulated annealing [Dias, Freitas and Maculan, 2013],
- constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
- graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],
Some of our results:

- simulated annealing [Dias, Freitas and Maculan, 2013],
- constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
- graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],
- integer programming [Dias, Freitas, Maculan and Michelon, 2016];
Our research

Some of our results:
- simulated annealing [Dias, Freitas and Maculan, 2013],
- constraint programming [Dias, Freitas, Maculan and Michelon, 2016];
- graph theoretical properties [Freitas, Dias, Maculan and Szwarcfiter, 2016],
- integer programming [Dias, Freitas, Maculan and Michelon, 2016];
- current work: polyhedral combinatorics [Dias, Freitas, Marenco and Maculan, 2017].
Our research

- Polyhedral Combinatorics
- Integer Programming
- Graph Theory
- Constraint Programming

- Nelson Maculan
  Brasil

- Javier Marenco
  Argentina

- Bruno Dias
  Brasil

- Rosiane de Freitas
  Brasil

- Philippe Michelon
  France

- Jayme Szwarcfiter
  Brasil
Happy 75 years old, Jayme!!
Happy 75 years old, Jayme!!

At Jayme’s 65 years old
Aconcagua (Andes) 2007/2008

Valeu.
Prof. JAYME
e amigos da UFRJ!

I owe a high mountain for your 75 years, Jayme!!

de Freitas, IComp/UFAM - Brazil
Happy 75 years old, Jayme!!

At Jayme’s 65 years old
Aconcagua (Andes) 2007/2008

At Jayme’s 70 years old
Elbrus (Russian Caucasus) 2012
Happy 75 years old, Jayme!!

At Jayme’s 65 years old
Aconcagua (Andes) 2007/2008

At Jayme’s 70 years old
Elbrus (Russian Caucasus) 2012

I owe a high mountain for your 75 years, Jayme!!
Integer programming models
Existing integer programming model [Koster, 1999]:

Variables:

- $x_{ik} = \begin{cases} 1 & \text{if color k is assigned to vertex i,} \\ 0 & \text{otherwise.} \end{cases}$

- $y_k = \begin{cases} 1 & \text{if color k is given to any vertex,} \\ 0 & \text{otherwise.} \end{cases}$

$z_{\text{max}} =$ maximum used color (channel).

$C =$ set of possible colors.
Integer programming models

Standard IP model

Existing integer programming model [Koster, 1999]:

- **Variables:**
  - \( x_{ik} = \begin{cases} 
  1 & \text{if color } k \text{ is assigned to vertex } i, \\
  0 & \text{otherwise.} 
  \end{cases} \)

\( y_k = \begin{cases} 
  1 & \text{if color } k \text{ is given to any vertex,} \\
  0 & \text{otherwise} 
  \end{cases} \)

\( z_{\text{max}} \) = maximum used color (channel).

\( C \) = set of possible colors.
Existing integer programming model [Koster, 1999]:

- Variables:
  - \( x_{ik} = \begin{cases} 
  1 & \text{if color } k \text{ is assigned to vertex } i, \\
  0 & \text{otherwise.} 
\end{cases} \)
  - \( y_k = \begin{cases} 
  1 & \text{if color } k \text{ is given to any vertex,} \\
  0 & \text{otherwise} 
\end{cases} \)
Existing integer programming model \cite{Koster,1999}:

- **Variables:**
  - $x_{ik} = \begin{cases} 
    1 & \text{if color } k \text{ is assigned to vertex } i, \\
    0 & \text{otherwise.} 
  \end{cases}$
  - $y_k = \begin{cases} 
    1 & \text{if color } k \text{ is given to any vertex,} \\
    0 & \text{otherwise.}
  \end{cases}$
  - $z_{max} = \text{maximum used color (channel)}$.

- $C = \text{set of possible colors.}$
Standard IP model

Minimize \( z_{\text{max}} \)
Subject to:

1. \[ \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \]
2. \[ x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j}) \]
3. \[ x_{ik} \leq y_k \quad (\forall i \in V; \forall k \in C) \]
4. \[ z_{\text{max}} \geq ky_k \quad (\forall k \in C) \]
5. \[ x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \]
6. \[ y_k \in \{0, 1\} \quad (\forall k \in C) \]
7. \[ z_{\text{max}} \in \mathbb{Z}_+ \]
Standard IP model

Minimize \( z_{\text{max}} \)

Subject to:

