Bispindle in strongly connected digraphs with large chromatic number

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Definition

• The *chromatic number*, $\chi(D)$, of a digraph $D$ is the chromatic number of its underlying graph.

• A directed path (dipath) is a path where all the arcs are oriented in the same direction.

Question

• What are the subgraphs of a digraph with large chromatic number?

• For a digraph $D$, can we bound the chromatic number of $D$-free digraphs?
Known results

Theorem (Gallai-Hasse-Roy-Vitaver, 60s)
\[ \chi(D) = k \Rightarrow D \text{ has a directed path with } k \text{ vertices.} \]
This can only be extended to acyclic digraphs:

Theorem (Erdős)
There are digraphs with arbitrarily large girth and chromatic number.

Conjecture (Burr, 1980)
Every digraph of chromatic number \(2k - 2\) contains all oriented tree of order \(k\)

Best bound is \(k^2/2\) by Addario-berry et al.
Definition
A subdivision of a digraph $D$ is obtained by replacing arcs by dipaths.

Theorem (Cohen, Havet, L. and Nisse)
For any oriented cycle $C$, there exists digraphs with arbitrarily large chromatic number and without a subdivision of $C$.

The question is different in strong digraphs:

Theorem (Bondy, 1976)
$D$ is strong, $\chi(D) = k \Rightarrow D$ has a directed cycle with at least $k$ vertices.
Definition
In an oriented path or cycle, a block is a maximal directed sub-path.

Theorem (Cohen, Havet, L. and Nisse)
Let $C$ be an oriented cycle with two blocks. There exists $f(C)$ s.t:
If $D$ is a strong digraph with $\chi(D) \geq f(C)$, then $D$ contains a subdivision of $C$. 
Disjoint path between a pair of vertices

A cycle on two blocks can be seen as two disjoint dipaths between a pair of vertices.

Question
Can we generalise this question for more dipaths?
Proposition

There exists strong digraphs with arbitrarily large chromatic number avoiding the following digraphs as subdivision:
Definition
Let $B(k_1, k_2; k_3)$ be the union of three dipaths, two of length $k_1$ and $k_2$ in one direction, and one of length $k_3$ in the other.

Theorem (Cohen, Havet, L. and Lopes)
For every $k$, there exists $f(k)$ such that every strong digraph $D$ with $\chi(D) \geq f(k)$ contains a subdivision of $B(k, 1; k)$.

I will present the proof for $B(k, 1; 1)$. 
Hamiltonian case

Theorem

Any Hamiltonian digraph on \( n > k + 3 \) vertices without a \( B(k, 1; 1) \) is \( 2k - 1 \) degenerate.

The maximum out degree of a vertex is \( k - 1 \).
Theorem

Any Hamiltonian digraph on \( n > k + 3 \) vertices without a \( B(k, 1; 1) \) is \( 2k - 1 \) degenerate.

The maximum out degree of a vertex is \( k - 1 \).
If $D$ is a $B(k, 1; 1)$-free strong digraph with large chromatic number:

- Bondy’s Theorem implies the existence of a large cycle.
- Large cycles have a simple structure.

So the idea is to contract long cycles such that:

- We control the chromatic number of the contracted structure.
- There are no long cycle remaining.
Lemma
Let $D$ be a digraph, $D_1 \ldots D_l$ be disjoint subdigraphs of $D$ and $D'$ the digraph obtained by contracting each $D_i$ into one vertex $d_i$. Then $\chi(D) \leq \chi(D') \cdot \max\{\chi(D_i) \mid i \in [l]\}$.

Proof.
Let $c'$ be a colouring of $D'$ and $c_i$ be colouring of the $D_i$. Define $c$ a colouring of $D$ as follows:

- if $x \in D_i$, $c(x) = (c'(d_i), c_i(x))$.
- else $c(x) = (c'(x), 1)$.
We say a set of cycles $\mathcal{C}$ of $D$ is *nice* if:

- All cycles of $\mathcal{C}$ are longer than $2k$.
- Any two distinct cycles $C_1, C_2$ of $\mathcal{C}$ intersect in at most one vertex.
Components of $\mathcal{C}$

**Definition**

- Two cycles of $\mathcal{C}$ are *adjacent* if they intersect.
- A *component* of $\mathcal{C}$ is a set of cycle forming a connected component in the adjacency graph of the cycle of $\mathcal{C}$.
Let $D$ be a strong digraph without $B(k, 1; 1)$.

Take $C$ a maximal collection of cycles and call $D'$ the digraph obtained by contracting each component into one vertex.

We will prove the two following lemmas:

**Lemma 1**
For each component $S$ of $C$, $\chi(D[S]) \leq 2k$

**Lemma 2**
$\chi(D') \leq 2k$

Which will prove that $\chi(D) \leq 4k^2$. 
Lemma

Let $C_1$ and $C_2$ be two cycles of the same component of $C$, then there is no dipath from $C_1$ to $C_2$ outside of $C$. 
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Lemma

Let $C$ be a nice collection of cycles and $S$ a component, then $\chi(D[S]) \leq 2k$

Proof.

By induction on the number of cycles in $S$. \qed
Proposition

There is no arcs between the new components.

This would contradict the headphone lemma.
Proposition

*Each new component intersects the blue cycle in one vertex.*

- Each cycle intersects the blue cycle in at most one vertex.
- Two cycles intersecting the blue cycle contradicts the headphone Lemma.
We can prove the result:

- Colour the blue cycle by degeneracy.
- Apply induction on each new component, where one vertex is already coloured.
Proposition

- $D'$ is strongly connected
- $D'$ has no cycle longer than $2k - 1$

Bondy’s result then implies that $\chi(D') \leq 2k$. 
Suppose there exists a cycle $C'$ longer than $2k$ in $D'$.
Each contracted vertex can be replaced by a path
This means we obtain a cycle $C$ of $D$. 
This means we obtain a cycle $C$ of $D$. 
Finally, it is easy to show that:

- $C$ is longer than $2k$.
- It intersect each other cycle of $C$ in at most 1 vertex.

Which contradicts the maximality of $C$!
The ideas are similar, but we allow cycles to intersect on a short path.

- Proving that the contracted digraph is without any long cycle is easy.
- Bounding the chromatic number of the components is way more difficult.
Thank you!