

More intensional versions of Rice's Theorem

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Rice's and Asperti-Rice's Theorems

Rice's Theorem

A cornerstone of computability.

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Proof.

$p \neq$ infinite loop, $p \in \mathcal{P}$, loop $\notin \mathcal{P}$.

$q'(x) = q(0); p(x)$.

$q' \in P \Leftrightarrow q(0)$ terminates. □

The power of Rice

Rice's Theorem allows to prove undecidability of a wide range of sets of programs:

- programs which (don't) terminate on input 0;
- programs which return 42 on input 54;
- programs which return an even result on any prime input;
- programs computing a total function;
- programs computing a bijection;
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But it cannot be used for *intensional* sets that depend on **program** behaviour (complexity, ...)

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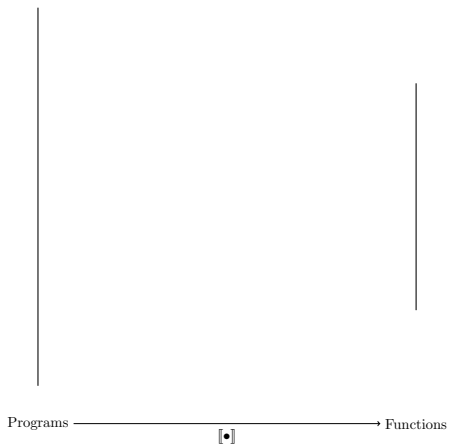
- \mathfrak{R} is undecidable;
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Theorem (Rice, again)

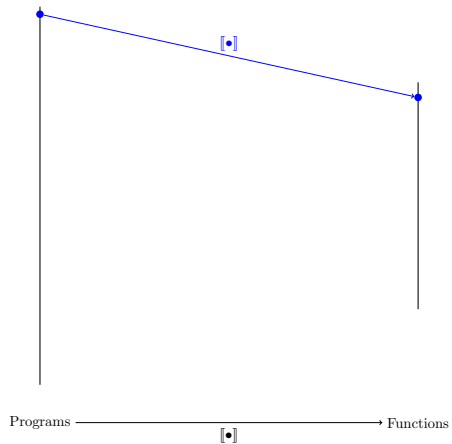
Any non-trivial set of programs which is the union of classes of \mathfrak{R} is undecidable.

What about equivalences more precise than \mathfrak{R} ?

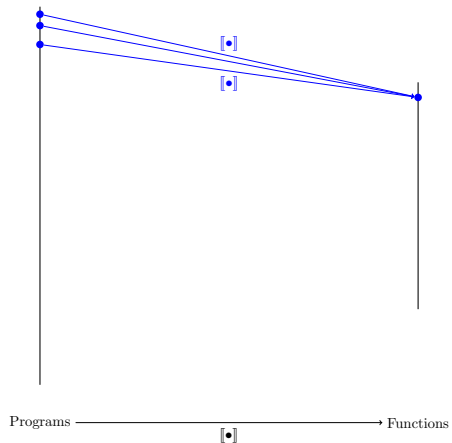
The semantics tunnel (1)



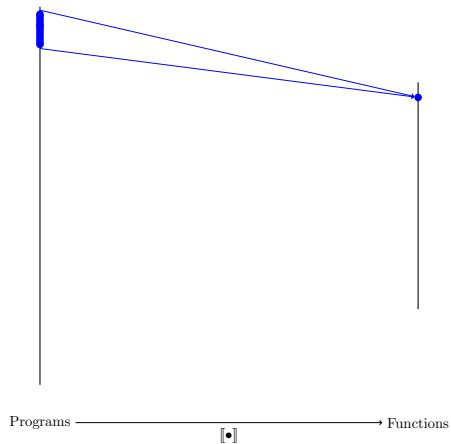
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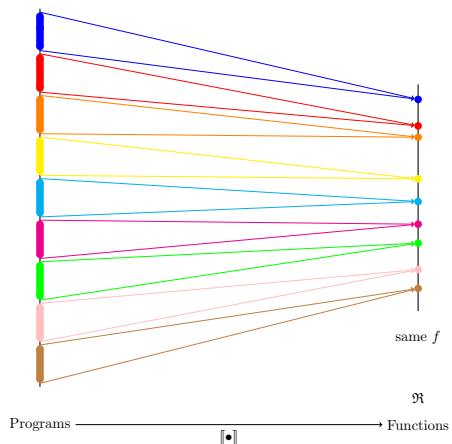
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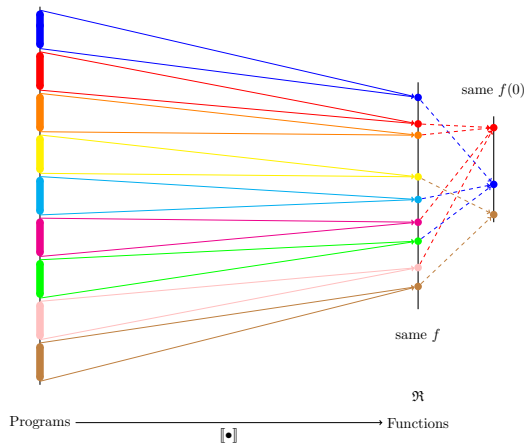
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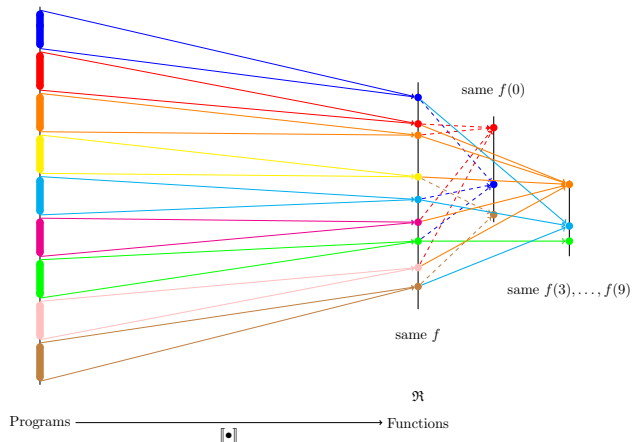
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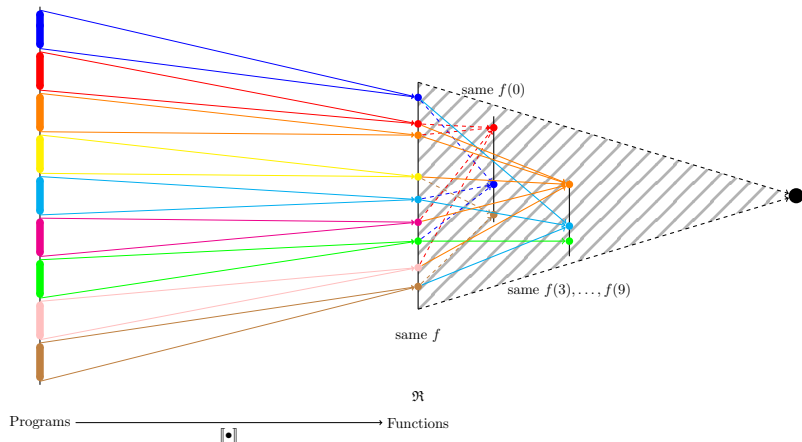
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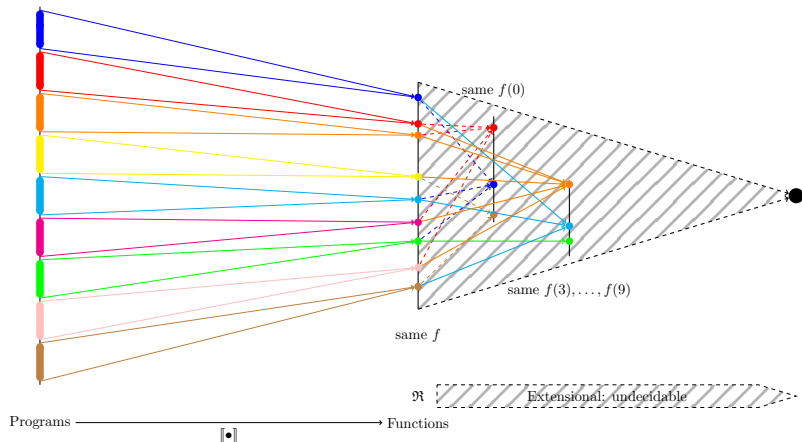
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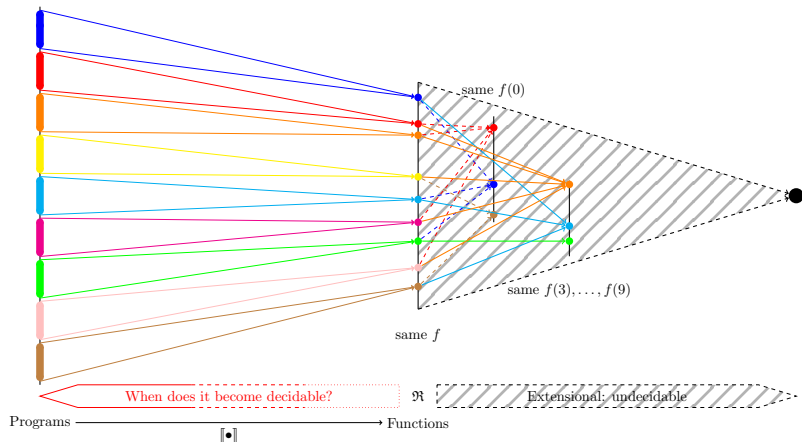
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Asperti-Rice's Theorem