1. \( \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \)  \( \triangleright \) Color demands
2. \( x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \ \forall k, m \in C : |k - m| < d_{ij}) \)
3. \( x_{ik} \leq y_k \quad (\forall i \in V; \ \forall k \in C) \)
4. \( z_{\text{max}} \geq ky_k \quad (\forall k \in C) \)
5. \( x_{ik} \in \{0, 1\} \quad (\forall i \in V; \ \forall k \in C) \)
6. \( y_k \in \{0, 1\} \quad (\forall k \in C) \)
7. \( z_{\text{max}} \in \mathbb{Z}_+ \)
Minimize \( z_{\max} \)

Subject to:

1. \[ \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \] **Color demands**
2. \[ x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E; \forall k,m \in C : |k - m| < d_{i,j}) \] **Distance constraints**
3. \[ x_{ik} \leq y_k \quad (\forall i \in V; \forall k \in C) \]
4. \[ z_{\max} \geq ky_k \quad (\forall k \in C) \]
5. \[ x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \]
6. \[ y_k \in \{0, 1\} \quad (\forall k \in C) \]
7. \[ z_{\max} \in \mathbb{Z}_+ \]
Standard IP model

Minimize \( z_{\text{max}} \)
Subject to:

1. \[ \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \]  \( \text{Color demands} \)

2. \[ x_{ik} + x_{jm} \leq 1 \quad (\forall (i,j) \in E; \forall k,m \in C : |k - m| < d_{ij}) \]  \( \text{Distance constraints} \)

3. \[ x_{ik} \leq y_k \quad (\forall i \in V; \forall k \in C) \]  \( \text{Color usage} \)

4. \[ z_{\text{max}} \geq ky_k \quad (\forall k \in C) \]

5. \[ x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \]

6. \[ y_k \in \{0, 1\} \quad (\forall k \in C) \]

7. \[ z_{\text{max}} \in \mathbb{Z}_+ \]
Standard IP model

Minimize \( z_{\text{max}} \)

Subject to:

1. \( \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \quad \text{Color demands} \)
2. \( x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{ij}) \quad \text{Distance constraints} \)
3. \( x_{ik} \leq y_k \quad (\forall i \in V; \forall k \in C) \quad \text{Color usage} \)
4. \( z_{\text{max}} \geq ky_k \quad (\forall k \in C) \quad \text{Maximum used color} \)
5. \( x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \)
6. \( y_k \in \{0, 1\} \quad (\forall k \in C) \)
7. \( z_{\text{max}} \in \mathbb{Z}_+ \)
Standard IP model

Minimize $z_{max}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$  
   - Color demands

2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j})$
   - Distance constraints

3. $x_{ik} \leq y_k \quad (\forall i \in V; \forall k \in C)$  
   - Color usage

4. $z_{max} \geq k y_k \quad (\forall k \in C)$  
   - Maximum used color

5. $x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C)$

6. $y_k \in \{0, 1\} \quad (\forall k \in C)$

7. $z_{max} \in \mathbb{Z}^+$

- Integrality constraints
Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize $z_{\text{max}}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$

2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j})$

3. $z_{\text{max}} \geq k x_{ik} \quad (\forall i \in V; \forall k \in C)$

4. $x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C)$
Improving the standard IP model

Improved IP model with less constraints and variables [Dias et al., 2016]:

Minimize \( z_{\text{max}} \)

Subject to:

1. \( \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \quad \text{Color demands} \)

2. \( x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j}) \)

3. \( z_{\text{max}} \geq kx_{ik} \quad (\forall i \in V; \forall k \in C) \)

4. \( x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \)
Improved IP model with less constraints and variables

[Dias et al., 2016]:

Minimize $z_{\text{max}}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$ \hspace{1cm} \text{Color demands}

2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j})$

3. $z_{\text{max}} \geq k x_{ik} \quad (\forall i \in V; \forall k \in C)$

4. $x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C)$
Improved IP model with less constraints and variables

[Dias et al., 2016]:

Minimize  $z_{\text{max}}$

Subject to:

1. $\sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V)$  \text{Color demands}

2. $x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{ij})$  \text{Distance constraints}

3. $z_{\text{max}} \geq k x_{ik} \quad (\forall i \in V; \forall k \in C)$  \text{Max. used color and color usage}

4. $x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C)$
Improved the standard IP model

Improved IP model with less constraints and variables

[Dias et al., 2016]:

Minimize \( z_{\text{max}} \)
Subject to:

