Relational type-checking of connected proof-structures

Giulio Guerrieri[1,2], Luc Pellissier[3], and Lorenzo Tortora de Falco[1]

1 **Dipartimento di Matematica e Fisica, Università Roma Tre**
Rome, Italy
{gguerrieri,tortora}@uniroma3.it
2 **Institut de Mathématiques de Marseille, UMR 7373, Aix-Marseille Université**
F-13453 Marseille, France
3 **LIPN, UMR 7030, Université Paris 13, Sorbonne Paris Cité**
F-93430 Villetaneuse, France
luc.pellissier@lipn.univ-paris13.fr

Abstract

It is possible to define a typing system for Multiplicative Exponential Linear Logic (MELL): in such a system, typing judgments are of the form $\vdash R : x : ⅋\Gamma$, where R is a MELL proof-structure, Γ is the list of types of the conclusions of R, and x an element of the relational interpretation of $⅋\Gamma$, meaning that x is an element of the relational interpretation of R (of type $⅋\Gamma$).

As relational semantics can be used to infer execution properties of the proof-structure, these judgment can be considered as forms of quantitative typing.

We provide an abstract machine that decides, if R satisfies a geometric condition, whether the judgment $\vdash R : x : ⅋\Gamma$ is valid. Also, the machine halts in bilinear time in the sizes of R and x.

1 Introduction

Intersection types have been introduced as a way of extending the λ-calculus' simple types with finite polymorphism. This is done by adding a new type constructor $\cap$ and new typing rules. A term of type $A \cap B$ can be used in an ulterior derivation both as data of type A or of type B. Contrarily to simple types (which are sound but incomplete), intersection types are a sound and complete characterization of strong normalization.

Intersection types were originally idempotent, that is, the equation $A \cap A = A$ held. This corresponds to an interpretation of a term typed as $M : A \cap B$ as *M can be used as data of type A or as data of type B*. In a non-idempotent setting (*i.e.* by dropping the equation $A \cap A = A$), the meaning of the typing judgment is strengthened in *M can be used once as data of type A and once as data of type B*. Non-idempotent intersection types have been used to get qualitative and quantitative information on the execution time of λ-terms [1, 5].

Relational semantics is one of the simplest semantics of λ-calculus (and linear logic). A type is interpreted by a set, and a λ-term (or linear logic proof-structure) by a relation between sets which is invariant under β-reduction (and cut-elimination). It happens that the relational semantics corresponds to a non-idempotent intersection types system, called System R in [1] (see also [7]): a type derivation of a λ-term in System R corresponds to an experiment (see [4]) of a linear logic proof-structure, and the conclusion of such a type derivation corresponds to the result of this experiment *i.e.* a point in the relational semantics. So, knowing that an element is or not in the relational interpretation of a λ-term (or linear logic proof-structure) already gives a lot of information on the execution of this λ-term (or linear logic proof-structure) [1, 2]. For instance, given two correct (*i.e.* arising from a derivation on the sequent calculus) MELL proof-structure π_1 and π_2 without cuts, it is possible to compute whether π_1 and π_2 can be composed and the length of the reduction to the normal form of this composition.

We introduce semantical typing judgments of the form $\vdash R : x : \parr\Gamma$, where R is a MELL (the multiplicative-exponential fragment of linear logic) proof-structure whose conclusion is the list of MELL formulæ Γ, and x in the interpretation in the relational model of the MELL formula $\parr\Gamma$. Our goal is to decide in a tractable way whether a judgment of this form is *valid* or not, *i.e.* whether x is a point of the relational semantics of R or not.

We thus define the *Relational Interaction Abstract Machine* (Section 5) able to decide such judgments on a fragment of all MELL proof-structures, that works by moving tokens embodying relational elements through the proof-structure. The machine moreover stops on a sequent $\vdash R : x : \parr\Gamma$ after a number of steps bilinear in the size of x and of R.

The class of MELL proof-structures on which our machine is sound and complete, defined in Section 4, is moreover quite natural and large enough to contain the λ-calculus.

As a corollary, we prove that languages decided by simply-typed λ-terms of type $\mathsf{Str}[A/X] \multimap \mathsf{Bool}$ are in **LinTIME** (deterministic linear time).

