Computational determination of the largest lattice polytope diameter

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based on joint works with: Nathan Chadder, McMaster
George Manoussakis, Paris Sud
Lionel Pournin, Paris XIII
Shmuel Onn, Technion
**Lattice polytopes with large diameter**

lattice \((d,k)\)-polytope: convex hull of points drawn from \(\{0,1,\ldots,k\}^d\)

**diameter** \(\delta(P)\) of polytope \(P\): smallest number such that any two vertices of \(P\) can be connected by a path with at most \(\delta(P)\) edges

\(\delta(d,k)\): largest diameter over all lattice \((d,k)\)-polytopes

ex. \(\delta(3,3) = 6\) and is achieved by a *truncated cube*
Lattice polytopes with large diameter

**lattice** \((d,k)-\text{polytope}\) : convex hull of points drawn from \(\{0,1,\ldots,k\}^d\)

**diameter** \(\delta(P)\) of polytope \(P\) : smallest number such that any two vertices of \(P\) can be connected by a path with at most \(\delta(P)\) edges

\(\delta(d,k)\): largest diameter over all lattice \((d,k)\)-polytopes

- \(\delta(P)\) : lower bound for the worst case number of iterations required by pivoting methods (simplex) to optimize a linear function over \(P\)

- **Hirsch conjecture** : \(\delta(P) \leq n - d\) \((n\ number\ of\ inequalities)\) was disproved [Santos 2012]
Lattice polytopes with large diameter

\( \delta(d,k) \): largest diameter of a convex hull of points drawn from \( \{0,1,\ldots,k\}^d \)

upper bounds:

\[
\delta(d,1) \leq d \quad \text{[Naddef 1989]}
\]

\[
\delta(2,k) = O(k^{2/3}) \quad \text{[Balog-Bárány 1991]}
\]

\[
\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k) \quad \text{[Thiele 1991]}
\]

\[
\delta(k,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k) \quad \text{[Acketa-Žunić 1995]}
\]

\[
\delta(d,k) \leq kd \quad \text{[Kleinschmid-Onn 1992]}
\]

\[
\delta(d,k) \leq kd - \lceil d/2 \rceil \quad \text{for } k \geq 2 \quad \text{[Del Pia-Michini 2016]}
\]

\[
\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k - 3) \quad \text{for } k \geq 3 \quad \text{[Deza-Pournin 2017]}
\]
Lattice polytopes with large diameter

$\delta(d,k)$: largest diameter of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds:

$\delta(d,1) \geq d$ [Naddef 1989]

$\delta(d,2) \geq \lceil 3d/2 \rceil$ [Del Pia-Michini 2016]

$\delta(d,k) = \Omega(k^{2/3}d)$ [Del Pia-Michini 2016]

$\delta(d,k) \geq \lceil (k+1)d/2 \rceil$ for $k < 2d$ [Deza-Manoussakis-Onn 2017]
**Lattice polytopes with large diameter**

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\( \delta(d,1) = d \)  

[Naddef 1989]
## Lattice polytopes with large diameter

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$\delta(d,1) = d$

$\delta(2,k)$ : close form

[Thiele 1991] [Acketa-Žunić 1995]

[Naddef 1989]
Lattice polytopes with large diameter

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$\delta(d,1) = d$

$\delta(2,k) : \text{close form}$

$\delta(d,2) = \lfloor 3d/2 \rfloor$

[Del Pia-Michini 2016]

[Naddef 1989]

[Thiele 1991] [Acketa-Žunić 1995]
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- $\delta(d,1) = d$
- $\delta(2,k)$ : close form
- $\delta(d,2) = \lfloor 3d/2 \rfloor$
- $\delta(4,3) = 8$
- $\delta(3,4) = 7$, $\delta(3,5) = 9$

[Naddef 1989]
[Thiele 1991] [Acketa-Žunić 1995]
[Del Pia-Michini 2016]
[Deza-Pournin 2017]
[Chadder-Deza 2017]
**Lattice polytopes with large diameter**

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> Conjecture [Deza-Manoussakis-Onn 2017] \(\delta(d,k) \leq \lceil (k+1)d/2 \rceil\)

and \(\delta(d,k)\) is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of \(\delta(d,k)\)
Q. What is $\delta(2,k)$ : largest diameter of a polygon which vertices are drawn form the $k \times k$ grid?