1. \( \sum_{k \in C} x_{ik} = w_i \quad (\forall i \in V) \)  \text{(Color demands)}
2. \( x_{ik} + x_{jm} \leq 1 \quad (\forall (i, j) \in E; \forall k, m \in C : |k - m| < d_{i,j}) \)  \text{(Distance constraints)}
3. \( z_{\text{max}} \geq kx_{ik} \quad (\forall i \in V; \forall k \in C) \)  \text{(Max. used color and color usage)}
4. \( x_{ik} \in \{0, 1\} \quad (\forall i \in V; \forall k \in C) \)  \text{(Integrality constraints)}
Orientation model for the BCP
Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.
Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- **Variables:**
  - $x_i =$ color assigned to vertex $i$ ($x_i \in \mathbb{N}$).
Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- Variables:
  - $x_i = \text{color assigned to vertex } i \ (x_i \in \mathbb{N})$.
  - $y_{ij} = \begin{cases} 
  1 & \text{if } x_i < x_j, \\
  0 & \text{otherwise}.
  \end{cases}$
Orientation IP formulation

Based on the model by [Borndörfer et al., 1998] for the classical vertex coloring problem.

- **Variables:**
  - $x_i =$ color assigned to vertex $i$ ($x_i \in \mathbb{N}$).
  - $y_{ij} = \begin{cases} 
  1 & \text{if } x_i < x_j, \\
  0 & \text{otherwise.}
\end{cases}$
  - $z_{\text{max}} =$ maximum used color (channel).

- $C =$ set of possible colors.
Orientation model for the BCP

Orientation IP formulation

Minimize \( z_{\text{max}} \)

Subject to:

1. \( x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \) \((\forall (i, j) \in E \mid i < j)\)

2. \( x_j + d_{i,j} \leq x_i + Uy_{ij} \) \((\forall (i, j) \in E \mid i < j)\)

3. \( z_{\text{max}} \geq x_i \) \((\forall i \in V)\)

4. \( x_i \in \mathbb{Z}_+ \) \((\forall i \in V)\)

5. \( y_{ij} \in \{0, 1\} \) \((\forall i, j \in V \mid i < j)\)
Orientation IP formulation

Minimize $z_{\text{max}}$

Subject to:

1. $x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \quad (\forall (i,j) \in E \mid i < j)$

   Distance constraint:
   If $x_j - x_i \geq d_{i,j}$, then $x_i < x_j$; set $y_{ij} = 1$

2. $x_j + d_{i,j} \leq x_i + Uy_{ij} \quad (\forall (i,j) \in E \mid i < j)$

3. $z_{\text{max}} \geq x_i \quad (\forall i \in V)$

4. $x_i \in \mathbb{Z}_+$ \quad (\forall i \in V)

5. $y_{ij} \in \{0, 1\} \quad (\forall i, j \in V \mid i < j)$
Orientation model for the BCP

Orientation IP formulation

Minimize \( z_{\text{max}} \)

Subject to:

1. \( x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \) \( (\forall (i, j) \in E \mid i < j) \)

Distance constraint:
If \( x_j - x_i \geq d_{i,j}, \) then \( x_i < x_j: \)
set \( y_{ij} = 1 \)

2. \( x_j + d_{i,j} \leq x_i + Uy_{ij} \) \( (\forall (i, j) \in E \mid i < j) \)

Distance constraint:
If \( x_i - x_j \geq d_{i,j}, \) then \( x_j < x_i: \)
set \( y_{ij} = 0 \)

3. \( z_{\text{max}} \geq x_i \) \( (\forall i \in V) \)

4. \( x_i \in \mathbb{Z}_+ \) \( (\forall i \in V) \)

5. \( y_{ij} \in \{0, 1\} \) \( (\forall i, j \in V \mid i < j) \)
Orientation model for the BCP

Orientation IP formulation

Minimize \( z_{\text{max}} \)

Subject to:

1.  \( x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \) \( (\forall(i,j) \in E \mid i < j) \)

   Distance constraint: If \( x_j - x_i \geq d_{i,j} \), then \( x_i < x_j \):
   set \( y_{ij} = 1 \)

2.  \( x_j + d_{i,j} \leq x_i + Uy_{ij} \) \( (\forall(i,j) \in E \mid i < j) \)

   Distance constraint: If \( x_i - x_j \geq d_{i,j} \), then \( x_j < x_i \):
   set \( y_{ij} = 0 \)

3.  \( z_{\text{max}} \geq x_i \) \( (\forall i \in V) \)

   Maximum used color

4.  \( x_i \in \mathbb{Z}_+ \) \( (\forall i \in V) \)

5.  \( y_{ij} \in \{0, 1\} \) \( (\forall i,j \in V \mid i < j) \)
Minimize \( z_{\text{max}} \)
Subject to:

1. \( x_i + d_{i,j} \leq x_j + U(1 - y_{ij}) \) \( (\forall (i, j) \in E \mid i < j) \)
   - Distance constraint: If \( x_j - x_i \geq d_{i,j} \), then \( x_i < x_j \): set \( y_{ij} = 1 \)