A first intensional version of Rice's Theorem.

$$p \mathcal{A} q \Leftrightarrow \llbracket p \rrbracket = \llbracket q \rrbracket \text{ and } \text{cplx}(p) = \Theta(\text{cplx}(q)) \quad (\text{"clique"})$$

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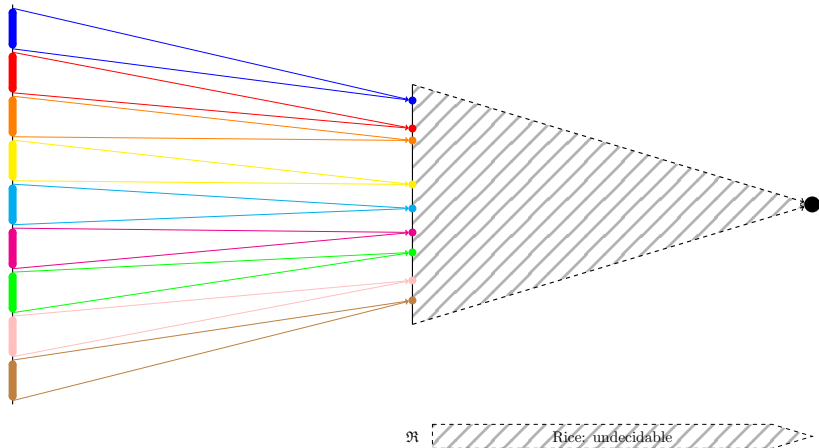
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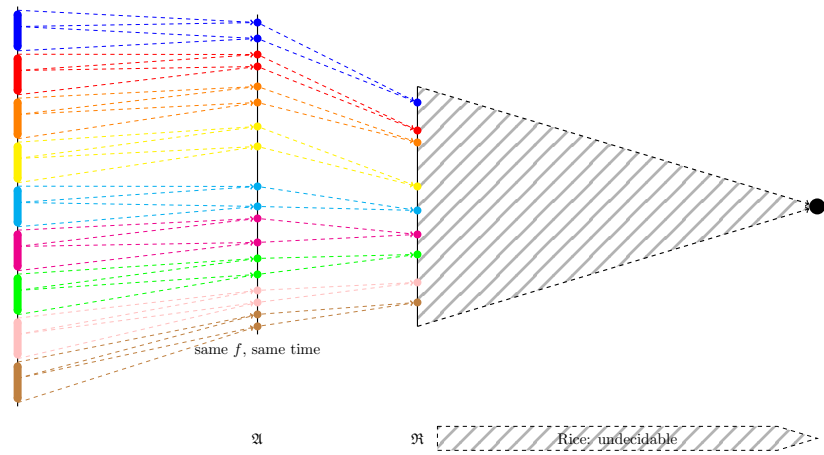
p not equivalent to infinite loop. $q'(x) = q(0); p(x)$.

If $q(0)$ terminates, it does so with a **fixed** complexity so p and q' have the same complexity up to an additive factor. \square

