Correspondence Homomorphisms

Pavol Hell
and
Tomás Feder

LAGOS 2017
Classical Colouring

A graph $G$ with $n$ vertices is $k$-colourable $\iff G \cong K^k_n$ has $n$ independent vertices.

Plesnivich + Vizing 1965

Cartesian product

Correspondence Homomorphisms
Graph \( G \) with \( n \) vertices

\[
G \text{ is } k\text{-colourable} \iff G \square K_k \text{ has } n \text{ independent vertices}
\]

Plesnevič + Vizing 1965
Classical Colouring

Graph $G$ with $n$ vertices

$G$ is $k$-colourable $\iff$ $G \square K_k$ has $n$ independent vertices

Plesnevič + Vizing 1965

Cartesian product

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{k}
\end{array}
\]

\[
\begin{array}{ccc}
G & G & G \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}
\]
Graph $G$ with $n$ vertices

$G$ is $k$-colourable $\iff G \square K_k$ has $n$ independent vertices

Plesnevič + Vizing 1965

Cartesian product

\[
\begin{array}{c}
G \\
1 \\
\end{array} \square
\begin{array}{c}
G \\
2 \\
\end{array} \square
\begin{array}{c}
G \\
k \\
\end{array}
\]
Graph $G$ with $n$ vertices

$G$ is $k$-colourable $\iff G \Box K_k$ has an independent transversal

Plesnevič + Vizing 1965

Cartesian product
$C_4$ is 2-colourable
$C_4$ is 2-colourable
$C_4$ is 2-colourable
List Colouring

Correspondence Homomorphisms
$C_4$ is 2-choosable
$C_4$ is 2-choosable
$C_4$ is 2-list-colourable
Correspondence Colouring

Forget the labels

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Correspondence Homomorphisms
Products $G \square^* K_k$

Put a partial matching between every two adjacent copies of $K_k$
Many possible products
Products $G \Box^* K_k$

Put a partial matching between every two adjacent copies of $K_k$

Many possible products

$G$ is $k$-correspondence-colourable

Each product $G \Box^* K_k$ has an independent transversal
Example $C_4 \square^* K_2$
Example $C_4 \square^* K_2$
Example $C_4 \square^* K_2$
$C_4$ is not 2-correspondence-colourable
$C_4$ is 3-correspondence-colourable
Correspondence chromatic number of $G$

Smallest $k$ such that $G$ is $k$-correspondence-colourable
Correspondence Colouring

Correspondence chromatic number of $G$

Smallest $k$ such that $G$ is $k$-correspondence-colourable

Correspondence chromatic number of $G$

$$\chi(G) \leq \chi_{list}(G) \leq \chi_{corr}(G)$$
Correspondence Colouring

Correspondence chromatic number of $G$
Smallest $k$ such that $G$ is $k$-correspondence-colourable

Correspondence chromatic number of $G$

$$\chi(G) \leq \chi_{\text{list}}(G) \leq \chi_{\text{corr}}(G)$$

Example results

- $\chi_{\text{corr}}(G) \leq k + 1$ for $k$-degenerate graphs
- $\chi_{\text{corr}}(G) \leq 3$ for triangle-free planar graphs
- $\chi_{\text{corr}}(G) \leq 5$ for planar graphs

Dvořák + Postle 2016
Fleischner + Steibitz 1992

Any cycle+triangles graph is 3-colourable

(Asked by Erdős 1990)
Useful Generalizations of Colouring

Fleischner + Steibitz 1992
Any cycle+triangles graph is 3-colourable
(Asked by Erdős 1990)
(Actually prove 3-choosability)
Useful Generalizations of Colouring

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Any cycle+triangles graph is 3-colourable
(Asked by Erdős 1990)
(Actually prove 3-choosability)

Dvořák Postle 2016
Every planar graph without 4-, 5-, 6-, 7-, and 8-cycles is 3-choosable.
(Asked by Borodin 2013)
Useful Generalizations of Colouring

**Fleischner + Steibitz 1992**
Any cycle+triangles graph is 3-colourable

(Asked by Erdős 1990)
(Actually prove 3-choosability)

