Thermodynamic formalism and k-bonacci substitution

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1. Thermodynamics formalism

Let (X,T) be a dynamical system with X a **compact** space and $T:X\to X$ a **continuous** function.

Definition: A **potential** is a **continuous** function $V:X\to\mathbb{R}_+$.

A potential defines a **pressure function** in the following way:

Definition: For any $\beta\geq 0$,

$$p(\beta)=\sup\left\{h_\mu-\beta\int_X V\,d\mu\right\}$$

where the sup is taken on all T-**invariant probability measures** and h_μ is the **Kolmogorov-Sinai entropy**.

2. Some properties of the pressure function

The pressure function is **decreasing** and **convex**.

Definition: A point β_0 such that p is **not analytic** in β_0 is called a **phase transition**.

The pressure function also has an **asymptote** of the form $y=a\beta+b$.

Definition: The smallest point β_0 such that for any $\beta\geq\beta_0$, $p(\beta)=a\beta+b$ is called, if it exists, a **freezing phase transition**.

Theorem: [Rue04] Let (X,σ) be a **subshift of finite type** on a finite alphabet (σ denotes the **shift**). If the potential V is **Hölder**, then p is **analytic**.

We will be interested in points of non analyticity. This is part of the motivation for the choice of potential (we do not want Hölder regularity for our potential).

3. Substitutions

Let $\mathcal{A}$ be a **finite alphabet**. $\mathcal{A}^*$ denotes the set of **finite words** on this alphabet. Then $\mathcal{A}^{\mathbb{N}}$ is the set of **unilateral infinite sequences** on this alphabet (an element of $\mathcal{A}^{\mathbb{N}}$ is a **configuration**). This set is endowed with the **product topology** which makes it a **Cantor set**.

Definition: A substitution $s:\mathcal{A}^*\to\mathcal{A}^*$ is a **non-erasing word morphism**.

Examples:

k-bonacci substitution:

$$\begin{array}{rcl} s_k:\{0,\dots,k-1\}^* & \to & \{0,\dots,k-1\}^* \\ 0 & \mapsto & 01 \\ 1 & \mapsto & 02 \\ \vdots & \vdots & \vdots \\ k-1 & \mapsto & 0 \end{array}$$

Thue-Morse substitution:

$$\begin{array}{rcl} H:\{0,1\}^* & \to & \{0,1\}^* \\ 0 & \mapsto & 01 \\ 1 & \mapsto & 10 \end{array}$$

Remark that s_2 is the Fibonacci substitution.

Computing the image of a word is achieved in the following way:

$$H(0110)=H(0)H(1)H(1)H(0)=01101001.$$

Definition: Let s be a **primitive aperiodic** substitution on a finite alphabet. We define the **language** of the substitution $\mathcal{L}_s=\{w\in\mathcal{A}^*\,|\,\exists n\in\mathbb{N},\,\exists a\in\mathcal{A},\,w\in s^n(a)\}$. We also define the **subshift** associated to the substitution as the set $\Sigma_s\subset\mathcal{A}^{\mathbb{N}}$ of configurations whose **every factor** is in $\mathcal{L}_s$.

4. Potential and Ruelle renormalization

For two different configurations x and y, define the **distance** $d(x,y)=\frac{1}{2^n}$ on $\mathcal{A}^{\mathbb{N}}$, where n is the smallest integer such that $x_n\neq y_n$, which is compatible with the product topology.
For $x\in\mathcal{A}^{\mathbb{N}}$, denote $\delta(x)$ the length of the **maximal prefix** of x in $\mathcal{L}_s$. Hence $d(x,\Sigma_s)=\frac{1}{2^{\delta(x)}}$.
We are interested in the following family of potentials:

$$\begin{array}{rcl} V:\mathcal{A}^{\mathbb{N}} & \to & \mathbb{R}_+ \\ x & \mapsto & \frac{g(x)}{\delta(x)}+\frac{h(x)}{\delta(x)}, \end{array}$$

with g a **positive continuous** function and h being 0 on Σ_s and **continuous**.

Definition: We define the **Ruelle renormalization operator** R on $\mathcal{C}(\mathcal{A}^{\mathbb{N}},\mathbb{R}_+)$ by:

$$\forall x\in\mathcal{A}^{\mathbb{N}},\,R(V)(x)=\sum_{j=0}^{|s(x_0)|-1}V\circ\sigma^j\circ s(x).$$

5. Previous results

Theorem: [BL15] There is a freezing phase transition for any potential in the family defined by the Fibonacci substitution.

Theorem: [BL13, BL15]
- For the Fibonacci substitution, for any configuration x, $R^n(V)(x)$ converges towards a **fixed point** of R.
- For the Thue-Morse substitution, for any configuration x, $R^n(V)(x)$ converges towards a **fixed point** of R in **Cesaro mean**.