2 Elements of MELL syntax

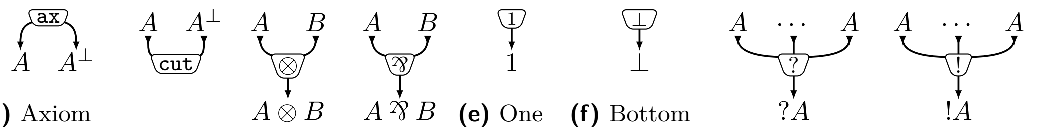
We set $\mathcal{L}_{\mathsf{MELL}} = \{1, \bot, \otimes, \parr, !, ?, ax, cut\}$. The MELL *connectives* are $1, \bot, \otimes, \parr, !, ?$. We say that 1, $\bot$, $\otimes$, $\parr$ (resp. $!$, $?$) are the *multiplicative* (resp. *exponential*) connectives, and 1, $\bot$ are the *units*.

The set of MELL *formulas* is generated by the grammar:

$$A, B, C ::= X \mid X^\bot \mid 1 \mid \bot \mid A \otimes B \mid A \parr B \mid !A \mid ?A\,.$$

where X ranges over a infinite countable set of *propositional variables*.

Proof-structures offer a syntax for MELL proofs. They are direct labelled graphs Φ built from the *cells*: We call *ports* the wires of such graphs, divided in *principal ports* (depicted



(a) Axiom **(b)** Cut **(c)** Tensor **(d)** Par **(e)** One **(f)** Bottom **(g)** Why Not? **(h)** Of Course!

Figure 1 The cells

down in the picture) and *auxiliary ports* (depicted up).

They are moreover endowed with a function box_Φ from the $?$ and cut cells to auxiliary ports of $!$ cells such that:

- all cells in the image of box_Φ have exactly one auxiliary port;
- inclusion of proof-structures obtained by chosing all cells which are above cells of the same image through box_Φ is a tree-like order.

We say that a proof-structure is a MELL proof-structure if all $!$-cells are in the image of box_Φ. We say that it is a DiLL_0 proof-structure if box_Φ is the empty function.

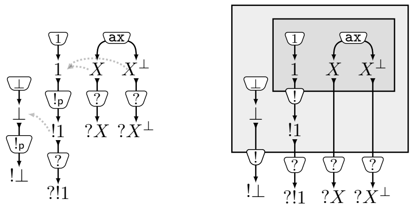


Figure 2 A MELL-ps, in our arrowed syntax and in the more traditionnal boxed syntax

3 Elements of relational semantics

We define relational experiments straightforwardly on DiLL_0 proof-structures (that is, proof-structures without boxes, but with arbitrary co-structural cells) by adapting the definition in [4]: a partial experiment is a function associating with a link an element of the interpretation of its type coherently with the structure. We define the relational semantics of a DiLL_0 proof-structure as the set of results of its experiments, *i.e.* the image of its conclusions through its experiments. The relational semantics of a MELL proof-structure R is just the union of the relational semantics of the DiLL_0 proof-structures in the Taylor expansion of R [3].

The Taylor expansion acts as a bridge between syntax and semantics, allowing to retain the simplicity of the multiplicative fragment while expanding it to the full MELL.

A drafted detailed of MELL syntax and relational semantics is available at `http://www.pps.univ-paris-diderot.fr/~giuliog/injtaylorLong.pdf`.

4 ?-connection

We now introduce the fragment on which our algorithm will act: ?-connected MELL proof-structures.

▶ **Definition 1** (?-path, ?-accessibility)**.** Let R be a MELL proof-structure.

A *?-path on R (from p_0 to p_n)* is a finite sequence $(p_0, \dots, p_n)$ of ports of R obtained by applying a finite number of times the following rules:

1. (p) is a ?-path for any p port of R;
2. if $\vec{p} = (p_0, \dots, p_n)$ is a ?-path where p_n is a port of a cell l of R of type not ?, then $\vec{p} \cdot q$ is a ?-path, for any q port of l;
3. if $\vec{p} = (p_0, \dots, p_n)$ is a ?-path where $p_n \neq p_0$ is a port of a ?-cell l of R, and if for all ports r of l, save at most one, there is a ?-path from p_0 to r, then $\vec{p} \cdot q$ is a ?-path, for any q port of l.

For every port p of R, the set of the *?-accessible ports from p in R* is

$\mathsf{acces}^?_R(p) = \{q \in \mathcal{P}_R \mid \text{there is a ?-path in } R \text{ from } p \text{ to } q\}$.

▶ **Definition 2** (?-path inside a box, ?-connectedness)**.** Let R be a MELL proof-structure.

Given a !-cell l, a ?-path $\vec{p} = (p_0, \dots, p_n)$ in R is *inside the box of l* if p_i is in the box of l for any $0 \leq i \leq n$.