**Dvořák Postle 2016**
Every planar graph without 4-, 5-, 6-, 7-, and 8-cycles is 3-choosable.

(Asked by Borodin 2013)
(Actually prove 3-correspondence colourability for certain cases that include list colouring)
Always consider perfect matchings

Get partial matching by restricting via lists
Always consider perfect matchings

Get partial matching by restricting via lists

Example
Change of Notation

Always consider perfect matchings
Get partial matching by restricting via lists

Example
$G$ is correspondence $k$-colourable

Each product $G \square^* K_k$ has an independent transversal

$C_4$ is 3-correspondence-colourable
Change of Notation

$G \square K_k$ versus $G \times K_k$
Change of Notation

$G \boxtimes K_k$ versus $G \times K_k$

$K_2 \times K_k$ is the complement of a matching
$G \square K_k$ versus $G \times K_k$

$K_2 \times K_k$ is the complement of a matching

Switch to complements of matchings
Change of Notation

\( G \square K_k \) versus \( G \times K_k \)

Independent transversal in \( G \square K_k = G \)-transversal in \( G \times K_k \)

Example

\[ \text{Diagram showing the comparison between } G \square K_k \text{ and } G \times K_k \]
Change of Notation

\[ G \square K_k \text{ versus } G \times K_k \]

\( G \text{ is } k\)-colourable \iff \( G \times K_k \) has a \( G \)-transversal

Example
Correspondence $k$-colouring

Consider products $G \times^* K_k$ with complements of matchings

Example
Change of Notation

$G$ is correspondence $k$-colourable

Each product $G \times^* K_k$ has $G$-transversal

Example
Change of Notation

$G$ is correspondence $k$-colourable

Each product $G \times^* K_k$ has $G$-transversal

Example
Homomorphism

$H$-colouring of $G$

A $G$-transversal of $G \times H$

Example

\[
\begin{array}{c}
G \\
\hline \\
\end{array} 
\hspace{2cm} 
\begin{array}{c}
G \times H \\
\hline \\
\end{array} 
\hspace{2cm} 
\begin{array}{c}
H \\
\hline \\
\end{array}
\]
**Homomorphism**

*H*-colouring of *G*

A *G*-transversal of *G* × *H*

**Example**

![Diagram of graphs](image)
$H$-colouring of $G$

A $G$-transversal of $G \times H$

Example

\begin{figure}
\centering
\begin{tikzpicture}
\node (G) at (0,0) [circle, draw] {$G$};
\node (H) at (1,0) [circle, draw] {$H$};
\node (GxH) at (2,0) [circle, draw] {$G \times H$};
\end{tikzpicture}
\end{figure}
Correspondence Homomorphism

Correspondence $H$-colouring of $G$

A $G$-transversal of some $G \times^* H$
Correspondence Homomorphism

Correspondence $H$-colouring of $G$

A $G$-transversal of some $G \times^* H$

Products $G \times^* H$

Put an isomorphic copy of $K_2 \times H$ between every two adjacent copies of $H$
Correspondence Homomorphism

Correspondence $H$-colouring of $G$

A $G$-transversal of some $G \times^* H$

Example

\[
\begin{align*}
G & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad H
\end{align*}
\]
Correspondence Homomorphism

Correspondence $H$-colouring of $G$

A $G$-transversal of some $G \times^* H$

Example

\[ G \quad \Rightarrow \quad G \times^* H \quad \Rightarrow \quad H \]
The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?
The Correspondence Homomorphism Problem

The correspondence $H$-colouring problem
Given some $G \times^* H$, is there a $G$-transversal?

The $H$-colouring problem
Given $G$, is there an $H$-colouring of $G$?
The correspondence $H$-colouring problem
Given some $G \times^* H$, is there a $G$-transversal?

The $H$-colouring problem
Given $G$, is there an $H$-colouring of $G$?
(I.e., given $G$, does $G \times H$ have a $G$-transversal?)
The correspondence $H$-colouring problem
Given some $G \times^* H$, is there a $G$-transversal?

The $H$-colouring problem
Given $G$, is there an $H$-colouring of $G$?
(I.e., given $G$, does $G \times H$ have a $G$-transversal?)

The $k$-colouring problem
Given $G$, is there a $k$-colouring of $G$?
The Correspondence Homomorphism Problem

The correspondence $H$-colouring problem
Given some $G \times^* H$, is there a $G$-transversal?