2. \( x_j + d_{i,j} \leq x_i + Uy_{ij} \) \( (\forall (i, j) \in E \mid i < j) \)
   - Distance constraint: If \( x_i - x_j \geq d_{i,j} \), then \( x_j < x_i \): set \( y_{ij} = 0 \)

3. \( z_{\text{max}} \geq x_i \) \( (\forall i \in V) \)
   - Maximum used color

4. \( x_i \in \mathbb{Z}_+ \) \( (\forall i \in V) \)
   - Integrality constraints

5. \( y_{ij} \in \{0, 1\} \) \( (\forall i, j \in V \mid i < j) \)
Orientation IP formulation

Minimize \( z_{\text{max}} \)

Subject to:

1. \( x_i + d_{i,j} \leq x_j + U (1 - y_{ij}) \) (\( \forall (i, j) \in E \mid i < j \))
   - Distance constraint: If \( x_j - x_i \geq d_{i,j} \), then \( x_i < x_j \): set \( y_{ij} = 1 \)

2. \( x_j + d_{i,j} \leq x_i + U y_{ij} \) (\( \forall (i, j) \in E \mid i < j \))
   - Distance constraint: If \( x_i - x_j \geq d_{i,j} \), then \( x_j < x_i \): set \( y_{ij} = 0 \)

3. \( z_{\text{max}} \geq x_i \) (\( \forall i \in V \))
   - Maximum used color

4. \( x_i \in \mathbb{Z}_+ \) (\( \forall i \in V \))
   - Integrality constraints

5. \( y_{ij} \in \{0, 1\} \) (\( \forall i, j \in V \mid i < j \))

This is a new formulation for BCP!
The formulation induces an orientation on the input graph, according to colors.

Values of variables in the optimal solution:

\[ \begin{align*}
\color{green}{1} & \quad \color{blue}{2} \\
\color{blue}{3} & \quad \color{red}{4} \\
\end{align*} \]

- \( \text{color}(1) = 1 \)
- \( \text{color}(2) = 3 \)
- \( \text{color}(3) = 5 \)
- \( \text{color}(4) = 4 \)

\[ \begin{align*}
x_1 &= 1 \\
x_2 &= 3 \\
x_3 &= 5 \\
x_4 &= 4 \\
y_{12} &= 1 \\
y_{13} &= 1 \\
y_{14} &= 1 \\
y_{24} &= 1 \\
y_{34} &= 0 \\
z_{\text{max}} &= 5
\end{align*} \]
Facet-defining inequalities for orientation model
The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.
The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.

If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.
The parameter \( U \in \mathbb{Z}_+ \) in the distance constraints must be an upper bound on available colors.

If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.

**Definition 2: BCP Polytope.**

We define \( PO(G, d, U) \) to be the convex hull of feasible solutions to the previous formulation.
The parameter $U \in \mathbb{Z}_+$ in the distance constraints must be an upper bound on available colors.

If no such a bound is imposed, then the convex hull of the resulting feasible solutions is not a polytope, and an integer programming formulation is not possible in this case.

**Definition 2: BCP Polytope.**

We define $PO(G, d, U)$ to be the convex hull of feasible solutions to the previous formulation.

**Theorem 1.** If $U \geq \chi(G, d) + 2d_{\text{max}}$, then the polytope $PO(G, d, U)$ is full-dimensional.
There are some valid inequalities (facets) proved to the Orientation model - k-coloring:
There are some valid inequalities (facets) proved to the Orientation model - k-coloring:

- Clique inequalities
There are some valid inequalities (facets) proved to the Orientation model - k-coloring:

- Clique inequalities
- Double clique inequalities
For the BCP: additional constraints and variables are needed to use the inequalities:
For the BCP: additional constraints and variables are needed to use the inequalities:

- Variables $y_{ji} = \begin{cases} 
1 & \text{if } x_i < x_j, \\
0 & \text{otherwise.} 
\end{cases}$
For the BCP: additional constraints and variables are needed to use the inequalities:

- Variables $y_{ji} = \begin{cases} 
1 & \text{if } x_i < x_j, \\
0 & \text{otherwise.} 
\end{cases}$

- Constraints $y_{ij} + y_{ji} = 1 \ (\forall i, j \in V)$. 
Definition 3. Let $i \in V$ and consider a clique $K \subseteq N(i)$. We define the clique inequality associated with $i$ and $K$ to be

$$\sum_{j \in K} y_{ji} \leq x_i.$$ 

The clique inequalities strengthen the bounds $x_i \geq 0$. 
Definition 4. Let \( i \in V \) and consider a clique \( K \subseteq N(i) \). For \( k \in K \), we define \( \delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt} \). We define the generalized clique inequality associated with the vertex \( i \) and the clique \( K \) to be

\[
\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.
\]