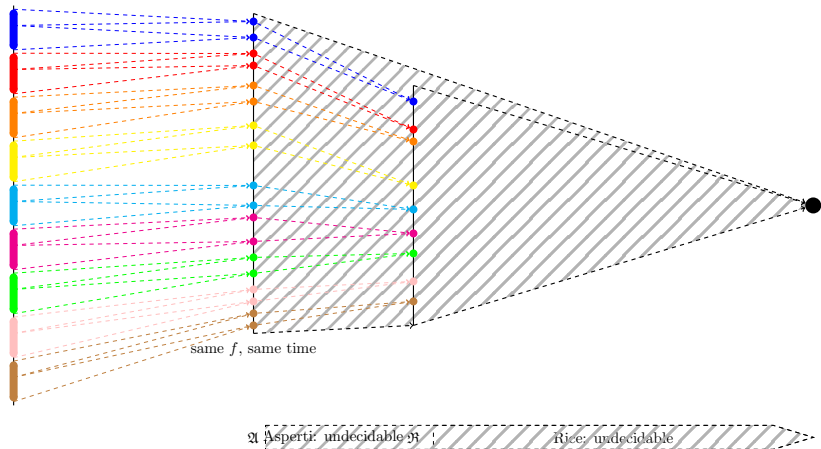
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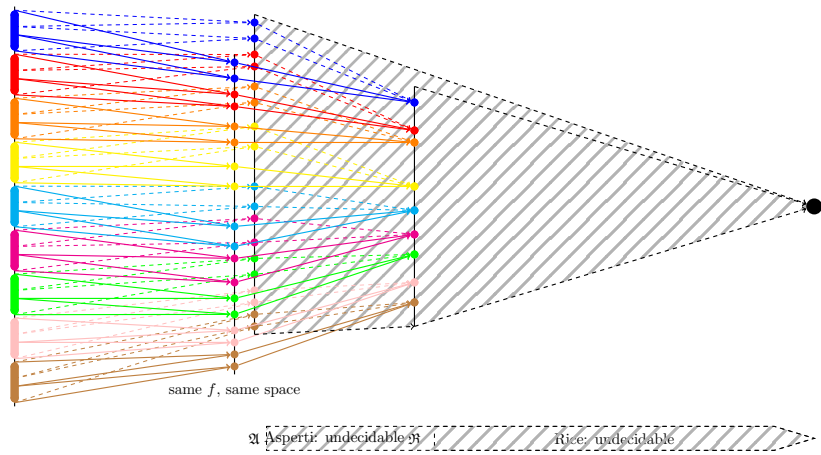
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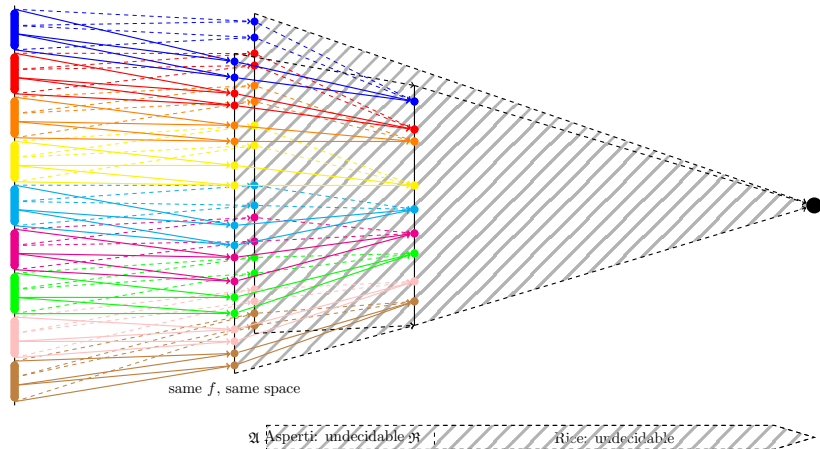
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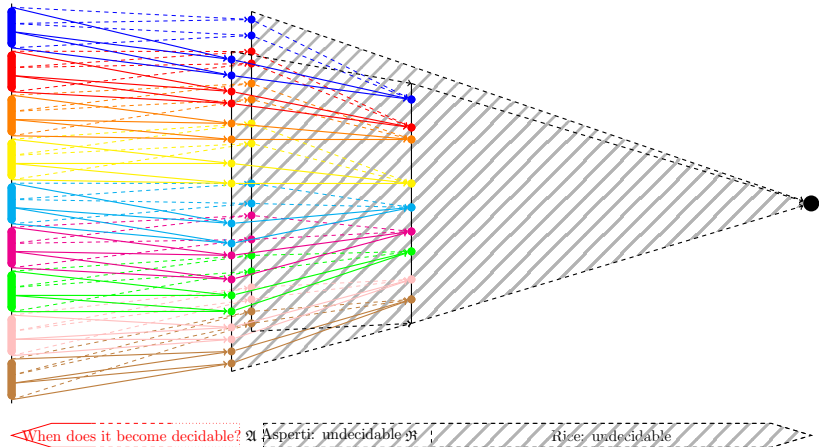
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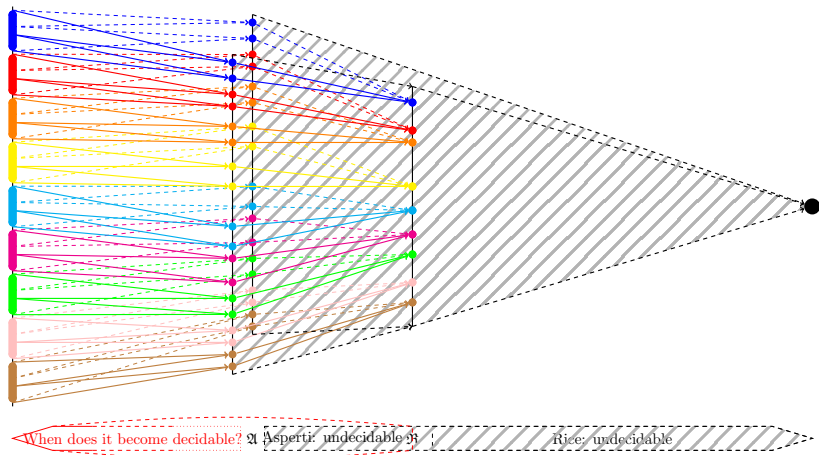
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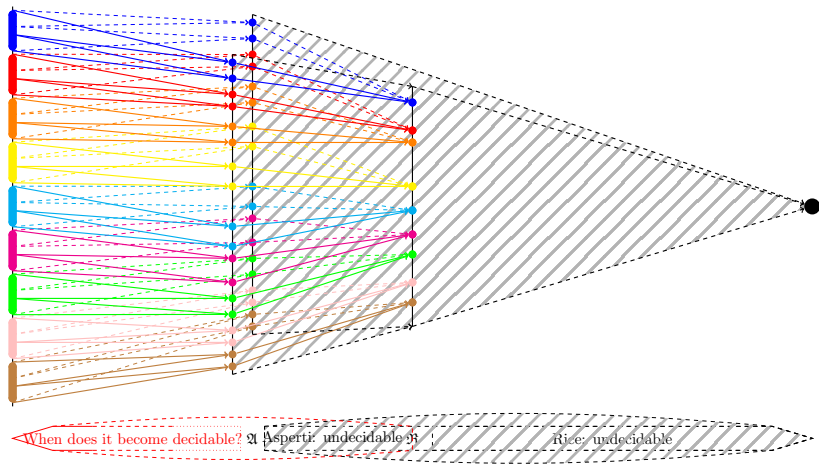
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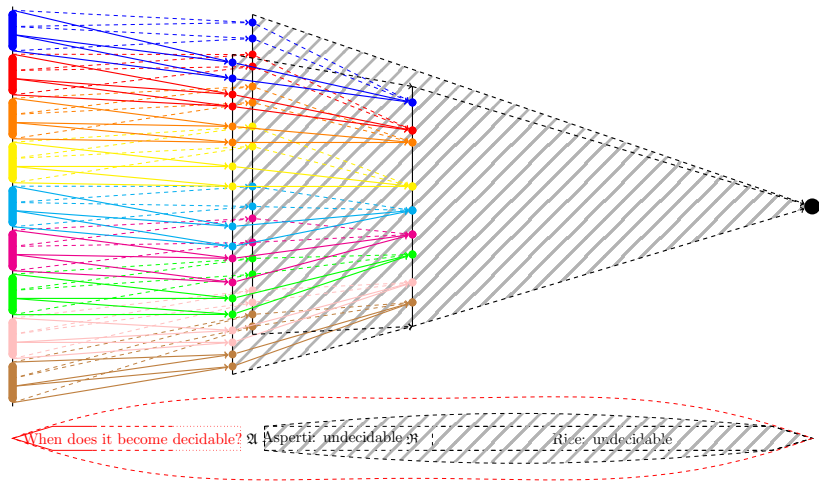
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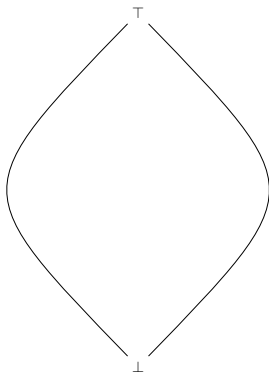
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The equivalences lattice