The $H$-colouring problem
Given $G$, is there an $H$-colouring of $G$?
(I.e., given $G$, does $G \times H$ have a $G$-transversal?)

The $k$-colouring problem
Given $G$, is there a $k$-colouring of $G$?
(I.e., given $G$, does $G \times K_k$ have a $G$-transversal?)
The Correspondence Homomorphism Problem

The correspondence $H$-colouring problem
Given some $G \times^* H$, is there a $G$-transversal?

The $H$-colouring problem
Given $G \times H$, is there a $G$-transversal? (i.e., is there a homomorphism $G \rightarrow H$?)

The $k$-colouring problem
Given $G \times K_k$, is there a $G$-transversal? (i.e., is there a $k$-colouring of $G$?)

The $k$-colouring problem
Polynomial-time solvable when $k \leq 2$, NP-complete otherwise
The correspondence $H$-colouring problem

Given some $G \times * H$, is there a $G$-transversal?

The $H$-colouring problem

Given $G \times H$, is there a $G$-transversal? (i.e., is there a homomorphism $G \rightarrow H$?)

The $H$-colouring problem

Polynomial-time solvable when $H$ has a loop or is is bipartite, NP-complete otherwise

H+Nesetril 1990
The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

The correspondence $H$-colouring problem

Is there a dichotomy classification?
The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

<table>
<thead>
<tr>
<th>Clauses</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + y + z$</td>
<td>$x, y, z$</td>
</tr>
</tbody>
</table>
The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

NP-complete when $H$ is reflexive $K_1 + K_2$

\[ \begin{array}{ccc}
  \mathcal{H} & b & \mathcal{H} \\
  a & & c \\
 \end{array} \quad H \]
The correspondence $H$-colouring problem

Given some $G \times \ast H$, is there a $G$-transversal?

NP-complete when $H$ is reflexive $K_1 + K_2$

Reduce from 1-in-3-SAT without negated variables
The Correspondence Homomorphism Problem

The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

NP-complete when $H$ is reflexive $K_1 + K_2$

$\begin{array}{ccc}
\emptyset & \quad & H \\
a & b & c
\end{array}$

Reduce from 1-in-3-SAT without negated variables

$$x + y + z$$
The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

NP-complete when $H$ is reflexive $K_1 + K_2$

\begin{align*}
\begin{array}{ccc}
a & b & c \\
& & H
\end{array}
\end{align*}

Reduce from 1-in-3-SAT without negated variables

\begin{align*}
\begin{array}{ccc}
G \times^* H & x & y + z \\
\downarrow & & \\
x & y & z
\end{array}
\end{align*}
The Correspondence Homomorphism Problem

The correspondence $H$-colouring problem

Given some $G \times^* H$, is there a $G$-transversal?

NP-complete when $H$ is reflexive $K_1 + K_2$

Reduce from 1-in-3-SAT without negated variables
The Correspondence Homomorphism Problem

Reduce from 1-in-3-SAT without negated variables

G \times H

x + y + z

x

y

z

Variables

Clauses

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Correspondence Homomorphisms
Reduce from 1-in-3-SAT without negated variables

\[ x + y + z \]

Variables

Clauses

G \times H

\[ x \]

\[ y \]

\[ z \]
The Correspondence Homomorphism Problem

Reduce from 1-in-3-SAT without negated variables

\[ x + y + z \]

\[ G \times^* H \]

Variables

Clauses

\[ x \]

\[ y \]

\[ z \]

\[ a \]

\[ b \]

\[ c \]

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Correspondence Homomorphisms
Reduce from 1-in-3-SAT without negated variables

\[ x + y + z \]

\[ G \times^* H \]

\begin{array}{c|c}
\text{Variables} & \text{Clauses} \\
\hline
x & a \\
y & b \\
z & c \\
\end{array}
The Correspondence Homomorphism Problem

Reduce from 1-in-3-SAT without negated variables

\[ x + y + z \]

Variables:
- \( x \)
- \( y \)
- \( z \)

Clauses:
- \( a \)
- \( b \)
- \( c \)

