Relational type-checking of MELL proof-structures

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Motivation
Motivation

Given a syntactic structure $R$, defines its semantics $\llbracket R \rrbracket$. Elements of $\llbracket R \rrbracket$ carry a lot of information on $R$:

- on its execution time (Carvalho, Pagani, and Tortora de Falco 2011),
- characterize it completely:
  - if $R$ is of type $\text{bool}$;
  - in general.

How tractable is $\exists x \in \llbracket R \rrbracket$?
Girard 1987, 3.16 Remark (ii) :

"Assume, just for a minute, that we try to compute the semantics of a proof $\pi$, i.e., we try to decide, given $z \in |A|$, whether or not $z \in \pi^*$. In all cases but [the cut] and [the existential quantification], the problem is immediately reduced to several simpler problems of the same type, with an effective bound on these problems. [...] This basic remark is the key to a semantic approach to computation, which will be undertaken somewhere else"
Multiplicative Linear Logic
Multiplicative proof-structures

\[ A, B, C ::= X \mid X^\perp \mid 1 \mid \bot \mid A \otimes B \mid A \parr B. \]

**Figure:** The formulæ

**Figure:** The cells
Proof-structures
Associate to every formula a set, to every proof-structure a relation between sets.

Definition (web of a formula)

\[ |\top| = |\bot| = \{()\} \]
\[ |X_i| = |X| = \mathcal{A}_i \text{, for any propositional variable } X; \]
\[ |1| = |\bot| = \{()\}; \]
\[ |A \otimes B| = |A \otimes B| = |A| \times |B| \]
Let $a \in |A|$, $b \in |B|$.

Figure: A proof-structure $R$
Let $a \in |A|, b \in |B|$.

**Figure:** An experiment of $R$

\[
[R] = \{((a, b), (a, b)), a \in |A|, b \in |B|\} 
\subseteq |A \otimes B| \times |A^\perp \otimes B^\perp|
\]
Intersection types

\[ \vdash M : A \cap B \] means M can be used both as an A and as a B.
Relational semantics are a non-idempotent type-system.
Our initial question \( \forall x \in [R] \) becomes \( \exists R \text{ typable at type } x \)?
Recognition
Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \not\exists B^\perp|\). Is \(((a, b), (a, b)) \in \llbracket R \rrbracket\)?

**Figure:** Successful recognition
Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A \perp \otimes B \perp|\).

¿Is \(((a, b), (a, b)) \in \mathbb{[}R]\)?

**Figure**: Successful recognition
Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \not\exists B^\perp|\).

Is \(((a, b), (a, b)) \in \llbracket R \rrbracket\)?

**Figure:** Successful recognition
Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \bowtie B^\perp|\). ¿Is \(((a, b), (a, b)) \in \llbracket R \rrbracket\)?
Recognition of the relational interpretation

Suppose I have $((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \otimes B^\perp|$.

Is $((a, b), (a, b)) \in \langle[R] \rangle$?

Figure: Successful recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \otimes B^\perp|\).

¿Is \(((a, b), (a, b)) \in \mathbb{R}\)?

**Figure:** Successful recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \otimes B^\perp|.

¿Is \(((a, b), (a, b)) \in [R]_i\)?

**Figure:** Successful recognition
Recognize the relational interpretation

Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A \perp \otimes B \perp|\). Is \(((a, b), (a, b)) \in [R]_x\)?

Figure: Successful recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, b)) \in |A \otimes B| \times |A \perp \otimes B \perp|\).

¿Is \(((a, b), (a, b)) \in \exists[R]\)?

Figure: Successful recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A \perp \otimes B \perp|\).

¿Is \(((a, b), (a, c)) \in [R]\)?

Figure: Failing recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \bowtie B^\perp|\).

¿Is \(((a, b), (a, c)) \in \Box R\)?

Figure: Failing recognition
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \not\exists B^\perp|\).

¿Is \(((a, b), (a, c)) \in \llbracket R \rrbracket\)?
Recognition of the relational interpretation

Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \otimes B^\perp|\).
¿Is \(((a, b), (a, c)) \in [R]\)?
Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \bowtie B^\perp|\). Is \(((a, b), (a, c)) \in [R]\)?

Figure: Failing recognition
Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \# B^\perp|\). Is \(((a, b), (a, c)) \in \llbracket R \rrbracket\)?
Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A \perp \gamma B\perp|\).
¿Is \(((a, b), (a, c)) \in \llbracket R \rrbracket\)?
Suppose I have \(((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \otimes B^\perp|\).  
¿Is \(((a, b), (a, c)) \in [R]\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).

¿Is \((a, a) \in [R]\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).

¿Is \((a, a) \in \lbrack R\rbrack\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).