Not the subject of today's talk!

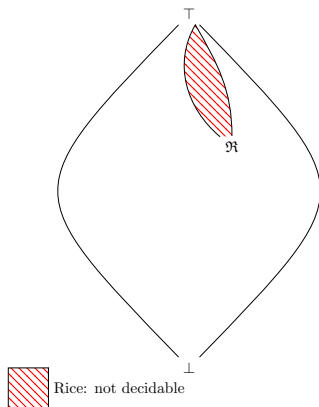
- The set of all equivalences is a complete lattice.
- \perp : equality, \top : one class with everything.



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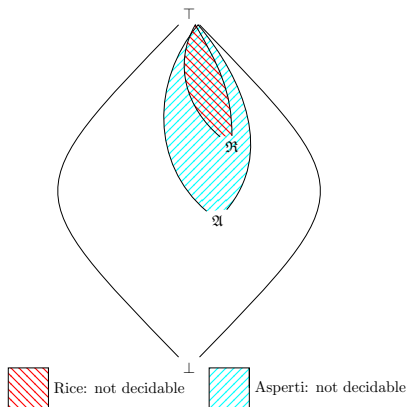
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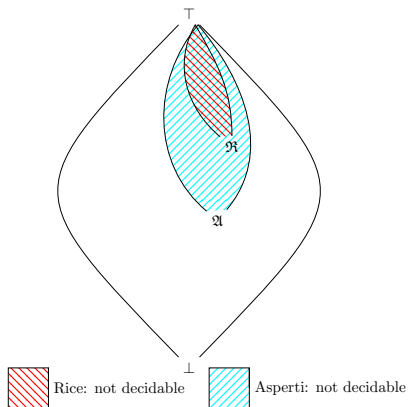
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Complicated and interesting structure.

Ongoing works with J. G. Simonsen and J. Avery.

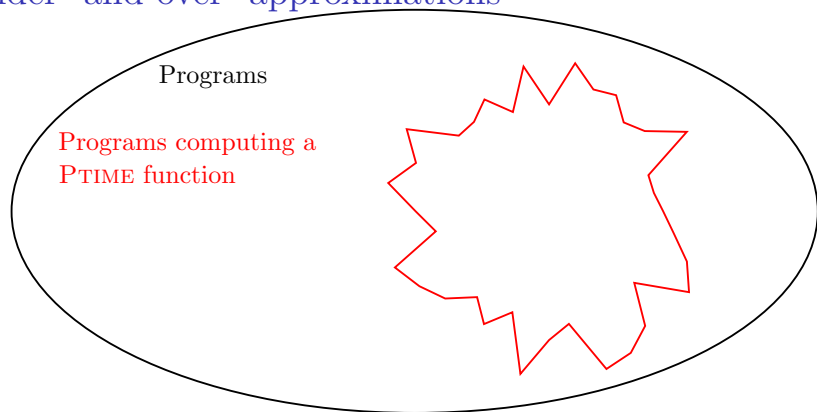
First generalisation


Today's talk

Two generalisations of Rice's Theorem relaxing the extensionality condition.

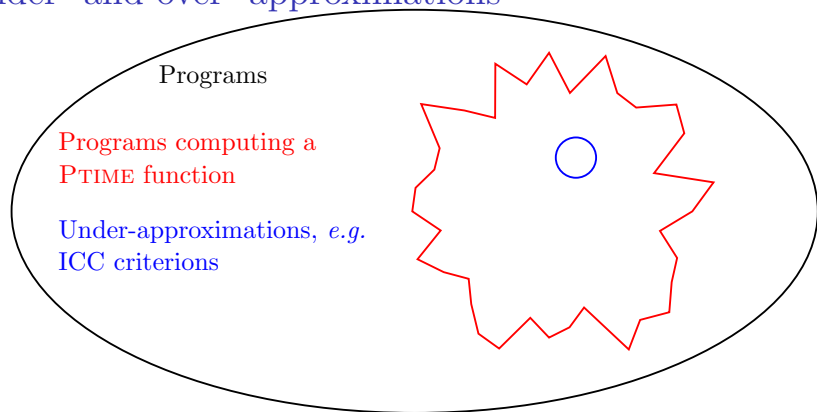
- 1 Rather than searching equivalences more precises than \mathfrak{R} , keep it but consider sets that are not just union of classes.
- 2 Try the same approach with a wide range of others equivalences.

Under- and over- approximations



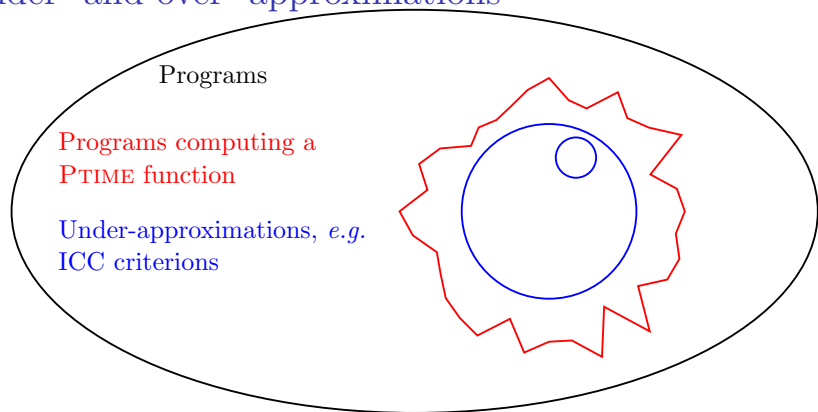
 is **not** PPTIME, the set of polytime **programs**. It is undecidable by Rice's Theorem.