Correspondence Homomorphisms
The Correspondence Homomorphism Problem

NP-complete when $H$ is reflexive $K_1 + K_2$
The Correspondence Homomorphism Problem

NP-complete when $H$ is reflexive $K_1 + K_2$

Polynomial-time solvable when $H$ is reflexive $K_2 + K_2$
The Correspondence Homomorphism Problem

Dichotomy classification

1. \( H \) is a reflexive \( K_2 + K_2 \)
2. \( H \) is a reflexive \( K_k \) or a reflexive \( kK_1 \)
3. \( H \) is an irreflexive \( K_2 \), \( 2 \)
4. \( H \) is an irreflexive \( pK_2 + qK_1 \)
5. \( H \) is an irreflexive \( pK_2 \) plus a reflexive \( qK_1 \)
6. \( H \) is a star with only one loop at the center

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Correspondence Homomorphisms
Dichotomy classification

The correspondence $H$-colouring problem is NP-complete except in the following polynomial-time solvable cases:

1. $H$ is a reflexive $K_2 + K_2$
2. $H$ is a reflexive $K_k$ or a reflexive $kK_1$
3. $H$ is an irreflexive $K_{2,2}$
4. $H$ is an irreflexive $pK_2 + qK_1$
5. $H$ is an irreflexive $pK_2$ plus a reflexive $qK_1$
6. $H$ is a star with only one loop at the center
The Correspondence List Homomorphism Problem

Dichotomy classification

The correspondence list $H$-colouring problem is NP-complete except in the following polynomial-time solvable cases:

1. $H$ is a reflexive $K_k$ or a reflexive $kK_1$
2. $H$ is an irreflexive $pK_2 + qK_1$
3. $H$ is an irreflexive $pK_2$ plus a reflexive $qK_1$
4. $H$ is a star with only one loop at the center

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Correspondence Homomorphisms
The Correspondence Homomorphism Problem

Polynomial-time solvable when $H$ is reflexive $K_2 + K_2$

Diagram:

- Vertices: $a$, $d$, $b$, $c$, $H$
- Edges: $a$ to $d$, $b$ to $c$
Polynomial-time solvable when $H$ is reflexive $K_2 + K_2$

Possible $K_2 \times^* (K_2 + K_2)$
The Correspondence Homomorphism Problem

Possible $K_2 \times^* (K_2 + K_2)$
The Correspondence Homomorphism Problem

Possible partitions of $K_2 + K_2$
Possible partitions of $K_2 + K_2$

\begin{align*}
10 & \quad \text{red circle} \\
01 & \quad \text{blue circle} \\
00 & \quad x = 0 \\
01 & \quad x = 1
\end{align*}
The Correspondence Homomorphism Problem

Possible partitions of $K_2 + K_2$

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<tbody>
<tr>
<td>10</td>
<td>11</td>
<td>$x+y=0$</td>
</tr>
<tr>
<td>00</td>
<td>01</td>
<td>$x+y=1$</td>
</tr>
</tbody>
</table>
### Possible partitions of $K_2 + K_2$

<table>
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<th>01</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>01</td>
</tr>
</tbody>
</table>

$y = 0, y = 1$
The Correspondence Homomorphism Problem

Example $K_2 \times^* (K_2 + K_2)$

\[
\begin{align*}
00 & \quad 00 \\
01 & \quad 01 \quad x_u = 0 \\
10 & \quad 10 \quad x_v + y_v = 0 \\
11 & \quad 11 \quad x_u = 1 \\
& \quad v \quad x_v + y_v = 1 \\
& \quad u \quad x_u + x_v + y_v = 1
\end{align*}
\]
The Correspondence Homomorphism Problem

Example $K_2 \times^* (K_2 + K_2)$

\[ \begin{array}{c c c}
00 & 00 & x_u = 0 \quad x_v + y_v = 0 \\
01 & 01 & x_u = 1 \quad x_v + y_v = 1 \\
10 & 10 & x_u = 1 \quad x_v + y_v = 1 \\
11 & 11 & x_u + x_v + y_v = 1 \\
\end{array} \]

Solved by Gaussian elimination

$G$-transversal $\iff$ satisfies all such equations
Happy Birthday!
Happy Birthday!

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Correspondence Homomorphisms