Bounds on Directed star arboricity in some digraph classes

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2 Known results and Conjectures

3 $k$-degenerate digraphs

4 Tournaments

5 Conclusion
A **directed star** is a star with all its arcs oriented towards its center. A **galaxy** is a set of vertex-disjoint stars.
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Introduction

Definitions

Directed star $k$-coloring problem:
Deciding whether or not there exists a partition of the arcs of a digraph $D$ into $k$ galaxies.

$NP$-complete for $k \geq 3$ even when restricted to different classes of digraphs (Amini et al. 2010, Baïou et al. 2013).

The directed star arboricity, $dst(D)$, of a digraph $D$, is the minimum number of galaxies needed to cover all the arcs of $D$. 
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Known results and Conjectures

For every digraph $D$, $dst(D) \leq 2\Delta^+ + 1$.

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Conjectures (O. Amini, F. Havet, F. Huc, and S. Thomassé (2010))

1. $dst(D) \leq 2\Delta^+$, if $\Delta^+ \geq 2$
2. $dst(D) \leq \Delta$, if $\Delta \geq 3$

$\Delta(D) = \max\{d^+(x) + d^-(x), x \in V(D)\}$
<table>
<thead>
<tr>
<th>Conjectures (O. Amini, F. Havet, F. Huc, and S. Thomassé (2010))</th>
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<tbody>
<tr>
<td>1. ( \text{dst}(D) \leq 2\Delta^+, \text{ if } \Delta^+ \geq 2 )</td>
</tr>
<tr>
<td>2. ( \text{dst}(D) \leq \Delta, \text{ if } \Delta \geq 3 )</td>
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- Both conjectures are true when restricted to acyclic digraphs.
- Both conjectures (if true) are tight.
- Conjecture 2 is true for \( \Delta = 3 \).
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A digraph $D$ is \textbf{$k$-degenerate} if its underlying graph is $k$-degenerate.
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$k$-degenerate graph

$G$ is empty, or
There is some $x$ with degree at most $k$ such that $(G - x)$ is $k$-degenerate.
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A graph $G$ is $k$-degenerate if

1. $G$ is empty, or
2. There is some vertex $x$ with degree at most $k$ such that $(G - x)$ is $k$-degenerate.
Theorem 1

Let $D$ be a $k$-degenerate digraph, $\text{dst}(D) \leq \Delta^+ + k$.

Sketch of proof

Proof by induction on the number of vertices of $D$. 
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Proof by induction on the number of vertices of $D$.

$H_n : \text{all } k\text{-degenerate oriented graphs on } n \text{ vertices are } (\Delta^+ + k)\text{-colorable with at most } k \text{ colors entering each vertex.}$
Illustration for $\Delta^+ = 2$ and $k = 2$. 

$$d^+(x) = 1$$
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2. Known results and Conjectures

3. $k$-degenerate digraphs

4. Tournaments

5. Conclusion
A **Tournament** is an orientation of a complete graph.

**Theorem 2**

Let $T$ be a tournament on $n$ vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

**Corollary:** if $n \geq 4$, then $dst(T) \leq 2\Delta^+$. 
A **Tournament** is an orientation of a complete graph.

### Theorem 2

*Let $T$ be a tournament on $n$ vertices, $n \geq 4$, then $\text{dst}(T) \leq \Delta$.***

**Corollary:** if $n \geq 4$, then $\text{dst}(T) \leq 2\Delta^+$. 

If $n$ is even there is nothing to prove (partition in $n - 1$ perfect matchings).
Proof

**Theorem 2**

Let $T$ be a tournament on $n$ vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

Sketch of the proof:

- remove one vertex,
- color the resulting even sub-tournament.
- extend the coloring using only one additional color.
Proof

Theorem 2

Let $T$ be a tournament on $n$ vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

We color the arc entering $u$ with the new color.

$$d^+(u) \geq d^-(u)$$
$$d^-(u) > 0$$
Proof

Theorem 2

Let $T$ be a tournament on $n$ vertices, $n \geq 4$, then $dst(T) \leq \Delta$.

Goal: color the arcs leaving $u$.

$d^+(u) \geq d^-(u)$

$d^-(u) > 0$
Assign a color to each arc leaving $u$. 

\[ N^+ \]

\[ C_1 \]

\[ C_{n-2} \]
Assign a color to each arc leaving $u$.

Find a maximum matching.
Color the remaining arcs.

\[ |N^+| = k \]
Color the remaining arcs.

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2. $\text{dst}(D) \leq \Delta, \text{ if } \Delta \geq 3$

- Conclusions of both conjectures are valid for tournaments.
- $k$-degenerate oriented graphs verify Conjecture (1) if $k \leq \Delta^+$.
- The main conjectures remain open.
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Thanks for your attention