- The generalized clique cuts introduce the distance constraints to the clique cuts.
**Definition 4.** Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the generalized clique inequality associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
Definition 4. Let \( i \in V \) and consider a clique \( K \subseteq N(i) \). For \( k \in K \), we define \( \delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt} \). We define the generalized clique inequality associated with the vertex \( i \) and the clique \( K \) to be
\[
\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.
\]
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the generalized clique inequality associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
**Definition 4.** Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the generalized clique inequality associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
**Definition 4.** Let \( i \in V \) and consider a clique \( K \subseteq N(i) \). For \( k \in K \), we define \( \delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt} \). We define the *generalized clique inequality* associated with the vertex \( i \) and the clique \( K \) to be
\[
\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.
\]
**Definition 4.** Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the *generalized clique inequality* associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
Definition 4. Let \( i \in V \) and consider a clique \( K \subseteq N(i) \). For \( k \in K \), we define \( \delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt} \). We define the \textit{generalized clique inequality} associated with the vertex \( i \) and the clique \( K \) to be
\[
\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.
\]
Definition 4. Let $i \in V$ and consider a clique $K \subseteq N(i)$. For $k \in K$, we define $\delta^i_K(j) := \min_{t \in K \cup \{i\} \setminus \{j\}} d_{jt}$. We define the generalized clique inequality associated with the vertex $i$ and the clique $K$ to be

$$\sum_{j \in K} \delta^i_K(j) y_{ji} \leq x_i.$$
Theorem 2. The generalized clique inequality is valid for $PO(G, d, U)$. If

(a) $U \geq \chi(G, d) + 3d_{\max}$,
(b) $d_{ij} = \delta^i_K(j)$ for every $j \in K$, and
(c) for every $t \in N(i) \setminus K$ there exists $j \in K$ with $jt \notin E$ and $d_{it} \leq d_{ij}$,

then the generalized clique inequality induces a facet of $PO(G, d, U)$. 
Definition 5. Let \((i, j) \in E\) and consider a clique \(K \subseteq N(i) \cap N(j)\). We define
\[
x_i + 1 + \sum_{v \in K} (y_{ik} - y_{jk}) \leq x_j + U(1 - y_{ij}).
\]
to be the double clique inequality associated with the edge \((i, j)\) and the clique \(K\).
Definition 5. Let \((i, j) \in E\) and consider a clique \(K \subseteq N(i) \cap N(j)\). We define
\[
x_i + 1 + \sum_{v \in K} (y_{ik} - y_{jk}) \leq x_j + U(1 - y_{ij}).
\]
to be the double clique inequality associated with the edge \((i, j)\) and the clique \(K\).

The double-clique inequalities strengthen the model constraints
\[
x_i + d_{i,j} \leq x_j + U(1 - y_{ij}).
\]
Definition 6. Let \((i, j) \in E\) and consider a clique \(K \subseteq N(i) \cap N(j)\). For \(v \in K\), define \(\delta_{K}^{ij}(v) := \min_{l \in K \cup \{i,j\} \setminus \{v\}} d_{v,l}\). Also, fix a vertex \(p \in K\). We define

\[
x_{i} + d_{i,j} + \sum_{v \in K} \gamma_{v} (y_{iv} - y_{jv}) \leq x_{j} + (U + d_{i,j} - \gamma(K)) y_{ji}
\]

to be the **generalized double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_{p} = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\}\), \(\gamma_{v} = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_{v}\).

The generalized clique cuts introduce the distance constraints to the double clique cuts.
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define

\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]

Fix a vertex \(p \in K\). We define

\[
x_i + d_{i, j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i, j} - \gamma(K)) y_{ji}
\]

to be the **double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where

\[
\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i, j}\},
\]

\[
\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i, j}\} \text{ for } v \in K \setminus \{p\}, \text{ and } 
\gamma(K) = \sum_{v \in K} \gamma_v.
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\}\), \(\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).
Definition 6. Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i, j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i, j} - \gamma(K)) y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i, j}\}\), \(\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i, j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).
Definition 6. Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{i,j}^K(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}
\]
to be the **double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{i,j}^K(p) - d_{i,j}\}\), \(\gamma_v = \max\{0, \delta_{i,j}^K(v) - d_{i,j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).
Definition 6. Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define

\[
\delta_{ij}^K(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]