Under- and over- approximations



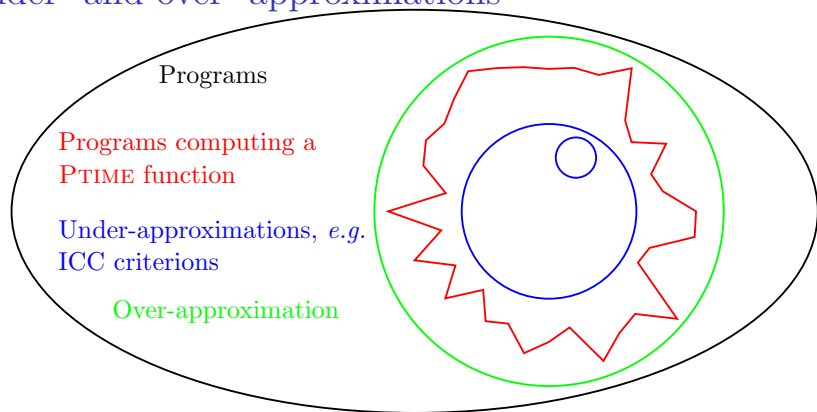
○ is an ICC criterion if it captures one program for each PTIME function.

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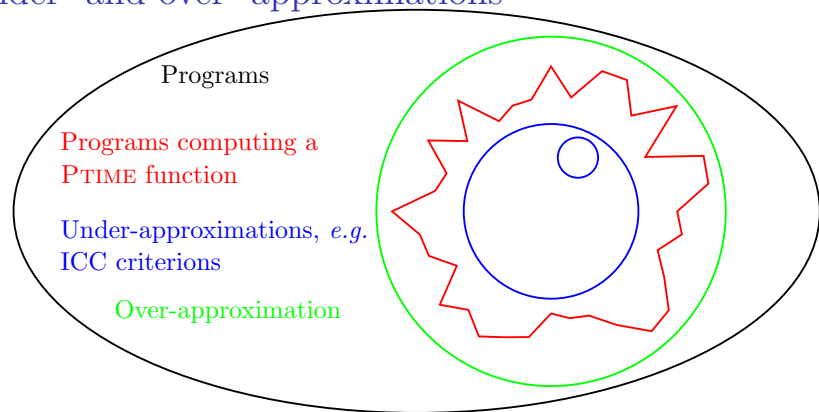
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Can \circ be decidable and “small enough”?

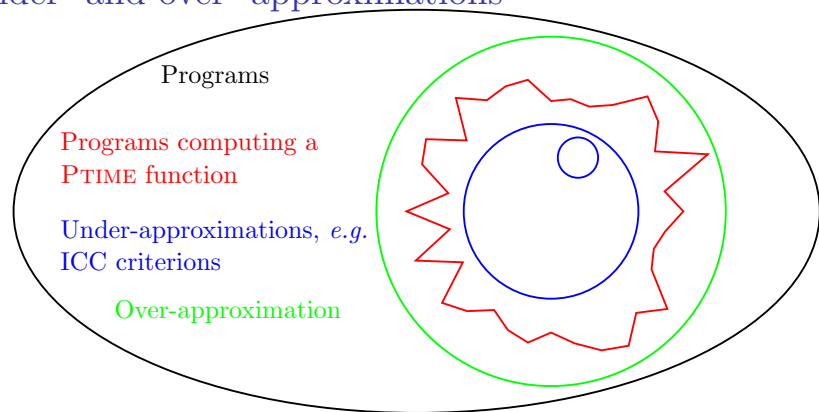
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Upper bound: $p \in \circ \Rightarrow \llbracket p \rrbracket \in \text{PTIME}$.

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Can \circ be decidable and “small enough”?

Upper bound: $p \in \circ \Rightarrow \llbracket p \rrbracket \in \text{PTIME}$.

Lower bound: $p \notin \circ \Rightarrow \llbracket p \rrbracket \notin \text{PTIME}$.

Vocabulary

A set of programs is:

- *non-trivial* if it is neither empty, nor the set of all programs.
- *extensional* if it is the union of classes of \mathfrak{R} ;
- *partially extensional* (for F) if it contains all the programs with $\llbracket \mathbf{p} \rrbracket \in F$ (over approximation).
- *extensionally complete* (for F) if it contains one program for each $f \in F$.
- *extensionally sound* (for F) if it contains only programs with $\llbracket \mathbf{p} \rrbracket \in F$ (under approximation).
- an *ICC characterisation* (of F) if it is both extensionally sound and complete for F .
- *extensionally universal* if it is extensionally complete for the set of computable partial functions.

First Result

Theorem

Any non-empty, partially extensional and decidable set is extensionally universal.

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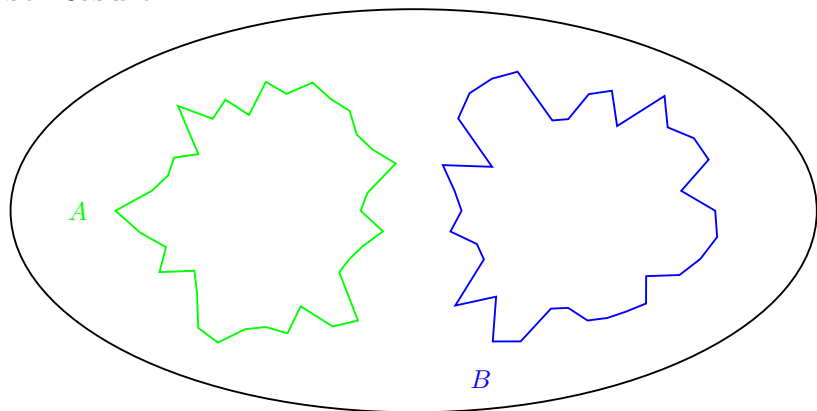
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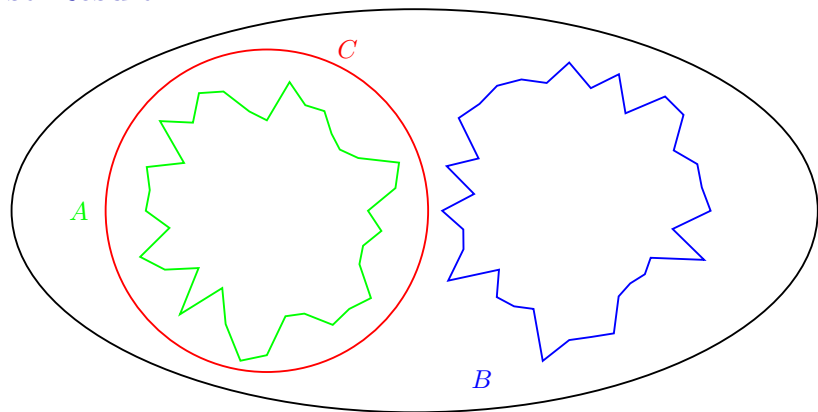
Definition

Two sets A and B are *recursively separable* if there exists C decidable with $A \subset C$ and $B \cap C = \emptyset$.

