Rauzy Dynamics, Old and New

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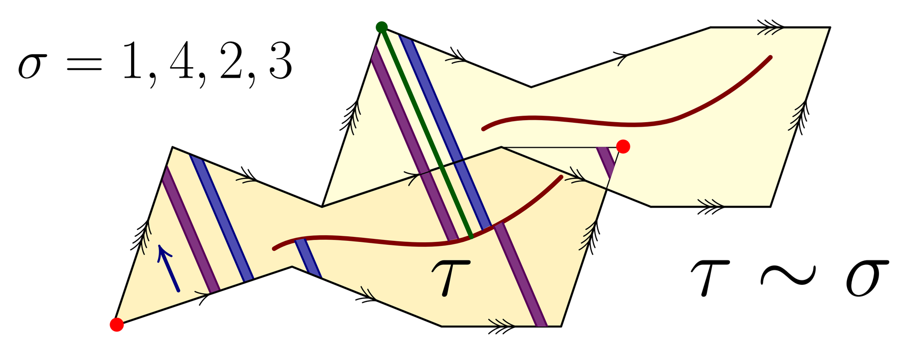
Some background

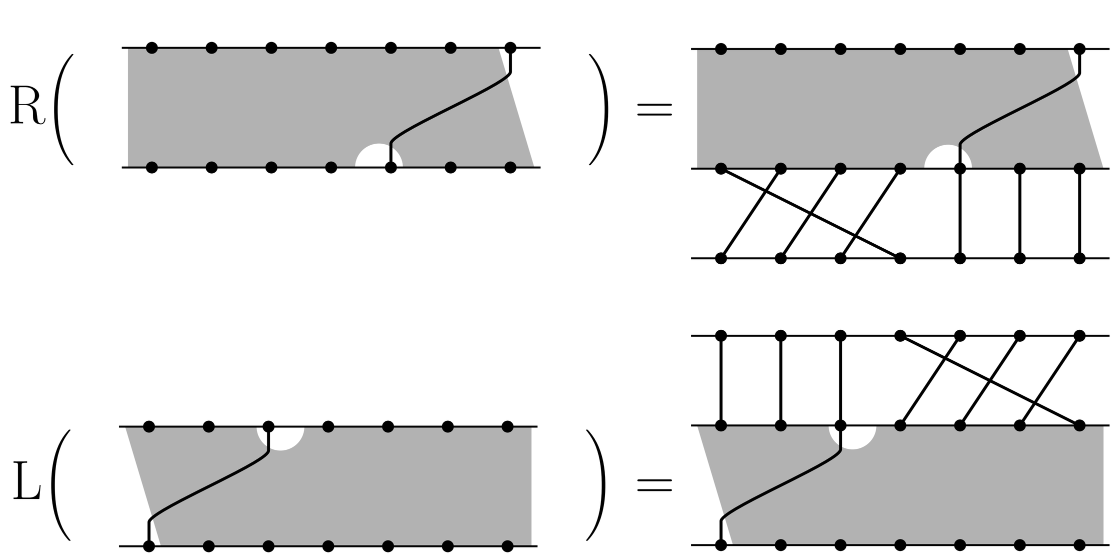
Given a permutation σ, we define a translation surface (i.e. a flat surface with conic singularities multiple of 2π). Then we consider all geodesics emitted from a segment X in a generic direction and let each of them come back to X. It gives a couple $(\tau, (d_i))$: the geodesics form intervals of given length (d_i) and τ encodes how the intervals are exchanged (when returning on X).



Due to some invariance and ergodic theorems we can define the equivalent relation : $\tau \sim \pi$ iff there exists two segment X and X' and directions such that X gives τ and X' gives π.
Notice that σ can be obtained by X the segment between the two red points.
This gives equivalent classes on permutations and translation surfaces.

The Rauzy-Veech induction

We can study the equivalence classes with a two operators combinatorial dynamic :



Clearly each operator is invertible and Veech proved that $\sigma \sim \tau$ iff $\exists S \in \{L, R\}^* \,/\, S(\sigma) = \tau$.
In term of surfaces, it corresponds to taking the left end of X and shortening it until the shorter interval between the first interval (that's a R) or the first return interval disappear (that's a L).
The classification was already found in [1] but it uses tools from algebraic geometry, dynamical system etc and a purely combinatorial proof was missing. That's what we present here.