Fix a vertex \(p \in K\). We define

\[
x_i + d_{i, j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i, j} - \gamma(K))y_{ji}
\]

to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{ij}^K(p) - d_{i, j}\}\), \(\gamma_v = \max\{0, \delta_{ij}^K(v) - d_{i, j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K)) y_{ji}
\]
to be the **double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where
\[
\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\},
\]
\[
\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}
\]
for \(v \in K \setminus \{p\}\), and
\[
\gamma(K) = \sum_{v \in K} \gamma_v.
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta^ij_K(v) = \min_{\ell \in K \cup \{i,j\}\setminus\{v\}} d_{v,\ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v(y_iv - y_jv) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where
\[
\gamma_p = \max\{0, 2\delta^ij_K(p) - d_{i,j}\},
\]
\[
\gamma_v = \max\{0, \delta^ij_K(v) - d_{i,j}\} \text{ for } v \in K\setminus\{p\}, \text{ and } \gamma(K) = \sum_{v \in K} \gamma_v.
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i, j} + \sum_{v \in K} \gamma_v(y_{iv} - y_{jv}) \leq x_j + (U + d_{i, j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where 
\[
\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i, j}\},
\]
\[
\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i, j}\}
\]
for \(v \in K \setminus \{p\}\), and 
\[
\gamma(K) = \sum_{v \in K} \gamma_v.
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_{v}(y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where
\[
\gamma_{p} = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\},
\]
\[
\gamma_{v} = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}
\]
for \(v \in K \setminus \{p\}\), and
\[
\gamma(K) = \sum_{v \in K} \gamma_{v}.
\]
Definition 6. Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_{i} + d_{i, j} + \sum_{v \in K} \gamma_{v}(y_{iv} - y_{jv}) \leq x_{j} + (U + d_{i, j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_{p} = \max\{0, 2\delta_{K}^{ij}(p) - d_{i, j}\}\),
\(\gamma_{v} = \max\{0, \delta_{K}^{ij}(v) - d_{i, j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_{v}\).
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{ij}^K(v) = \min_{\ell \in K \cup \{i, j\}\setminus\{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v (y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]
to be the double clique inequality associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{ij}^K(p) - d_{i,j}\}\), \(\gamma_v = \max\{0, \delta_{ij}^K(v) - d_{i,j}\}\) for \(v \in K\setminus\{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).

\[
x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma(K))y_{61}
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define

\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]

Fix a vertex \(p \in K\). We define

\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_{v}(y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]

to be the **double clique inequality**

associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where

\[
\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\},
\]

\[
\gamma_{v} = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}
\]

for \(v \in K \setminus \{p\}\), and

\[
\gamma(K) = \sum_{v \in K} \gamma_{v}.
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define

\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]

Fix a vertex \(p \in K\). We define

\[
x_i + d_{i,j} + \sum_{v \in K} \gamma_v(y_{iv} - y_{jv}) \leq x_j + (U + d_{i,j} - \gamma(K))y_{ji}
\]

to be the **double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where \(\gamma_p = \max\{0, 2\delta_{K}^{ij}(p) - d_{i,j}\}\), \(\gamma_v = \max\{0, \delta_{K}^{ij}(v) - d_{i,j}\}\) for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_v\).

\[
x_1 + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_6 + (U + 2 - \gamma_K)y_{61}
\]
**Definition 6.** Let \((i, j) \in E, K \subseteq N(i) \cap N(j)\) be a clique. For \(v \in K\), define
\[
\delta_{K}^{ij}(v) = \min_{\ell \in K \cup \{i, j\} \setminus \{v\}} d_{v, \ell}.
\]
Fix a vertex \(p \in K\). We define
\[
x_{i} + d_{i, j} + \sum_{v \in K} \gamma_{v}(y_{iv} - y_{jv}) \leq x_{j} + (U + d_{i, j} - \gamma(K))y_{ji}
\]
to be the **double clique inequality** associated with the edge \((i, j)\), the clique \(K\), and the vertex \(p\), where
\[
\gamma_{p} = \max\{0, 2\delta_{K}^{ij}(p) - d_{i, j}\},
\]
\[
\gamma_{v} = \max\{0, \delta_{K}^{ij}(v) - d_{i, j}\}
\]
for \(v \in K \setminus \{p\}\), and \(\gamma(K) = \sum_{v \in K} \gamma_{v}\).

\[
x_{1} + 2 + 2(y_{13} - y_{63}) + 0(y_{14} - y_{64}) + 1(y_{15} - y_{65}) \leq x_{6} + (U + 2 - 3)y_{61}
\]
Theorem 3. The generalized double clique inequality is valid for \( PO(G, d, U) \). If