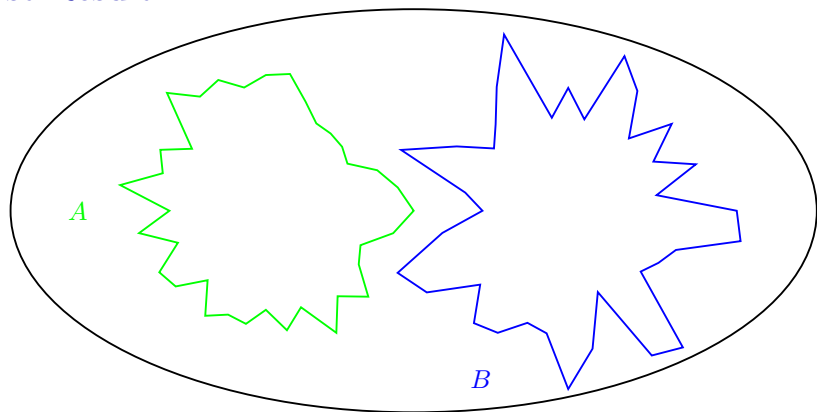
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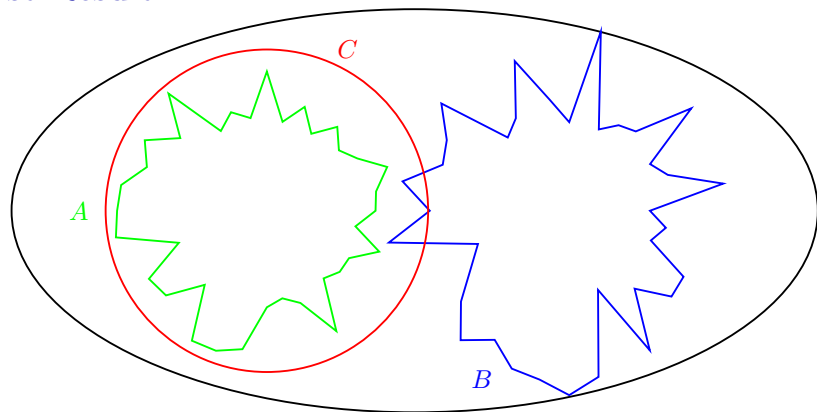
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Example

$$\left. \begin{aligned} A &= \{ p : \llbracket p \rrbracket(0) = 0 \} \\ B &= \{ p : \llbracket p \rrbracket(0) \notin \{0, \perp\} \} \end{aligned} \right\} \text{ recursively inseparable}$$

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\mathcal{P} decidable, partially extensional for $\llbracket p \rrbracket$, \mathcal{P} contains no program for $\llbracket q \rrbracket$.

$r'(x) = \text{if } r(0)=0 \text{ then } p(x) \text{ else } q(x)$



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$\llbracket r \rrbracket(0) = 0 \Rightarrow r' \in \mathcal{P}$

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Example (Rice)

Any non-empty extensional set is partially extensional. Hence, if decidable, must be extensionally universal, and thus trivial.

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Any computable function is computed by programs of arbitrarily large size.

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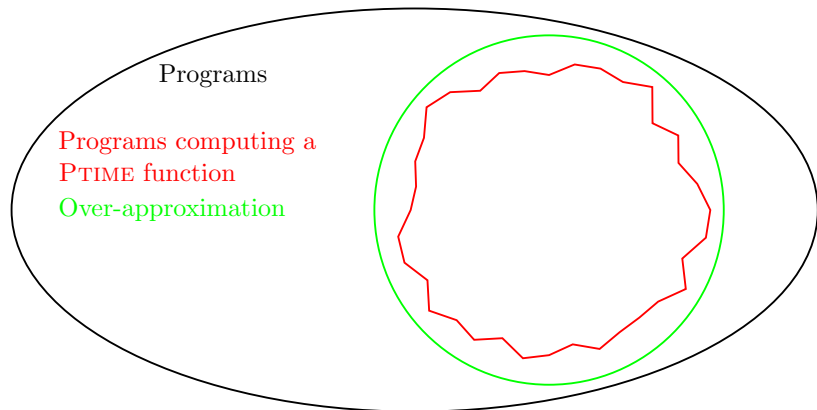
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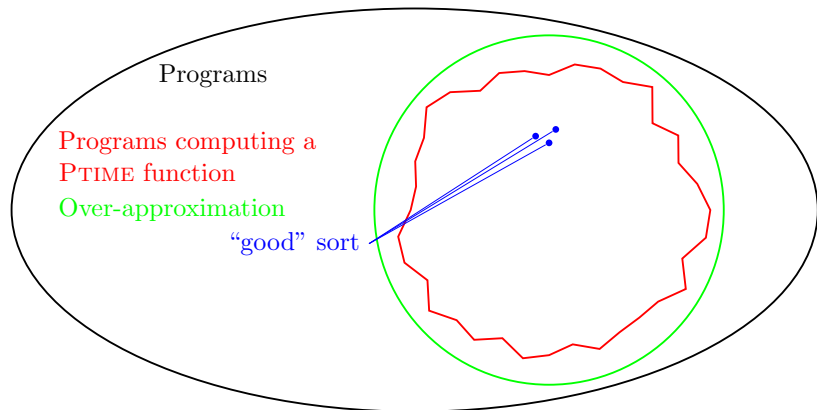
Example

Any decidable set containing all programs for PTIME functions contains programs for any computable function.