Invariants

• A permutation σ is irreducible iff it doesn't send any interval $\{1 \dots, k\}$ onto $\{n, \dots, n-k\}$.

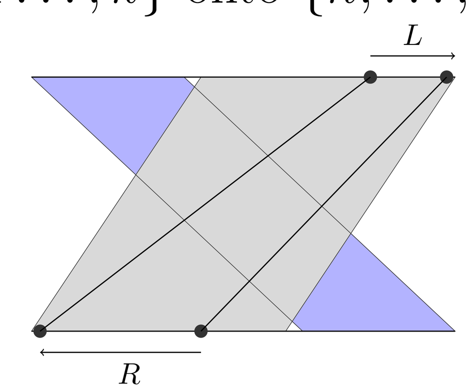


Figure 1: A reducible permutation σ ; the dynamic acts only the gray part.

• The cycle invariant is defined as below :

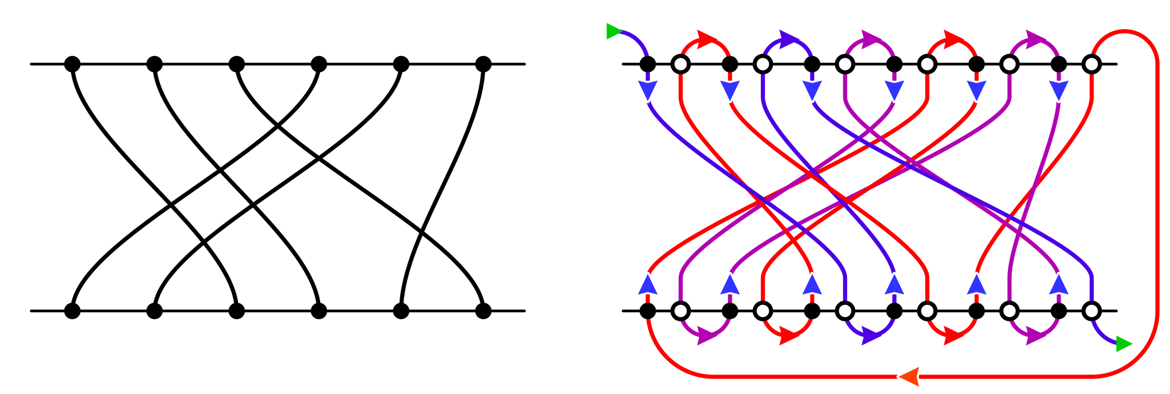
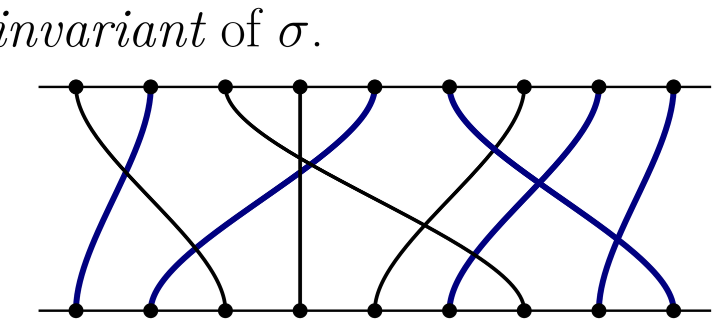


Figure 2: We count the number of bottom arcs of a given color. The blue path is called the rank. The pair ($\lambda = \{2, 2\}$, $r = 1$) is invariant

• For every I a set of edges, define $\chi(I)$ as the number of pairs $\{i', i''\} \subseteq I$ of non-crossing edges. Call

$$A(\sigma) = \sum_{I \subseteq [n]} (-1)^{|I| + \chi(I)} / 2^{c(\sigma)}$$

the *Arf invariant* of σ.



$\chi(I) = 8, |I| = 5, (-1)^{|I|+\chi(I)} = -1$

$A(\sigma) = \pm 1$ if no even cycle, $A(\sigma) = 0$ otherwise.

