On the (di)graphs with (directed) proper connection number two

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Proper colorings

Well known...

- four color map problem [Appel and Haken (1989)]
- chromatic number problem - NP-complete
- for vertices / edges
- many applications - avoid conflicts
  - register allocation
  - scheduling problems
  - interference, security in communication networks
Proper path colorings

Relaxed constrains

- Not necessary to impose constrains on the colors for all pairs of adjacent edges/vertices in order avoid conflicts, but:
  - assure paths between any pair of vertices on which communication is safe
  - Possible advantages: less colors, algorithms
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  - assure paths between any pair of vertices on which communication is safe
  - Possible advantages: less colors, algorithms
- **Proper path coloring** - proper connection number
  - Borozan et al. (2012)
Proper path colorings

Definition (Borozan et al. (2012); Andrews et al. (2016))

\[ c : E(G) \rightarrow \{1, 2, \ldots, k\} \]

- **proper path** \( P \): \( c|_{E(P)} \) proper edge-coloring
- **proper connection number** of \( G \): \( pc_e(G) \)
- similar - **proper vertex-connection number** of \( G \): \( pc_v(G) \)
- **proper connection for strong digraphs**: for every ordered pair \( u,v \) of vertices exists a proper (di)path from \( u \) to \( v \).

Related to rainbow coloring - [Chartrand et al. (2008)] computing the rainbow connection number is NP-hard and not FPT for any fixed \( k \geq 2 \) [Chakraborty et al. (2011)]

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Known results

**Survey** - [Li and Magnant (2015)]

Combinatorial results:
- Existence problems [Andrews et al. (2016)]
  - difference between chromatic number and pc can be arbitrary large

- Connection to structure properties of graph
  - minimum degree, domination, connectivity [Li et al. (2015)]
Known results

Combinatorial results:

- Extremal graphs [Laforge et al. (2016)]
  - graphs with $pc(G) = m - 1, m - 2$
  - graphs with $pc = 2$ - no complete characterization or algorithmic results
    - 3-connected, 2-connected with diameter 2, bipartite bridgeless
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- Classes of graphs
  - random graphs [Gu et al. (2016)]
    - Almost all graphs have proper connection number 2
  - bipartite graphs [Borozan et al. (2012); Huang et al. (2015, 2016)]
    - sufficient conditions to have proper connection number 2
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- **Digraphs** [Magnant et al. (2016)]
  - $\overrightarrow{pc_e}(D) \leq 3$
  - **Conjecture**: A strong digraph with no even dicycle has $\overrightarrow{pc_e}(D) = 3$. 
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- Proper vertex-connection for undirected graphs - trivial
Our results

- No algorithmic results
- Bipartite graphs with $pc = 2$ - no complete characterization or algorithms

Polynomial-time recognition algorithms for bounded-treewidth graphs and bipartite graphs with $pc = 2$

Characterization of bipartite graphs with $pc = 2$
Our results

- **Digraphs**
  - \(\overrightarrow{pc_e}(D) \leq 3\)

  Deciding whether \(\overrightarrow{pc_e}(D) \leq 2\) is NP-complete.
  - Reduction from Positive NAE-SAT

- **Conjecture**: A strong digraph with no even dicycle has \(\overrightarrow{pc_e}(D) = 3\)

  There exists an infinite family of digraphs with no even dicycles that also have properly connected 2-colorings.
Our results

- **Digraphs**
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- **Proper vertex-connection** for undirected graphs - trivial

  - Initiate study of proper vertex-connection for digraphs
  - $\overrightarrow{pc_v}(D) \leq 3$
  - Deciding whether $\overrightarrow{pc_v}(D) \leq 2$ is NP-complete
    - Reduction from 3-SAT
Bipartite graphs

Lemma (Huang et al. (2015))

If $G$ is a connected bipartite bridgeless graph, then $pc_e(G) \leq 2$. Furthermore, such a coloring can be produced with the strong property.

Coloring with strong property - ear decomposition
Bipartite graphs

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Remark

- $pc_e(G) \geq b(G) = \text{maximum number of bridges incident in a vertex}$
  
  [Andrews et al. (2016)]

- $pc_e(G) \leq 2 \implies b(G) \leq 2$
Bipartite graphs

Any bipartite graph with \( b(G) \leq 2 \) has \( pc_e(G) \leq 2 \)?
Bipartite graphs

Any bipartite graph with $b(G) \leq 2$ has $pce(G) \leq 2$?