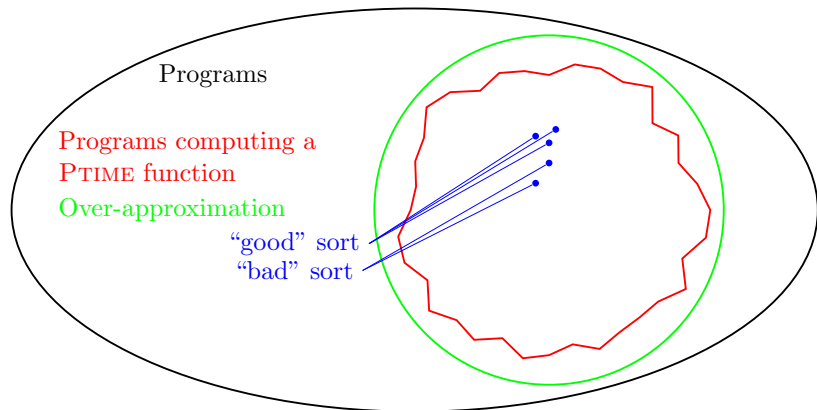
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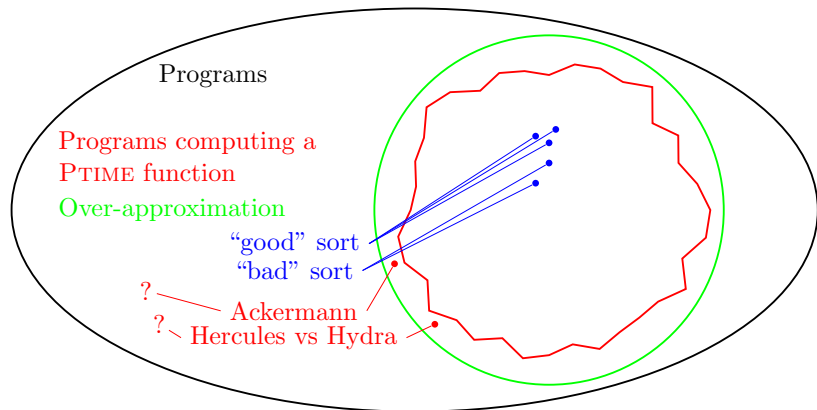
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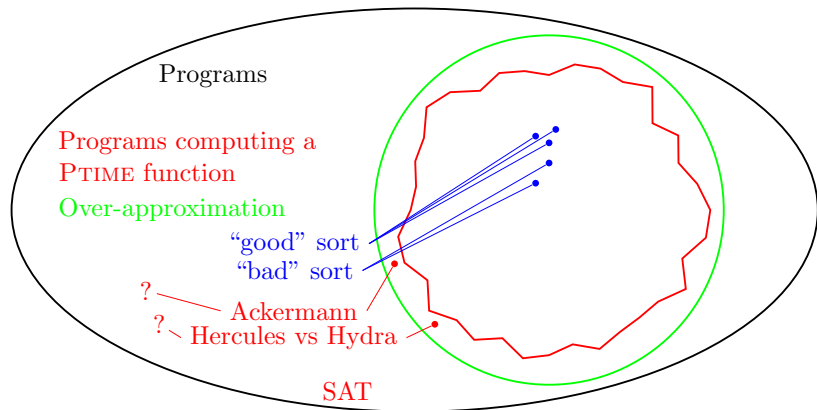
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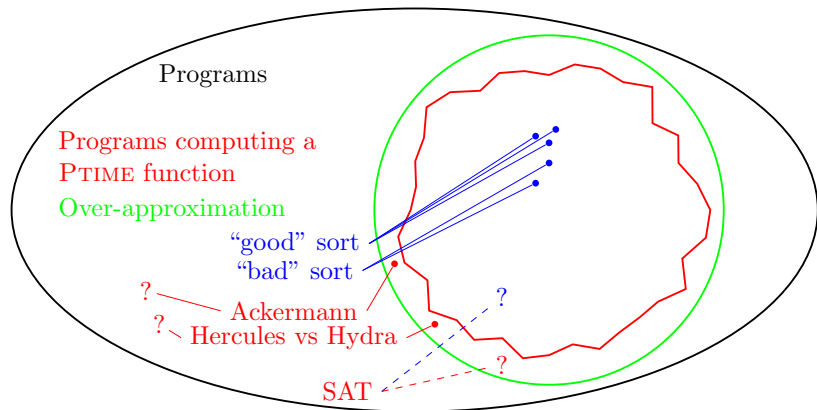
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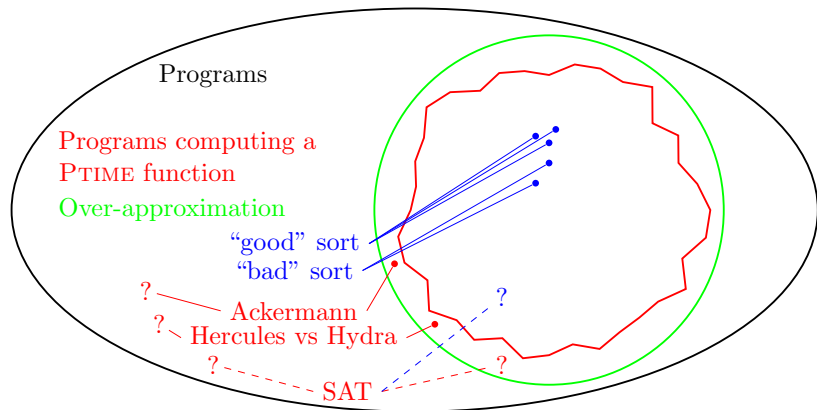
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Second generalisation

Switching families

Definition

(S, \approx) : a set and an equivalence.

switching family compatible with \approx : a family $I = (\pi_s)_{s \in S}$ of computable total functions $\pi_s : S \times S \rightarrow S$

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$$A_I = \{ s \in S : \forall x, y. \pi_s(x, y) \approx x \}$$

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 $B_I = \{ s \in S : \forall x, y. \pi_s(x, y) \approx y \}$

} recursively inseparable.

$\pi_s(x, y) \approx \left(\begin{array}{c} x \\ y \\ ??? \end{array} \right)$ for all or some x, y .

Switching families

Definition

(S, \approx) : a set and an equivalence.

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Example

Projections can form a switching family.

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Example

Projections can form a switching family.

Example (Standard switching family)

$r'(x) = \pi_r(p, q)(x) = \text{if } r(0)=0 \text{ then } p(x) \text{ else } q(x).$

Compatible with \mathfrak{R} (and many others).

Vocabulary

- \mathfrak{B} : equivalence on programs. A set of programs is:
- ~~extensional~~ *compatible* if it is the union of blocks of \mathfrak{B} ;
 - ~~partially-extensional~~ *partially compatible* if it contains one block of \mathfrak{B} ;
 - ~~extensionally-complete~~ *complete* (for a set of blocks) if it intersects each of these;
 - ~~extensionally-sound~~
 - ~~an ICC-characterisation~~
 - ~~extensionally-universal~~ *universal* if it intersects each single block of \mathfrak{B} .

Second Result

Theorem

Let \mathfrak{P} be a partition of a set S and $I = (\pi_s)_{s \in S}$ be a switching family compatible with it.

Any non-empty decidable partially compatible subset of S is universal.

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Any non-empty decidable partially compatible subset of S is universal.