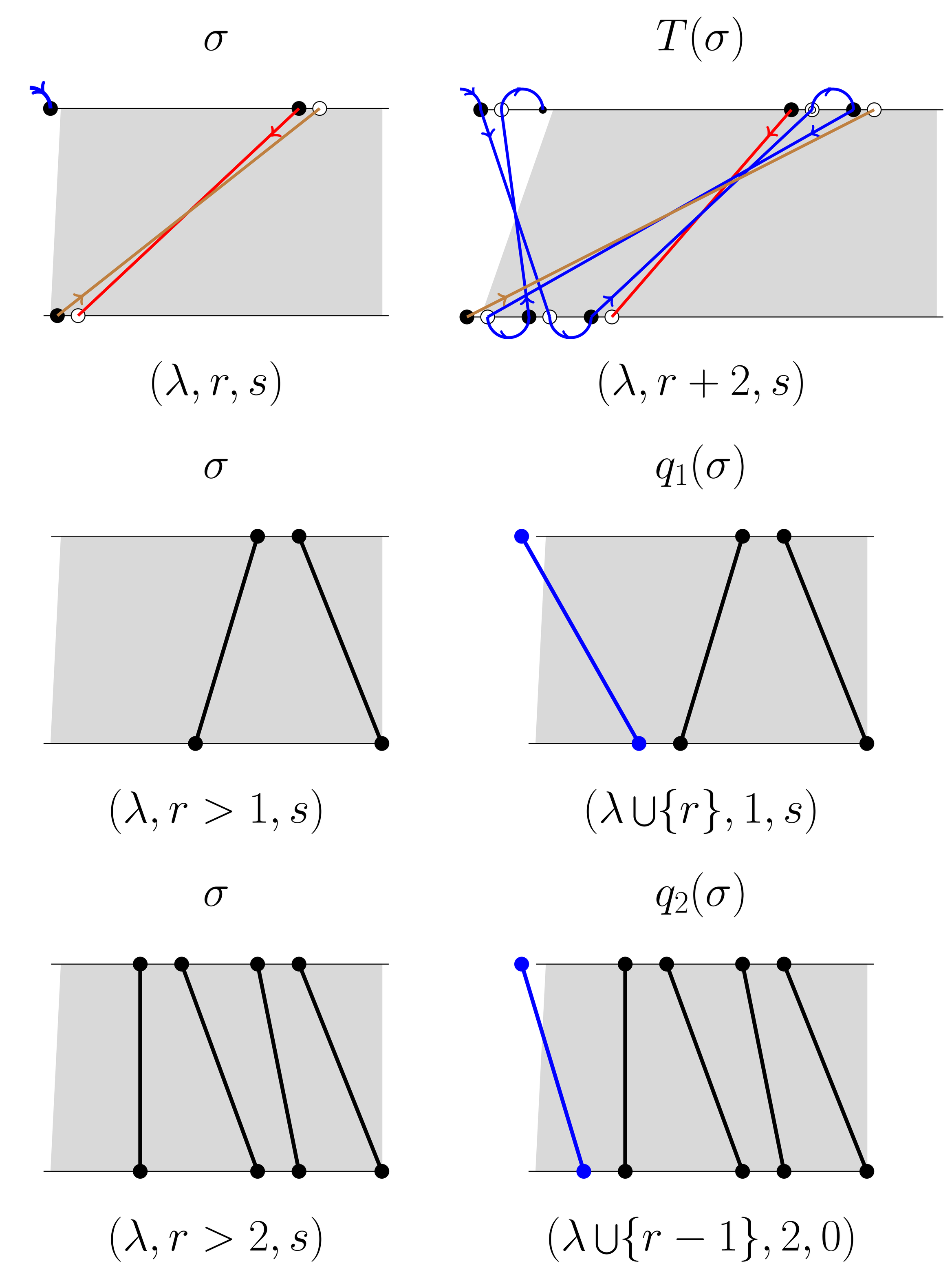
Main Theorem

The number of irreducible classes with cycle invariant (λ, r) (no $\lambda_i = 1$) depends on the number of even elements in the list $\{\lambda_i\} \cup \{r\}$, and is

• 0 : if there is an odd number of even elements;
• 1 : if there is a positive even number of even elements; the class then has sign 0.
• 2 : if there are no even elements at all. The two classes then have non-zero opposite sign invariant.