R is *?-connected* if

- for any !-cell l and any port p inside the box of l, there is a ?-path inside the box of l from the principal door of the box of l to p;
- all the ports at depth 0 are ?-accessible from the conclusions.

This technical condition arises from the algorithm presented next. Nonetheless, the fragment of ?-connected proof-structures is quite general: all MELL proof-structures which are translations of λ-terms are ?-connected.

5 Recognition of the relational interpretation

We now introduce the main object of this article: the Relational Interaction Abstract Machine that decides the semantic sequents. The notation is inspired by Danos, Régnier, Mackie and Laurent's Interaction Abstract Machine [6]. Indeed, this work has a distinct Geometry of Interaction flavour.

The definition of the machine is in the Appendix. The main idea behind it is that its state is composed of tokens containing a relational element that travel through the proof-structure, obeying type-directed rules. For instance, whenever a token goes up through a $\otimes$ cell, it splits into two tokens, one going left, one going right. A token going up and one going down containing the same relational element anihilate when they meet. The only thing that could cause non-determinism are the contractions, where we don't know how to split a multiset between the different branches: that's why the ?-connection condition restricts the way contractions arise in the structures.

▶ **Lemma 3.** *Let R be a* MELL *proof-structure whose conclusions are ordered.*
A successful run of M^R defines a partial experiment of R.
Reciprocally, an experiment of R defines a successful run of M^R.

▶ **Theorem 4.** *Let R be a ?-connected* MELL *proof-structure whose conclusions Γ are ordered, and let $x \in |\parr\Gamma|$.*

The point x is in the relational interpretation of R iff M^R runs successfully on x.

Moreover, if R is acyclical, if we write $|x|$ the number of atoms appearing in x and $\mathsf{size}(R)$ *the number of links in R, the machine halts after $O(|x| \times \mathsf{size}(R))$.*

The Relational Interaction Abstract Machine decides sequents of the form $\vdash R : x : \parr\Gamma$, when R is acyclical and ?-connected, in bilinear time in the sizes of R and x.

In particular, the machine decides in bilinear times such sequents for correct proof-structures.

This result can be used in the following special case: we know (from the aforelinked long version) that a certain point (an injective 2-point) of a ?-connected MELL proof-structure characterizes entirely the proof-structure. So our algorithm can answer the following question: given a ?-connected MELL proof-structure R (of conclusions Γ) and a cut-free MELL proof-structure S (with the same conclusions), is S the normal form of R?

In the general case, there is no better algorithm than performing the cut-elimination on R and verifying whether the resulting proof-structure is isomorphic to S. In the box-connected case, it suffices to compute an injective 2-point of S (which faithfully represents S) and to verify that it is an element of the interpretation of R.

▶ **Definition 5** (Injective 2-point)**.** An *injective 2-point* is a point x of the relational interpretation of a MELL-proof structure R such that:

- each atom appearing in x appears exactly twice;

- every multiset in x corresponding to a co-contraction in the difnet from which it arose is of cardinality (counted with its multipicity) 2.

Every MELL proof-structure has injective 2-points. Theye are moreover all equivalent under the substitution of atoms.

▶ **Theorem 6.** *If R and S are two ?-connected* MELL *proof-structures of same (ordered) conclusions, and S is moreover cut-free, M^R runs successfully on any 2-point of S if and only if S is isomorphic to the normal form of R.*

We use here ?-connection twice: the recognition algorithm requires R to be ?-connected, and ?-connection allows us to limit ourselves to having to check the 2-point of S.

The main theorem also have an interesting corollary, proven but unpublished by Terui:

▶ **Theorem 7** (Terui, 2012). *Let*

$$\begin{aligned}\mathsf{Str} &:= !(X \multimap X) \multimap !(X \multimap X) \multimap X \multimap X\\ \mathsf{Bool} &:= !X \multimap !X \multimap X\end{aligned}$$

be the linear-logic translations of Church binary strings and booleans.

Let R be a simply-typed MELL*-proof structure of type* $\mathsf{Str}[A/X] \multimap \mathsf{Bool}$*, for arbitrary A. It decides a language $\mathcal{L}$.*

If R is ?-connected, then $\mathcal{L}$ is in **LinTIME** *(deterministic linear time).*

The result is surprising, as ?-connected proof-structures encompass the call-by-name translation of simply-typed λ-calculus, and simply-typed λ-terms of type $\mathsf{Nat}[A/X] \multimap \mathsf{Nat}$ can represent a function of complexity an arbitrary tower of exponentials.

Acknowledgements

The authors thank Damiano Mazza for making them aware of Terui's theorem. Work partially supported by ANR projects Coquas ANR-12-JS02-006-01 and Elica ANR-14-CE25-0005.