Proof.

$$\left. \begin{array}{l} [x] \subset S', [y] \cap S' = \emptyset \quad s' = \pi_s(x, y) \\ \pi_s(x, y) \mathfrak{P} x \Rightarrow s' \in S' \\ \pi_s(x, y) \mathfrak{P} y \Rightarrow s' \notin S' \end{array} \right\} \text{recursively inseparable.} \quad \square$$

Example (1)

Theorem

Any non-empty decidable partially compatible set of programs is universal.

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Example (Complexity)

Φ : complexity measure (Blum). $p \equiv_{\Phi} q$ iff $\Phi_p \in \Theta(\Phi_q)$.

The standard switching family is compatible with \equiv_{Φ} .

$r'(x) = \pi_r(p, q)(x) = \text{if } r(0)=0 \text{ then } p(x) \text{ else } q(x)$.

when $r(0)$ terminates it does so with a constant complexity.

Any non-empty decidable set of programs partially compatible with \equiv_{Φ} is universal and must contain programs of arbitrarily high complexity.

Example (2)

Theorem

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Any non-empty decidable partially compatible set of programs is universal.

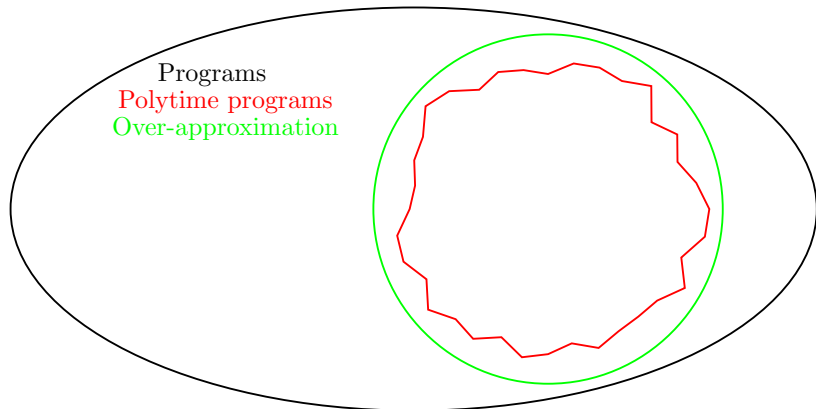
Example (Polynomial time)

Φ : time complexity. PPTIME: set of polytime *programs* (**not** all programs computing PTIME functions); it is undecidable and partially compatible with \equiv_{Φ} .

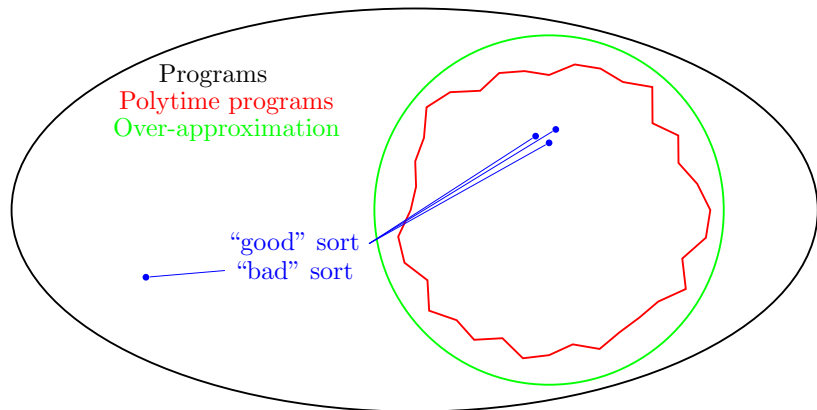
Any decidable set of programs including all polytime programs also includes programs of arbitrarily high time complexity.

Any attempt at finding a decidable over-approximation of PPTIME is doomed to also contain many extremely “bad” programs.

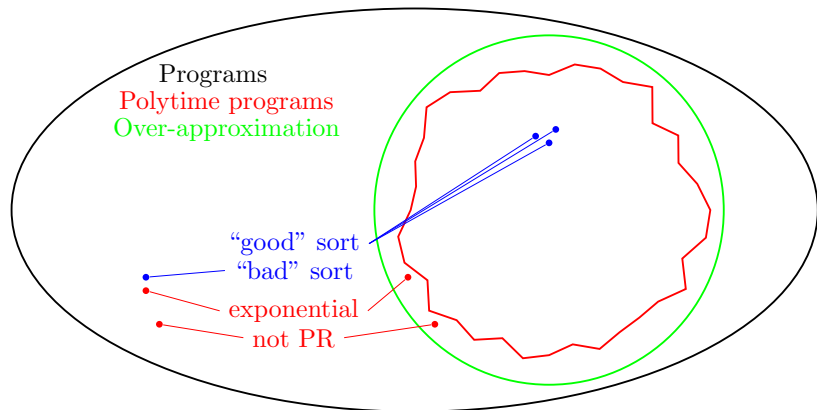
Example (2)



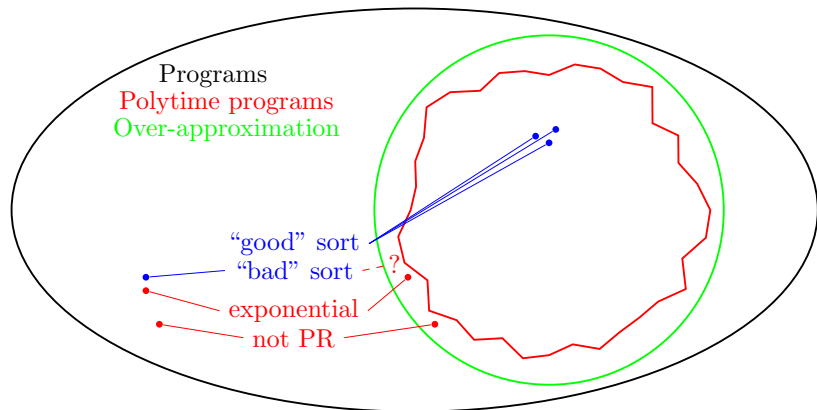
Example (2)



Example (2)



Example (2)



Example (3)

Theorem

Any non-empty decidable partially compatible set of programs is universal.

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Theorem

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Example (Linear space (not closed under composition))

Φ : space complexity. PLINSPACE: set of *programs* computing in linear space; it is partially compatible with \equiv_{Φ} .

Any decidable set of programs including all linear space programs also contains programs of arbitrarily high space complexity.

Example (Asperti-Rice)

Theorem

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Example (Asperti-Rice)

The standard switching family is compatible with $\mathfrak{A} = \mathfrak{R} \cap \equiv_{\Phi}$.

Any decidable non-empty set partially compatible with \mathfrak{A} is universal.

Especially, the only decidable unions of blocks of \mathfrak{A} are the trivial ones.

Going further

Example (Spambot)

$p \equiv q$ if they send the same number of mails (**not** a Blum complexity measure). The standard switching family is compatible with it.

Any decidable set containing all the programs that never send mail also contains spambots.

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Other equivalences?

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