(a) \( U \geq \chi(G, d) + 4d_{\text{max}} \), and

(b) \( d_{i,v} = d_{j,v} = \delta_{K}^{ij}(v) \) for every \( v \in K \),

(c) \( d_{p,v} = d_{p,j} \) for every \( v \in K \setminus \{p\} \),

(d) \( d_{i,j} \leq \delta_{K}^{ij}(v) \) for every \( v \in K \setminus \{p\} \), and

(e) \( (t, p) \not\in E \) and \( d_{i,t} + d_{t,j} \leq d_{i,j} \) for every \( t \in [N(i) \cap N(j)] \setminus K \)

then the generalized double clique inequality induces a facet of \( PO(G, d, U) \).
Computational experiments
We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.
We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.

Experiments executed on a computer with:
- Intel Core i7-3770 (3.4GHz), 8 cores.
- 8GB of RAM.
- Ubuntu Linux 16.04.2 LTS.
We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.

Experiments executed on a computer with:

- Intel Core i7-3770 (3.4GHz), 8 cores.
- 8GB of RAM.
- Ubuntu Linux 16.04.2 LTS.

We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.

Experiments executed on a computer with:
- Intel Core i7-3770 (3.4GHz), 8 cores.
- 8GB of RAM.
- Ubuntu Linux 16.04.2 LTS.


Time limit: 3600 seconds (1 hour).
We implemented a preliminary cut-and-branch (C&B) algorithm in order to test these inequalities.

Experiments executed on a computer with:
- Intel Core i7-3770 (3.4GHz), 8 cores.
- 8GB of RAM.
- Ubuntu Linux 16.04.2 LTS.


Time limit: 3600 seconds (1 hour).

Instances used: GEOM set (without multicoloring demands) [Trick et al., 2002].
Cut-and-branch for BCP - pseudocode

**Require:** graph $G = (V, E)$, distances $d : E \rightarrow \mathbb{Z}_{\geq 0}$.

**function** CUTANDBRANCH-BCP-ORIENTATION($G = (V, E), d$)

1. $lpOrient \leftarrow ASSEMBLELPMODEL-RELAXATION(G)$
2. $(x, y, z_{max}) \leftarrow LPSOLVER(lpOrient)$
3. If $(x, y, z_{max})$ is not integer then
   1. $H \leftarrow GENERATEMAXIMALCLIQUES(G)$
   2. For each clique $K \in V$ do
      1. ADD-GENERALCLIQUECUT($lpOrient, (x, y, z_{max}), K$)
      2. ADD-GENERALDBLCliQUECUT($lpOrient, (x, y, z_{max}), K$)
      3. $(x, y, z_{max}) \leftarrow LPSOLVER(lpOrient)$
   4. If $(x, y, z_{max})$ is integer then
      1. Break
4. If $(x, y, z_{max})$ is not integer then
   1. $mipOrient \leftarrow CHANGEVARSTOINT(lpOrient)$
   2. $(x, y, z_{max}) \leftarrow B&C-MIPSOLVER(mipOrient)$
5. Return $(x, y, z_{max})$
# Computational experiments - Orientation model

The orientation model has advantages on most problems in comparison with the standard IP model.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Standard model</th>
<th>Orientation model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Best</td>
<td>Time</td>
</tr>
<tr>
<td>GEOM20</td>
<td>21</td>
<td>0.33</td>
</tr>
<tr>
<td>GEOM30</td>
<td>28</td>
<td>0.88</td>
</tr>
<tr>
<td>GEOM40</td>
<td>28</td>
<td>1.97</td>
</tr>
<tr>
<td>GEOM50</td>
<td>28</td>
<td>21.44</td>
</tr>
<tr>
<td>GEOM60</td>
<td>33</td>
<td>45.73</td>
</tr>
<tr>
<td>GEOM70</td>
<td>38</td>
<td>533.53</td>
</tr>
<tr>
<td>GEOM80</td>
<td>41</td>
<td>3019.18</td>
</tr>
<tr>
<td>Instance</td>
<td>Standard model</td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td></td>
<td>Best</td>
<td>Time</td>
</tr>
<tr>
<td>GEOM20</td>
<td>21</td>
<td>0.33</td>
</tr>
<tr>
<td>GEOM30</td>
<td>28</td>
<td>0.88</td>
</tr>
<tr>
<td>GEOM40</td>
<td>28</td>
<td>1.97</td>
</tr>
<tr>
<td>GEOM50</td>
<td>28</td>
<td>21.44</td>
</tr>
<tr>
<td>GEOM60</td>
<td>33</td>
<td>45.73</td>
</tr>
<tr>
<td>GEOM70</td>
<td>38</td>
<td>533.53</td>
</tr>
<tr>
<td>GEOM80</td>
<td>41</td>
<td>3019.18</td>
</tr>
</tbody>
</table>

The orientation model has advantages on most problems in comparison with the standard IP model.
The orientation model has advantages on most problems in comparison with the standard IP model.
The clique and double-clique inequalities are quite useful within the cut-and-branch procedure.