References

1 Daniel de Carvalho. Execution time of lambda-terms via denotational semantics and intersection types. To appear in *Mathematical Structures in Computer Science*, 2009. Available at `http://arxiv.org/abs/0905.4251`.

2 Daniel de Carvalho, Michele Pagani, and Lorenzo Tortora de Falco. A semantic measure of the execution time in Linear Logic. *Theoretical Computer Science, Special issue Girard's Festschrift*, 412(20):1884–1902, 2011.

3 Thomas Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

4 Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

5 Stéphane Graham-Lengrand and Alexis Bernadet. Non-idempotent intersection types and strong normalisation. *Logical Methods in Computer Science*, Volume 9, Issue 4(4), October 2013.

6 Olivier Laurent. A token machine for full geometry of interaction. *Typed Lambda Calculi and Applications*, 2001.

7 Luca Paolini, Mauro Piccolo, and Simona Ronchi Della Rocca. Essential and relational models. *Mathematical Structures in Computer Science*, FirstView:1–25, September 2015.

A The Machine: Formally

▶ **Definition 8** (Relational Interaction Abstract Machine). Let R be a MELL proof-structure where Γ is the list of its (ordered) conclusions.

A *state* of the *machine* M^R associated with the MELL proof-structure R is a multiset of *tokens* $A^{\updownarrow}(p, x, s)$ where

- A is a MELL formula,
- $\updownarrow \in \{\uparrow, \downarrow\}$,
- p is a port of R of type A,
- $x \in |A|$ or $x = \mathbf{0}$, with $\mathbf{0}$ neutral for multiset sum,
- s is a stack of box-cells of R.

The machine follows the transitions of Figures 3, 4 and 5: a rule of the form $\cdot \xrightarrow[P]{P'} \cdot$ removes from the state the tokens on the left and adds to it the tokens on the right, if the guard condition P and P' are verified. Notations of the Figures:

- c (respectively d) is always the cell of R such that p is its principal (respectively auxiliary) port, when it exists and is unique;
- $\mathsf{P}^{\mathsf{pri}}_R(c)$ denotes the principal ports of c; it is either a set (denoted by curly brackets $\{\cdot\}$) or an ordered pair (denoted by angle brackets $\langle\cdot\rangle$);
- $\mathsf{P}^{\mathsf{aux}}_R(c)$ denotes the auxiliary ports of c; it is either a set (denoted by curly brackets $\{\cdot\}$) or an ordered pair (denoted by angle brackets $\langle\cdot\rangle$);
- $\mathsf{auxd}_R(l)$ denotes the auxiliary doors of a box rooted in the cell l;
- tp is the type of a port.

A *run* of M^R on $(x_1, \ldots, x_n) \in \Gamma$ is any succession of transitions with the machine initialized in the state

$$\left[A_i^{\uparrow}(\mathsf{concl}_R(i), x_i, \varepsilon), 1 \leqslant i \leqslant n\right].$$

where $\mathsf{concl}_R(i)$ is an enumeration of the conclusions of R.

We say that M^R *accepts* $(x_1, \ldots, x_n)$ if there exists a run of M^R on $(x_1, \ldots, x_n)$ that halts on the empty state.

$$
\begin{aligned}
A^{\uparrow}(p,a,s) &\xrightarrow[\mathsf{P}^{\mathsf{pri}}_{\Phi}(c)=\{p,p'\}]{c:ax} A^{\downarrow}(p',a,s)\\
A^{\downarrow}(p,a,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{\Phi}(d)=\{p,p'\}]{d:cut} A^{\uparrow}(p',a,s)\\
A^{\downarrow}(p,a,s), A^{\uparrow}(p,a,s) &\to \varnothing\\
1^{\uparrow}(p,(),s) &\to \varnothing\\
\bot^{\uparrow}(p,(),s) &\to \varnothing\\
A\otimes B^{\uparrow}(p,(a,b),s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{\Phi}(c)=\langle p_A,p_B\rangle]{c:\otimes} A^{\uparrow}(p_A,a,s), B^{\uparrow}(p_B,b,s)\\
A\parr B^{\uparrow}(p,(a,b),s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\parr} A^{\uparrow}(p_A,a,s), B^{\uparrow}(p_B,b,s)\\
A^{\downarrow}(p_A,a,s), B^{\downarrow}(p_B,b,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\otimes} A\otimes B^{\downarrow}(p,(a,b),s)\\
A^{\downarrow}(p_A,a,s), B^{\downarrow}(p_B,b,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\parr} A\parr B^{\downarrow}(p,(a,b),s)
\end{aligned}
$$