¿Is \((a, a) \in \llbracket R \rrbracket\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).
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Let \((a, a) \in |A^\perp| \times |A|\).
¿Is \((a, a) \in \llbracket R \rrbracket\)?
Semantically hidden cycles

Let $(a, a) \in |A^\perp| \times |A|$.
¿Is $(a, a) \in \llbracket R \rrbracket$?
Let \((a, a) \in |A^\perp| \times |A|\).
Is \((a, a) \in \llbracket R \rrbracket\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).
¿Is \((a, a) \in \llbracket R \rrbracket\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).

¿Is \((a, a) \in \llbracket R \rrbracket\)?
Semantically hidden cycles

Let \((a, a) \in |A^\perp| \times |A|\).
¿Is \((a, a) \in \llbracket R \rrbracket\)?
Variant of a (coloured) Vector Addition System:

- one counter per port;
- each counter’s value is a formal sum of relational elements with coefficients $\pm 1$;
- displacement transitions, that moves tokens from counters to counters;
- unification transitions.

Reduce the problem to the accessibility of 0 by a machine.

**Theorem**

$x \in \llbracket \Phi \rrbracket$ if and only if the machine associated with $\Phi$, initialized on $x$, stops with zero on all counters.

*It can do so in $O(|\text{cells of } \Phi|)$.*
The exponentials
Syntactically

We add the following cells:

and the ability to box entire proof-structures:

Figure: A box
Figure: A MELL-ps R
\[ |!A| = |?A| = \mathcal{M}_{\text{fin}}(|A|) \]

An experiment assigns the multiset of its assignments on the auxiliary ports to the principal port of a ?-cell.

An experiment of a box is any number of experiments of its contents.
Example

Figure: A MELL-ps $R$
Weakenings
Weakenings
Derelictions
Boxes

Figure: A box
Figure: A box
Boxes

Figure: A box
Figure: A box
Contractions?

Either:

• try everything;
• restrict structures geometrically.
Example
Example

\[
([a_1, a_2], [[]])^\uparrow
\]

\[
([[], [b_1]], ([a_1, a_2], [b_2]))^\uparrow
\]
Example

\[
\left[ \left[ a_1, a_2 \right], \emptyset \right]^\uparrow
\]

\[
\left[ \left[ \emptyset, b_1 \right], \left[ a_1, a_2, b_2 \right] \right]^\uparrow
\]
Example
Example
Example

\[
[([a_1, a_2], [[]])^\uparrow, ([[]], [b_1]), ([a_1, a_2], [b_2])]^\uparrow
\]
Example

\[
[(\cdot, [b_1]), ([a_1, a_2], [b_2])]^\uparrow
\]

\[
[[a_1, a_2], [\cdot]]^\uparrow
\]
Example
Example

\[
[(\emptyset, [b_1]), ([a_1, a_2], [b_2])]
\]

\[
[[a_1, a_2], []]
\]

\[
[([], [b_1]), ([a_1, a_2], [b_2])]
\]

\[
[([], [], [[a_1, a_2], []])]
\]
Example
Example

\[
\begin{align*}
\text{ax} & \quad \text{cut} \\
\Gamma & \quad \\
\text{?} & \quad \text{\&} \\
\text{?} & \quad (\cdot, [b_1, b_2])^\downarrow \\
\text{?} & \quad \\
\text{?} & \quad [([a_1, a_2], [\cdot])]\bigwedge \\
\text{?} & \quad \\
\text{?} & \quad ((([], [b_1]), ([a_1, a_2], [b_2])))^\uparrow
\end{align*}
\]
Failure...
Example
Example

\[ \left( [], [b_1] \right), \left( [a_1, a_2], [b_2] \right) \]
Example
Example

\[(\emptyset, [b_1]), ([a_1, a_2], [b_2])]^\uparrow\]
Example

\[
[(\emptyset, [b_1]), ([a_1, a_2], [b_2])]^\uparrow
\]
Example

\[(\emptyset, [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}\]
Example

\[ ([], [b_1]) , ([a_1, a_2], [b_2])^\dagger \]
Example
Example
Example
Example
Adapting the machine

Easy:

- add new rules corresponding to new cells
- coefficients of relational elements of counters now in $\mathbb{Z}$.

But does not work on all MELL structures...
We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere:
We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ??
We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ???
  • Something to do with polarity;
We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ???

• Something to do with polarity;
• contains the \( \lambda \)-calculus.
Theorem

In a fragment of MELL containing the $\lambda$-calculus, a machine $M^Φ$ decides $x \in \llbracket Φ \rrbracket$ in time $O(|x| \times |\text{cells of } Φ|)$.

Corollary (Terui, 2012)

Let

$$Str := !(X \rightarrow X) \dashrightarrow !(X \rightarrow X) \rightarrow X \rightarrow X$$
$$Bool := !X \rightarrow !X \rightarrow X$$

be the linear-logic translations of Church binary strings and booleans.

Let $t$ be a simply-typed $\lambda$-term of type $Str[A/X] \rightarrow Bool$, for arbitrary $A$. It decides a language $\mathcal{L}$.

$\mathcal{L}$ is in LinTIME (deterministic linear time).