On the Existence of Critical Clique-Helly Graphs

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Conjecture [Dourado, Protti and Szwarcfiter]

Every clique-Helly graph $G$ (the family of maximal cliques of the graph satisfies the Helly property) contains a vertex $v$ such that $G - v$ is a clique-Helly graph.

A **complete set** of $G$ is a subset of $V(G)$ inducing a complete subgraph. A **clique** is a maximal complete set (with respect to the inclusion relation).
A set family $\mathcal{F}$ satisfies the **Helly property** if the intersection of all the members of any pairwise intersecting subfamily of $\mathcal{F}$ is non-empty. When the cliques family of $G$, $\mathcal{C}(G)$, satisfies the Helly property, we say that $G$ is a **clique-Helly graph**.
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The way we will build a counterexample to the conjecture:

\[ \text{Icosahedron} \]

\[ \downarrow \]

\[ \text{Icosahedron} \times K_3 \quad \text{(tensor product)} \]

\[ \downarrow \]

\[ K(\text{Icosahedron} \times K_3) \quad \text{(clique graph)} \]
The icosahedron $I$ is one of the platonic graphs.
\* Every vertex has degree 5.
* The open neighborhood of each vertex induces a $C_5$. 
**The cliques are all triangles.**
* Every vertex is in exactly 5 cliques.
The tensor product $I \times K_3$ is the graph with:

- vertices $(i, j)$ where $i \in V(I)$ and $j \in V(K_3)$ and
- two vertices $(i, j)$ and $(i', j')$ adjacent in $P$ if and only if
  
  $i$ is adjacent to $i'$ in $I$ and
  $j$ is adjacent to $j'$ in $K_3$. 
The tensor product $I \times K_3$

* Every vertex of $P$ has degree 10.
* The open neighborhood of each vertex of $P$ induces a $C_{10}$.
* The cliques of $P$ are triangles $\{(i, 1), (j, 2), (k, 3)\}$ for any triangle of $I$.
* Every vertex of $P$ is in exactly ten cliques.
The clique graph $K(I \times K_3)$

- The **clique graph** of $I \times K_3$ is the intersection graph of the cliques family of $I \times K_3$.

It has 120 vertices.....
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A example of clique graph

![Clique Graph Example](image-url)
Main result of this work

Theorem

The graph $G = K(I \times K_3)$ is clique-Helly and for each $v \in G$, $G - v$ is not clique-Helly.

Those graphs satisfying the conditions of the theorem are called critical clique-Helly.
To prove this we will use

**Theorem (Larrión, Neumann-Lara, Pizaña)**

*If the local girth of the graph $G$ is greater than 6 (i.e. $lg(G) \geq 7$) then $K(G)$ is clique-Helly.*

The **local girth** of $G$ **at a vertex** $v \in V(G)$ is the length of a shortest chordless cycle of the subgraph induced by the open neighborhood of $v$ in $G$.  
And the **local girth** of $G$ is the minimum of the local girth at all the vertices $v$. 
As we saw in the properties of $P$ the open neighborhood of each vertex induces a $C_{10}$
$K(I \times K_3) - v$ is not clique-Helly for all $v \in V(K(I \times K_3))$

- Every vertex $v$ of $G = K(I \times K_3)$ represents a clique $Q_v$ of $I \times K_3$, say $Q_v = \{x, y, z\}$.
- Each of these vertices is the center of a 10-wheel.
- Each pair of those wheels have two triangles in common.
- And the total intersection between them is $Q_v$.
- So, if we remove $v$ of $G$ the corresponding cliques of the wheels are pairwise intersecting with empty total intersection.
The graph \( K(I \times K_3) - v \) is not clique-Helly for all \( v \in V(K(I \times K_3)) \)

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Future work

- Show that there is an infinite family of critical clique-Helly graphs.
- Show that there is an infinite family of critical clique-Helly self clique graphs.
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Thank you!