No

Lemma

Let $G = (V, E)$ be a connected graph, $B$ be a bridge-block of $G$ that is bipartite. If $B$ is incident to at least three bridges then $pce(G) \geq 3$. 

![Diagram of a bipartite graph with $pce(G) = 3$]
Bipartite graphs
Bipartite graphs

paths of same parity

(Di)graphs with proper connection 2
Bipartite graphs

paths of same parity

(Di)graphs with proper connection 2
Bipartite graphs

paths of same parity

even length
Bipartite graphs

**Theorem**

Let $G = (V, E)$ be a connected bipartite graph. We have $pc_e(G) \leq 2$ if and only if the bridge-block tree of $G$ is a path. Furthermore, if $pc_e(G) \leq 2$, then such a coloring can be computed in linear-time.
Bipartite graphs

If bridge-block tree of $G$ is a path $\Rightarrow$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color blocks in this order (with strong property)

![Diagram showing bridge-blocks $B_0, B_1, B_2, B_3, B_4$.](image)
Bipartite graphs

If bridge-block tree of $G$ is a path $\implies$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color $B_0$ (with strong property)
Bipartite graphs

If bridge-block tree of $G$ is a path $\implies$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color bridge between $B_0$ and $B_1$ arbitrary
Bipartite graphs

If bridge-block tree of $G$ is a path $\implies$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color $B_1$ (with strong property)
Bipartite graphs

If bridge-block tree of $G$ is a path $\implies$ linear ordering $B_0, B_1, \ldots, B_l$ over the bridge-blocks.

- Color bridge between $B_i$ and $B_{i+1}$ according to color of bridge between $B_{i-1}$ and $B_i$ and parity of paths length

![Diagram of Bipartite Graphs]

$B_0$ $B_1$ $B_2$ $B_3$ $B_4$
Bipartite graphs

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\begin{figure}
\centering
\includegraphics[width=\textwidth]{bipartite_graph.png}
\end{figure}
Bipartite graphs

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Conclusions

- Complete characterization of bipartite graphs with $pc = 2$
- First algorithmic and complexity results on proper connection

Open problems
- NP-completeness results and algorithms for undirected case and related type of colorings
  - characterize graphs with $pc \leq 2$
  - proper connection number of bipartite graphs
- Conditions for strong digraphs to have proper edge/vertex connection number 2 or 3.


Thank You!
Proposition (McCuaig (2015))

There is only one strongly 2-connected digraph with no even dicycle (up to an isomorphism), digraph $D_7$.

Figure: $D_7$. 
Strongly 2-connected digraphs with no even dicycle

**Lemma**

\[ \overrightarrow{pc}(D_7) = 2. \]
Theorem

There is an infinite family of strongly connected digraphs with no even dicycle having proper connection number equal to 2
NP-completeness result

**Theorem**

Deciding whether $\overrightarrow{pc}_e(D) \leq 2$ for a given digraph $D$ is NP-complete.

**Proof.**

- **NP-hard**: reduction from Positive NAE-SAT

**Problem (Positive NAE-SAT)**

**Input**: A propositional formula $\Phi$ in conjunctive normal form, with unnegated variables.

**Question**: Does there exist a truth assignment satisfying $\Phi$ in which no clause has all its literals valued 1?
Figure: $D_\Phi$ for $\Phi = (x_1 \lor x_2 \lor x_3) \land (x_2 \lor x_3 \lor x_4)$
Clause gadget

**Figure:** Gadget representing the clause $C_j = x_{i_1} \lor x_{i_2} \lor x_{i_3}$. 

(Bibliography)
Clause gadget

- unique paths: $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$ and $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j]$

**Figure:** Gadget representing the clause $C_j = x_{i_1} \lor x_{i_2} \lor x_{i_3}$.
Clause gadget

- unique paths: $[\alpha_j, \beta_j, C_j, \gamma_j, \delta_j]$ and $[\gamma_j, \delta_j, C_j, \alpha_j, \beta_j] \implies$ arcs with same color

Figure: Gadget representing the clause $C_j = x_{i_1} \lor x_{i_2} \lor x_{i_3}$. 
Proper vertex connection number of digraphs

Figure: $D_\Phi$ for $\Phi = (x_1 \lor x_2 \lor x_3) \land (x_2 \lor \overline{x_3})$