Figure 3 Multiplicative transitions

$$
\begin{aligned}
A^{\downarrow}(p_A,a,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\otimes} A\otimes B^{\downarrow}(p,(a,\bullet),s), B^{\uparrow}(p_B,\bullet,s)\\
B^{\downarrow}(p_B,b,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\otimes} A\otimes B^{\downarrow}(p,(\bullet,b),s), A^{\uparrow}(p_A,\bullet,s)\\
A^{\downarrow}(p_A,a,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\parr} A\parr B^{\downarrow}(p,(a,\bullet),s), B^{\uparrow}(p_B,\bullet,s)\\
B^{\downarrow}(p_B,b,s) &\xrightarrow[\mathsf{P}^{\mathsf{aux}}_{R}(c)=\langle p_A,p_B\rangle]{c:\parr} A\parr B^{\downarrow}(p,(\bullet,b),s), A^{\uparrow}(p_A,\bullet,s)
\end{aligned}
$$

Figure 4 Cyclicity transitions. $\bullet$ denotes a fresh variable.

$$!A^{\uparrow}(p,[a_1,\dots,a_n],s) \xrightarrow[\substack{\mathsf{P}^{\mathsf{aux}}_R(c)=\{p'\}\\ n\neq 0}]{c:!} A^{\uparrow}(p',a_1,s\cdot p),\dots,A^{\uparrow}(p',a_n,s\cdot p)$$

$$!A^{\uparrow}(p,[],s) \xrightarrow[\mathsf{auxd}_R(l)=\{p_1,\dots,p_n\}]{c:!} \mathsf{tp}(p_1)^{\downarrow}(p_1,\mathbf{0},s),\dots,\mathsf{tp}(p_n)^{\downarrow}(p_n,\mathbf{0},s)$$

$$?A^{\uparrow}(p,[],s) \xrightarrow[\mathsf{P}^{\mathsf{aux}}_R(c)=\varnothing]{c:?} \varnothing$$

$$\left.\begin{array}{c} A^{\downarrow}(p_1,a_1^1,s\cdot l_1^1\cdots l_1^{k_1}) \\ \dots \\ A^{\downarrow}(p_1,a_1^{m_1},s\cdot l_1^1\cdots l_1^{k_1}) \\ \dots \\ A^{\downarrow}(p_n,a_n^1,s\cdot l_n^1\cdots l_n^{k_n}) \\ \dots \\ A^{\downarrow}(p_n,a_n^{m_n},s\cdot l_1^1\cdots l_n^{k_n}) \end{array}\right\} \xrightarrow[\substack{\mathsf{P}^{\mathsf{aux}}_R(d)=\{p_1,\dots,p_n\}\\ s\neq s'\cdot l' \text{ with } p_i\in\mathsf{auxd}_R(l')\\ p_i\in\mathsf{auxd}_R(l_i^j)}]{d:?} ?A^{\downarrow}\left(p,\left[a_1^1,\dots,a_n^{m_n}\right],s\right)$$

$$\left.\begin{array}{c} A^{\downarrow}(p_1,a_1^1,s\cdot l_1^1\cdots l_1^{k_1}) \\ \dots \\ A^{\downarrow}(p_1,a_1^{m_1},s\cdot l_1^1\cdots l_1^{k_1}) \\ \dots \\ A^{\downarrow}(p_n,a_n^1,s\cdot l_n^1\cdots l_n^{k_n}) \\ \dots \\ A^{\downarrow}(p_n,a_n^{m_n},s\cdot l_n^1\cdots l_n^{k_n}) \\ ?A^{\downarrow}\left(p,\left[a_1^1,\dots,a_n^{m_n},a'_1,\dots,a'_p\right],s\right) \end{array}\right\} \xrightarrow[\substack{\mathsf{P}^{\mathsf{aux}}_R(d)=\{p_1,\dots,p_n\}\cup\{p'\}\\ s\neq s'\cdot l' \text{ with } p_i\in\mathsf{auxd}_R(l')\\ \forall i,\forall j,p_i\in\mathsf{auxd}_R(l_i^j)\\ \forall i,p'\in\mathsf{auxd}_R(l'_i)}]{d:?} \left\{\begin{array}{c} A^{\uparrow}(p',a'_1,s\cdot l'_1\cdots l'_k) \\ \dots \\ A^{\uparrow}(p',a'_p,s\cdot l'_1\cdots l'_k) \end{array}\right.$$

Figure 5 Exponential transitions.