A polygon can be associated to a set of vectors (edges) *summing up to zero*, and *without a pair of positively multiple vectors*

$\delta(2,3) = 4$ is achieved by the 8 vectors : $(\pm 1,0), (0,\pm 1), (\pm 1,\pm 1)$.
$\delta(2,2) = 2$ ; vectors : $(\pm 1,0)$, $(0,\pm 1)$
Lattice polygons with many vertices

\[ \delta(2,2) = 2 ; \text{vectors : } (\pm 1,0), (0,\pm 1) \]
Lattice polygons with many vertices

δ(2,2) = 2; vectors: (±1,0), (0,±1)
δ(2,3) = 4; vectors: (±1,0), (0,±1), (±1,±1)
Lattice polygons with many vertices

\[ \delta(2, 2) = 2 ; \text{vectors: } (\pm 1, 0), (0, \pm 1) \]
\[ \delta(2, 3) = 4 ; \text{vectors: } (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1) \]
\[ \delta(2, 9) = 8 ; \text{vectors: } (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 1, \pm 2), (\pm 2, \pm 1) \]
Lattice polygons with many vertices

\[ \delta(2, k) = 2 \sum_{i=1}^{p} \varphi(i) \text{ for } k = \sum_{i=1}^{p} i \varphi(i) \]

\[ \varphi(p) : \text{Euler totient function} \] counting positive integers less or equal to \( p \) relatively prime with \( p \)

\[ \varphi(1) = \varphi(2) = 1, \varphi(3) = \varphi(4) = 2, \ldots \]
## Lattice polygons

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\[
\delta(2,k) = 2 \sum_{i=1}^{p} \varphi(i) \quad \text{for} \quad k = \sum_{i=1}^{p} i\varphi(i)
\]

\( \varphi(p) : \text{Euler totient function} \) counting positive integers less or equal to \( p \) relatively prime with \( p \)

\( \varphi(1) = \varphi(2) = 1, \varphi(3) = \varphi(4) = 2, \ldots \)
$H_1(2,p)$ : Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p, \gcd(x)=1, x \succ 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2 \sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by $(1,0), (0,1), (1,1), (1,-1)$ (fits, up to translation, in 3x3 grid)

$x \succ 0$ : first nonzero coordinate of $x$ is nonnegative
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

\[ H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0) \]

\[ Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0) \]

\[ x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative} \]

Given a set $G$ of $m$ vectors (generators)

**Minkowski** $(G)$ : convex hull of the $2^m$ sums of the $m$ vectors in $G$

**Zonotope** $(G)$ : convex hull of the $2^m$ signed sums of the $m$ vectors in $G$

up to translation $Z(G)$ is the image of $H(G)$ by an homothety of factor 2

*Primitive zonotopes*: zonotopes generated by short integer vectors which are pairwise linearly independent
**Primitive zonotopes**
*(generalization of the permutahedron of type \(B_d\))*

\[ H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

\[ Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

\( x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative} \)

- \( H_q(d, 1) : [0, 1]^d \text{ cube for } q \neq \infty \)
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p) :$ Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p) :$ Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0 :$ first nonzero coordinate of $x$ is nonnegative

$Z_1(d,2) :$ permutahedron of type $B_d$
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d, p) : $ Minkowski $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \geq 0)$

$Z_q(d, p) : $ Zonotope $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \geq 0)$

$x \geq 0 : $ first nonzero coordinate of $x$ is nonnegative

- $H_1(3,2) : $ truncated cuboctahedron  
  *(great rhombicuboctahedron)*
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p)$: Minkowski ($x \in \mathbb{Z}^d : \|x\|_q \leq p$, gcd($x$)=1, $x \succeq 0$)

$Z_q(d,p)$: Zonotope ($x \in \mathbb{Z}^d : \|x\|_q \leq p$, gcd($x$)=1, $x \succeq 0$)