Thus two irreducible permutations σ and τ are in the same class iff $(\lambda(\sigma), r(\sigma), A(\sigma)) = (\lambda(\tau), r(\tau), A(\tau))$

Proof overview

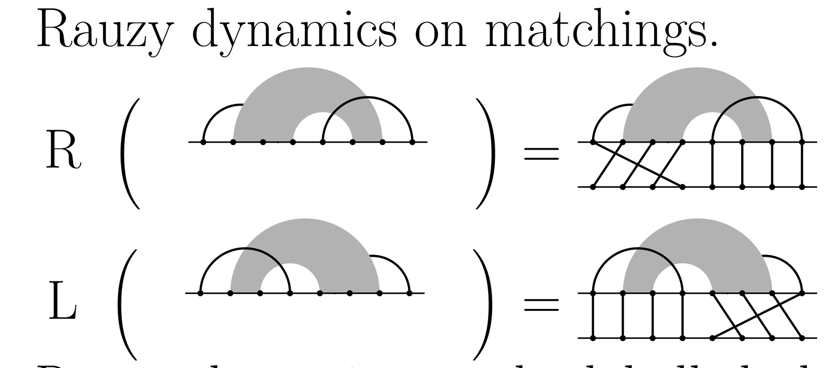


We ask that :
• If $\sigma \sim \tau$ then $T(\sigma) \sim T(\tau)$, $q_1(\sigma) \sim q_1(\tau)$ and $q_2(\sigma) \sim q_2(\tau)$. Thus for a class C, we define $\bar{T}(C)$, $\bar{q}_1(C)$ and $\bar{q}_2(C)$
• If B has invariant $(\lambda, r > 2, s)$ then the unique C with invariant $(\lambda, r-2, s)$ has $\bar{T}(C) = B$.
• If B has invariant $(\lambda, 1, s)$ then for all C with invariant (λ', r', s) such that $\bar{q}_1(C)$ has invariant $(\lambda, 1, s)$ we have $\bar{q}_1(C) = B$. $(\lambda', r'$ has the form $\lambda' = \lambda \setminus \{i\}$ and $r = i$ for some i).
• If B has invariant $(\lambda, 2, 0)$ then for all C with invariant (λ', r', s) such that $\bar{q}_2(C)$ has invariant $(\lambda, 2, 0)$ we have $\bar{q}_2(C) = B$. $(\lambda', r'$ has the form $\lambda' = \lambda \setminus \{i\}$ and $r = i + 1$ for some i).

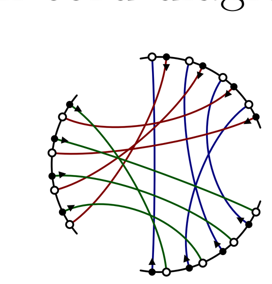
With those properties the induction proof goes as follows : true for $n, \dots, 1$ thus a class of size $n+1$ with invariant (λ, r, s) exists (apply either T, q_1, q_2 on the correct (λ', r', s')) and it is the only one with this invariant. Indeed, (we do the case with q_1) suppose there are two classes B_1, B_2 with invariant $(\lambda, 1, s)$ then all the unique classes C_i with invariant $(\lambda \setminus \{i\}, i, s)$ have $\bar{q}_1(C_i) = B_1$ and $\bar{q}_1(C_i) = B_2$ thus $B_1 = B_2$

Futher results

We found models with the same type of invariants (cycle and sign). Here is a list :
Rauzy dynamics on matchings.



Rauzy dynamics on the labelled objects (permutation and matchings).
Rauzy dynamics on cord diagram ($2k$ operators).



Reverse operators dynamics on matching (for every arc we have the involutive operator that reverses all the endpoint inside the arc)

References

[1] Connected components of the moduli spaces of Abelian differentials with prescribed singularities (M.Kontsevich, A.Zorich), Invent. Math. 153 (2003), 631-678.
[2] Flat Surfaces. A.Zorich, Frontiers in number theory, physics, and geometry. I, 437-583, Springer, Berlin, 2006

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