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Theorem: [BHL15] For any **primitive aperiodic marked** substitution, there is a freezing phase transition.

Theorem: [BHL15] For any **primitive aperiodic marked 2-full** substitution, there exists U **fixed point** for R such that for any configuration x, $\lim_{n\to+\infty}R^n(V)(x)=U(x)$.

6. The k-bonacci substitution case

Theorem: (E.) Let $k\geq 2$, denote s_k the k-bonacci substitution, there exists a potential $U\in C\left(\mathcal{A}^{\mathbb{N}},\mathbb{R}_+\right)$ such that $RU=U$ given by:

$$\forall x\in\mathcal{A}^{\mathbb{N}},\,U(x)=\log\left(1+\frac{v_{x_0}}{\frac{\lambda}{\lambda-1}+\sum_{l\in\mathcal{A}_k}v_l|x_{[0..\delta_{s_k}(x)-1]}|_l-v_{x_0}}\right)$$

where λ is the **dominant root** of the polynomial $X^k-\sum_{j=0}^{k-1}X^j$ and

$$\forall l\in\mathcal{A},\,v_l=\frac{1}{\lambda^{k-1-l}}\sum_{j=0}^{k-1-l}\lambda^j.$$

Moreover, if $V:\mathcal{A}^{\mathbb{N}}\to\mathbb{R}_+$ is of the form:

$$\forall x\in\mathcal{A}^{\mathbb{N}},\,V(x)=\frac{g(x)}{\delta_{s_k}(x)^\alpha}+\frac{h(x)}{\delta_{s_k}(x)^\alpha}$$

with g a **positive continuous** function, h being 0 on Σ_{s_k} and **continuous** and $\alpha>0$, then, for any x in $\mathcal{A}^{\mathbb{N}}$:

$$\lim_{n\to+\infty}R^nV(x)=\begin{cases}0 & \text{if } \alpha>1\\ +\infty & \text{if } \alpha<1\\ \int g\,d\mu\cdot U(x) & \text{if } \alpha=1\end{cases}$$

where μ is the **only ergodic probability measure** on Σ_{s_k}.

Theorem: (E.) For the k-bonacci substitution, **there is a freezing phase transition.**

Remark that for any integer n, for any configuration x:

$$R^n(V)(x)=\sum_{j=0}^{|s^n(x_0)|-1}V\circ\sigma^j\circ s^n(x).$$

Lemma: For any configuration x and for any integer n, the **maximal prefix** of $s_k^n(x)$ in $\mathcal{L}_{s_k}$ is:

$$s_k^n\left(x_{[0..\delta(x)-1]}\right)s_k^{n-1}(0)\dots s_k(0)0.$$

Remark that we could expect, for any $j<|s_k^n(x_0|$, to have the relation:

$$\delta\left(\sigma^j\left(s_k^n\left(x\right)\right)\right)=\delta\left(s_k^n\left(x\right)\right)-j.$$

But there might be some integer j_0 such that

$$\delta\left(\sigma^{j_0-1}\left(s_k^n\left(x\right)\right)\right)-\delta\left(\sigma^{j_0}\left(s_k^n\left(x\right)\right)\right)\neq 1.$$

Such points are called **accidents**.

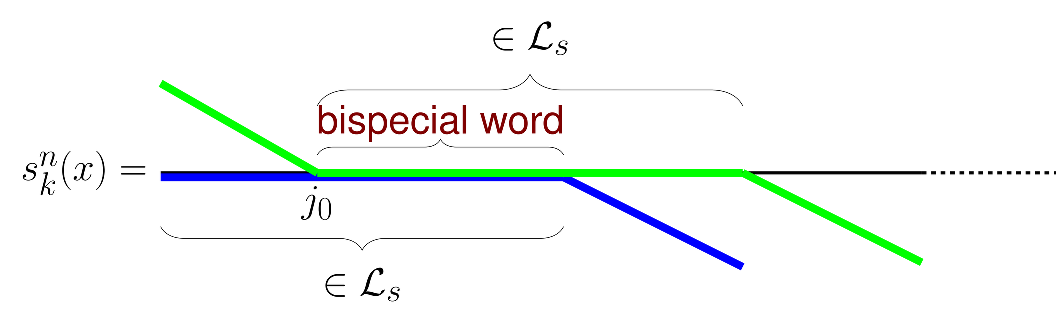

Figure 1: *Illustration of an accident at point j_0.*

Lemma: For any $n\geq k$, **there is no accident**.

References

[BHL15] Nicolas Bédaride, Pascal Hubert, and Renaud Leplaideur. Thermodynamic formalism and substitutions. Preprint, (arXiv:1511.03322), 2015.

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