$x \succeq 0$: first nonzero coordinate of $x$ is nonnegative

- $H_\infty(3,1)$: truncated small rhombicuboctahedron
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p):$ Minkowski $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p):$ Zonotope $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0:$ first nonzero coordinate of $x$ is nonnegative

$H^+ / Z^+:$ **positive** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+:$ Minkowski sum of the permutahedron with the $\{0,1\}^d$, i.e.,
  graphical zonotope obtained by the $d$-clique with a loop at each node

**graphical** zonotope $Z_G$: Minkowski sum of segments $[e_i,e_j]$ for all *edges* $\{i,j\}$ of a given graph $G$
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$$H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succ 0)$$

$$Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succ 0)$$

- $x \succ 0$ : first nonzero coordinate of $x$ is nonnegative
- $H^+ / Z^+$: **positive** primitive lattice polytope $x \in \mathbb{Z}_d^+ $

- For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2)$ is, up to translation, a lattice $(d,k)$-polytope with diameter $\lfloor (k+1)d/2 \rfloor$
Lattice polytopes with large diameter

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Conjecture [Deza-Manoussakis-Onn 2017] \( \delta(d,k) \leq \lfloor (k+1)d/2 \rfloor \)

and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors. The conjecture holds for all known entries of \( \delta(d,k) \).
A conjecture [Deza-Manoussakis-Onn 2017] states that:

\[ \delta(d,k) \leq \left\lfloor \frac{(k+1)d}{2} \right\rfloor \]

and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors. The conjecture holds for all known entries of \( \delta(d,k) \).
**Computational determination of** $\delta(d,k)$

Given a lattice $(d,k)$-polytope $P$, two vertices $u$ and $v$ such that $\delta(P) = d(u,v)$, then $d(u,v) \leq \delta(d-1,k) + k$ and $d(u,v) < \delta(d-1,k) + k$ unless:

- $u + v = (k,k,...,k)$,
- any edge of $P$ with $u$ or $v$ as vertex is $\{-1,0,1\}$-valued,
- any intersection of $P$ with a facet of the cube $[0,k]^d$ is a $(d-1)$-dimensional face of $P$ of diameter $\delta(d-1,k)$.

These conditions, combined with combinatorial properties, drastically reduce the search space for a lattice $(d,k)$-polytope $P$ such that $\delta(P) = \delta(d-1,k) + k$.

Computationally ruling out $\delta(d,k) = \delta(d-1,k) + k$ and using $\delta(d,k) \leq \left\lfloor (k+1)d / 2 \right\rfloor$ for $k < 2d$ yields: $\delta(3,4) = 7$ and $\delta(3,5) = 9$.

i.e. : $\delta($great rhombicuboctahedron$) = \delta(3,5)$
A034997  Number of Generalized Retarded Functions in Quantum Field Theory.
2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)
OFFSET 1,1
COMMENTS
a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.
Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES
Number of Generalized Retarded Functions in Quantum Field Theory.

A034997

Number of Generalized Retarded Functions in Quantum Field Theory.

1, 1

A(d) is the number of parts into which d-dimensional space \( (x_1, \ldots, x_d) \) is split by a set of \((2^d - 1)\) hyperplanes \( c_1 x_1 + c_2 x_2 + \ldots + c_d x_d = 0 \) where \( c_j \) are 0 or +1 and we exclude the case with all \( c=0 \).

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T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.


Table of \( A(n) \) for \( n=1 \ldots 8 \).

Computational determination of the number of vertices of primitive zonotopes

Sloane OEI sequences

$H_\infty(d,1)^+ $ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $d=8$)

$H_\infty(d,1)$ vertices : A009997 = number of regions of hyperplane arrangements with $\{-1,0,1\}$-valued normals in dimension $d$ (determined till $d=7$)

Estimating the number of vertices of $H_\infty(d,1)^+$

[Odlyzko 1988], [Zuev 1992], [Kovijanić-Vukićević 2007]

$$d^2 \ (1-o(1)) \leq \log_2 | H_\infty(d,1)^+ | \leq d^2$$
Lattice polytopes with large diameter and many vertices

\( \delta(d,k) \): largest diameter over all lattice \((d,k)\)-polytopes

- **Conjecture**: \( \delta(d,k) \leq \lfloor (k+1)d/2 \rfloor \) and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known \( \delta(d,k) \))

\[ \Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \quad \text{for} \quad k < 2d \]

- Determination of \( \delta(3,k) \) and of \( \delta(d,3) \)? (\( \delta(d,3) = 2d \)?)


- Answer to [Colbourn-Kocay-Stinson 1986] question: Deciding if a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]
Lattice polytopes with large diameter and many vertices

\( \delta(d,k) \): largest diameter over all lattice \((d,k)\)-polytopes

- **Conjecture**: \( \delta(d,k) \leq \lfloor (k+1)d/2 \rfloor \) and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known \( \delta(d,k) \))

\[ \Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \text{ for } k < 2d \]

- Determination of \( \delta(3,k) \) and of \( \delta(d,3) \)? \( (\delta(d,3) = 2d ?) \)


- Answer to [Colbourn-Kocay-Stinson 1986] question: Deciding if a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]

✓ thank you
Convex Matroid Optimization

The optimal solution of \( \max \{ f(Wx) : x \in S \} \) is attained at a vertex of the projection integer polytope in \( \mathbb{R}^d : \text{conv}(WS) = W\text{conv}(S) \)

\( S \) : set of feasible point in \( \mathbb{Z}^n \) (in the talk \( S \in \{0,1\}^n \))
\( W \) : integer \( d \times n \) matrix (\( W \) is mostly \( \{0,1,\ldots,p\} \)-valued)
\( f \) : convex function from \( \mathbb{R}^d \) to \( \mathbb{R} \)

Q. What is the maximum number \( v(d,n) \) of vertices of \( \text{conv}(WS) \) when \( S \in \{0,1\}^n \) and \( W \) is a \( \{0,1\} \)-valued \( d \times n \) matrix?

obviously \( v(d,n) \leq |WS| = O(n^d) \)
in particular \( v(2,n) = O(n^2) \), and \( v(2,n) = \Omega(n^{0.5}) \)
Convex Matroid Optimization

The optimal solution of $\max \{ f(Wx) : x \in S \}$ is attained at a vertex of the projection integer polytope in $\mathbb{R}^d : \text{conv}(WS) = W\text{conv}(S)$

$S$ : set of feasible point in $\mathbb{Z}^n$ (in the talk $S \in \{0,1\}^n$ )
$W$ : integer $d \times n$ matrix ($W$ is mostly $\{0,1,\ldots,p\}$-valued)
$f$ : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number $v(d,n)$ of vertices of $\text{conv}(WS)$ when $S \in \{0,1\}^n$ and $W$ is a $\{0,1\}$-valued $d \times n$ matrix ?

obviously $v(d,n) \leq |WS| = O(n^d)$
in particular $v(2,n) = O(n^2)$, and $v(2,n) = \Omega(n^{0.5})$

[Melamed-Onn 2014] Given matroid $S$ of order $n$ and $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, the maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is independent of $n$ and $S$
Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid $S$ of order $n$ and $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, the maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is independent of $n$ and $S$.

Ex: maximum number $m(2,1)$ of vertices of a planar projection $\text{conv}(WS)$ of matroid $S$ by a binary matrix $W$ is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$W = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$S = U(3,8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\text{conv}(WS)$
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[Deza-Manoussakis-Onn 2016] Given matroid $S$ of order $n$, $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is equal to the number of vertices of $H_\infty(d,p)$.

$$m(d,p) = |H_\infty(d,p)|$$
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$$m(d,p) = |H_\infty(d,p)|$$

[Melamed-Onn 2014]

$$d \cdot 2^d \leq m(d,1) \leq 2 \sum_{i=0}^{d-1} \binom{3^d - 3}{i}/2$$

- $m(2,1) = 8$
- $24 \leq m(3,1) \leq 158$
- $64 \leq m(4,1) \leq 19840$

[Deza-Manoussakis-Onn 2016]

$$d! \cdot 2^d \leq m(d,1) \leq 2 \sum_{i=0}^{d-1} \binom{3^d - 3}{i}/2 - f(d)$$

- $m(3,1) = 96$
- $m(4,1) = 5376$

$$m(2,p) = 8 \sum_{i=1}^{p} \varphi(i)$$
Primitive Zonotopes
(complexity questions)

For fixed $p$ and $q$, linear optimization over $Z_q(d,p)$ is polynomial-time solvable, even in variable dimension $d$ (polynomial number of generators)

$\Rightarrow$ for fixed positive integers $p$ and $q$, the following problems are polynomial time solvable:

- **extremality**: given $x \in \mathbb{Z}^d$, decide if $x$ is a vertex of $Z_q(d,p)$

- **adjacency**: given $x_1,x_2 \in \mathbb{Z}^d$, decide if $[x_1,x_2]$ is an edge of $Z_q(d,p)$

- **separation**: given rational $y \in \mathbb{R}^d$, either assert $y \in Z_q(d,p)$, or find $h \in \mathbb{Z}^d$ separating $y$ from $Z_q(d,p)$ i.e., satisfying $h^Ty > h^Tx$ for all $x \in Z_q(d,p)$
**Primitive Zonotopes**

*(complexity questions)*

For *fixed* \( p \) and \( q \), linear optimization over \( Z_q(d,p) \) is polynomial-time solvable, even in *variable* dimension \( d \) (polynomial number of generators)

\[ \Rightarrow \text{for fixed positive integers } p \text{ and } q, \text{ the following problems are polynomial time solvable:} \]

- **extremality**: given \( x \in Z^d \), decide if \( x \) is a vertex of \( Z_q(d,p) \)
- **adjacency**: given \( x_1, x_2 \in Z^d \), decide if \([x_1, x_2]\) is an edge of \( Z_q(d,p) \)
- **separation**: given rational \( y \in R^d \), either assert \( y \in Z_q(d,p) \), or find \( h \in Z^d \) separating \( y \) from \( Z_q(d,p) \) i.e., satisfying \( h^T y > h^T x \) for all \( x \in Z_q(d,p) \)

Q. existence of a *direct* algorithm for fixed \( p \) and \( q \)

- existence of an algorithms for fixed \( p \) and \( q = \infty \)
- existence of *hole* : \( x \in H_q(d,p)^+ \cap Z^d \) which can not be written as a sum of a subset of generators of \( H_q(d,p)^+ \)
**Primitive Zonotopes**
*(complexity questions)*

$D_d :$ convex hull of the degree sequences of all hypergraphs on $d$ nodes

$D_d = H_\infty(d,1)+$

$D_d(k) :$ convex hull of the degree sequences of all $k$-uniform hypergraphs on $d$ nodes

**Q:** check whether $x \in D_d(k) \cap Z^d$ is the degree sequence of a $k$-uniform hypergraph. Necessary condition: sum of the coordinates of $x$ is multiple of $k$.

[Erdős-Gallai 1960]: for $k = 2$ (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for $k = 3$ (Klivans-Reiner Q.)

- Do $H_q(d,p)+$ have hole: $x \in H_q(d,p)+ \cap Z^d$ which can not be written as a sum of a subset of generators of $H_q(d,p)+$

- complexity of deciding whether $x$ is a hole?
Lattice polytopes with large diameter and many vertices

\( \delta(d,k) \): largest diameter over all lattice \((d,k)\)-polytopes

- **Conjecture**: \( \delta(d,k) \leq \lceil (k+1)d/2 \rceil \) and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known \( \delta(d,k) \))

\[ \Rightarrow \delta(d,k) = \lceil (k+1)d/2 \rceil \text{ for } k < 2d \]

- \( m(d,p) = |H_\infty(d,p)| \)

- determination of \( \delta(3,k) \) and of \( \delta(d,3) \) ? \( (\delta(d,3) = 2d ?) \)

- complexity issues, e.g. decide whether a given point is a vertex of \( Z_\infty(d,1) \)

- existence of **hole**: \( x \in H_q(d,p)^+ \cap \mathbb{Z}^d \) which can not be written as a sum of a subset of generators of \( H_q(d,p)^+ \)