<table>
<thead>
<tr>
<th>Instance</th>
<th>CPLEX</th>
<th>C&amp;B: CPLEX + cuts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Best</td>
<td>Time (s)</td>
</tr>
<tr>
<td>GEOM20</td>
<td>21</td>
<td>0.03</td>
</tr>
<tr>
<td>GEOM30</td>
<td>28</td>
<td>0.22</td>
</tr>
<tr>
<td>GEOM40</td>
<td>28</td>
<td>0.19</td>
</tr>
<tr>
<td>GEOM50</td>
<td>28</td>
<td>4.26</td>
</tr>
<tr>
<td>GEOM60</td>
<td>33</td>
<td>149.60</td>
</tr>
<tr>
<td>GEOM70</td>
<td>38</td>
<td>121.61</td>
</tr>
<tr>
<td>GEOM80</td>
<td>41</td>
<td>2167.43</td>
</tr>
</tbody>
</table>
## Computational experiments - Orientation model: cuts

| Instance | CPLEX | | | |
| --- | --- | --- | --- | --- | |
| | Cliques | Imp.Bnd. | MI Round. | 0-Half | Gomory |
| GEOM20 | 0 | 7 | 14 | 1 | 1 |
| GEOM30 | 0 | 12 | 52 | 8 | 2 |
| GEOM40 | 0 | 13 | 49 | 14 | 4 |
| GEOM50 | 0 | 53 | 123 | 40 | 4 |
| GEOM60 | 0 | 45 | 233 | 54 | 1 |
| GEOM70 | 0 | 135 | 213 | 51 | 1 |
| GEOM80 | 0 | 182 | 318 | 69 | 1 |

| Instance | C&B: CPLEX + cuts | | | |
| --- | --- | --- | --- | --- | |
| | Cliques | Imp.Bnd. | MI Round. | 0-Half | Gomory |
| GEOM20 | 0 | 6 | 1 | 4 | 2 |
| GEOM30 | 1 | 24 | 13 | 3 | 7 |
| GEOM40 | 3 | 33 | 37 | 5 | 6 |
| GEOM50 | 17 | 47 | 72 | 16 | 22 |
| GEOM60 | 21 | 85 | 102 | 23 | 26 |
| GEOM70 | 22 | 94 | 161 | 44 | 23 |
| GEOM80 | 16 | 98 | 170 | 43 | 30 |

More CPLEX clique and Gomory cuts are added when the valid inequalities are included.
Computational experiments

Orientation model and cut-and-branch: discussions

The clique and double-clique inequalities proved to be useful within a cut-and-branch procedure. More CPLEX clique and Gomory cuts are added when the valid inequalities are included. The orientation model seems to be a better platform than the standard model for tackling BCP with integer programming. Further cuts are possible, which indicates that the orientation model is a very competitive approach to BCP.
The clique and double-clique inequalities proved to be useful within a cut-and-branch procedure.

More CPLEX clique and Gomory cuts are added when the valid inequalities are included.

The orientation model seems to be a better platform than the standard model for tackling BCP with integer programming.
The clique and double-clique inequalities proved to be useful within a cut-and-branch procedure.

More CPLEX clique and Gomory cuts are added when the valid inequalities are included.

The orientation model seems to be a better platform than the standard model for tackling BCP with integer programming.

Further cuts are possible, which indicates that the orientation model is a very competitive approach to BCP.
Concluding remarks
We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.
Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.

- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.

Current work:

- Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.
- Try to apply the distance model for the BCP.
Concluding remarks

- We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.

- We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.

- **Current work:**
  - Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.
  - Try to apply the distance model for the BCP.
We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.

We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.

**Current work:**

- Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.
We proposed a new orientation-based IP formulation for the Bandwidth Coloring Problem, which seems to be a better platform than the standard model for tackling BCP with integer programming.

We made a preliminary study of the polytope associated to the new model, including the definition of facet-defining cutting planes for it.

**Current work:**
- Search for more facet-inducing inequalities, and implement a branch-and-cut procedure for BCP.
- Try to apply the distance model for the BCP.
References
References


Merci!
Thank you!

Rosiane de Freitas
rosiane@icomp.ufam.edu.br

Supported by: