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DISSÉQUER LES SÉMANTIQUES DÉNOTATIONNELLES :

DU BIEN ÉTABLI $\mathcal{H}^*$ AUX PLUS RÉCENTS COEFFETS QUANTITATIFS

DISSECTING DENOTATIONAL SEMANTICS:

FROM THE WELL-ESTABLISHED $\mathcal{H}^*$ TO THE MORE RECENT QUANTITATIVE COEFFECTS

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Résumé

Disséquer les sémantiques dénotationnelles :

Du bien établi $\mathcal{H}^*$ aux plus récents coeffets quantitatifs

À travers la résolution de deux problèmes ouverts bien différents, cette thèse présente quatre approches distinctes permettant d'observer finement et de classer des modèles dénotationnels.

Dans un premier temps, l'on s'intéressera au λ-calcul non typé et à sa théorie observationnelle $\mathcal{H}^*$ (en appel de tête). Nous donnons une caractérisation, au sein d'une large classe de modèles du λ-calcul non typé, des modèles pleinement adéquats pour $\mathcal{H}^*$: un K-modèle extentionnel D est pleinement adéquat si et seulement s'il est hyperimmune, *i.e.*, les chaines mal fondées d'éléments de D ne sont pas capturées par aucune fonction récursive. Nous allons présenter deux techniques permettant de prouver cette même caractérisation de deux manières indépendantes: l'une purement sémantique et l'autre purement syntaxique. La preuve sémantique consiste en l'utilisation d'un modèle syntaxique, les arbres de Böhm, et de propriétés d'approximation filtrant le plongement de ces derniers dans nos modèles. La preuve syntaxique consiste en l'utilisation de syntaxes sémantiques, les λ-calculs avec tests, qui transcrivent les propriétés internes des modèles à un niveau syntaxique.

Dans un second temps, l'on s'intéressera aux $B_{\mathcal{S}}LL$, qui sont des raffinements de la logique linéaire où les exponentielles sont paramétrées par les éléments d'un semi-anneau $\mathcal{S}$. Ces semi-anneaux capturent une notion de coeffet, *i.e.*, les hypothèses requises par un programme sur son contexte (accessibilité d'une ressource, prérequis sur une entrée...). Nous faisons ici la première analyse dénotationnelle de $B_{\mathcal{S}}LL$. En particulier, nous décrivons deux manières d'extraire de tels modèles. L'une est *"a priori"*: il s'agit de choisir un semi-anneau $\mathcal{S}$ le long duquel on va stratifier l'exponentielle usuelle de la logique linéaire. L'autre est au contraire *"a posteriori"*: elle montre qu'à un modèle de la logique linéaire est naturellement associé une sorte de semi-anneau sémantique résultant en une interprétation de $B_{\mathcal{S}}LL$.

Abstract

Dissecting denotational semantics:

From the well-established $\mathcal{H}^*$ to the more recent quantitative coeffects

Throughout the resolution of two different open problems, this thesis presents four distinct approaches enabling refine observations and classifications of denotational models.

First, we treat the untyped λ-calculus and its observational theory $\mathcal{H}^*$ (wrt head reduction). We give a characterization, with respect to a large class of models of untyped λ-calculus, of those models that are fully abstract for $\mathcal{H}^*$: an extensional K-model D is fully abstract if and only if it is hyperimmune, *i.e.*, non-well founded chains of elements of D cannot be captured by any recursive function. In fact, we will present two different techniques leading to two different proofs of the same characterization: one purely semantic and another one purely syntactic. The semantic proof consists in the use of a syntactical model, the Böhm trees, together with aproximation properties that can filter their embeddings into the considered models. The syntactic proof consists in the use of sementical calculi, the λ-calculi with tests, which are translating the properties of the considered model at syntactical level.

In the second part of the thesis, we consider $B_{\mathcal{S}}LL$, a refinement of linear logic, where the exponential connective is parametrized by elements of a semi-ring $\mathcal{S}$. These semirings allow to express *coeffects*, *i.e.*, specific requirements of a program with respect to the environment (availability of a resource, some prerequisite of the input, etc.). We give the fist denotational analysis of $B_{\mathcal{S}}LL$. In particular we give two ways for extracting a model of $B_{\mathcal{S}}LL$ from a model of usual linear logic. One way is "a priori": it uses the semi-ring S in order to impose a stratification to the usual linear logic exponential. The other way is instead "a posteriori": it shows that any model of linear logic is naturally associated with (kind of) semantical semi-ring R, by which one can derive an interpretation of $B_{\mathcal{S}}LL$.

Introduction

Content and contributions of this thesis

My research branches in two directions: the study of the untyped λ-calculus, and the study of functional languages with quantitative operators. Despite both studies being focused on denotational semantics, a particular attention is always given to the computational meaning of our models.

The first topic, *i.e.*, the untyped λ-calculus, is an old and deep subject; it is the subject of a whole theory that is well understood and serves as a basis for the denotational semantics of higher-order functional languages. On the contrary, the second topic, *i.e.*, the quantitative languages, is an emerging new topic; there, the simple notion of what is a model or not is still a matter of debate. The fact that my work splits into these two lines actually results from a long term strategy aiming at transferring knowledge and methods from the first line to the second.

The full abstraction of the untyped λ-calculus

My first area of expertise stems from back to the 70's and the foundations of denotational semantics. We aim at taking back questions from the origins of the subject and reinterpreting them in a more modern way. This methodology has two objectives. The first one is to verify that no shortcut nor any middling concession regarding generality has been done while the theory was being built up. The second one is to reforge traditional tools into modern ones with the same purpose but not the same domain of application.

Denotational semantics fundations have been much studied and is the subject of many publications. However, most of these publications focus on algebraic and categorical axiomatization of those well known fundamentals. My approach is slightly different as it consists in finding technical holes and misconceptions by looking at the limits of the theory.

In this thesis, my analysis more specifically concerns full abstraction, *i.e.*, the absolute correspondence between the model and the given behavior (or observation). This is the strongest notion of adequation and is generally invoked to show that a model is "the best possible one".

In this prospect, I reversed the usual existential approach of full abstraction to a universal one. The usual approach is fundamentaly existential: it consists in finding a fully abstract model for a specific calculus/theory. The universal approach consists in gathering all models that are fully abstract for a simple calculus. I choose to work with the untyped λ-calculus with head reducibility as observation (*i.e.*, the theory $\mathcal{H}^*$). The corresponding existential problem has been solved in 1976 by Hyland [Hyl76] and Wadsworth [Wad76] (full abstraction of Scott's D_∞); it is by now particularly well understood.

The universal approach already appeared in Milner's uniqueness theorem for fully abstract models of PCF [Mil77]. But the case of PCF is intrinsically different from the theory $\mathcal{H}^*$.

Indeed $\mathcal{H}^*$ has various fully abstract but not isomorphic models. The quest for a general characterization of the fully abstract models of $\mathcal{H}^*$ started by successive refinements of a sufficient but unnecessary condition [DGFH99, X.G95, Man09], improving the proof techniques from 1976 [Hyl76, Wad76].

All along these studies, you could feel a general intuition gravitating around well foundedness. Indeed, for a model to be fully abstract for $\mathcal{H}^*$, it seemed that it had to be stratified over a well founded set.[1] In terms of game semantics, this means that the semantics should forbid non-productive plays with both player and opponent that keep answering questions by asking deeper and deeper questions without any productive answer. In terms of filter models, this means that points of the model are well (pre)ordered so that any equation $\alpha = \beta \to \gamma$ respect $\beta < \alpha$ and $\gamma \leq \alpha$ (except for the bottom $\omega = \omega \to \omega$).

In [Bre14], I showed that this intuition had a flaw. In fact, such degenerated models with non-well founded chains are potentially allowed providing that those chains cannot be caught by a recursive function (*i.e.* by a λ-term).

Up to restricting our study to a specific class of models,[2] I even achieved a full characterization [Bre14]: a model D is fully abstract for $\mathcal{H}^*$ iff D is *hyperimmune* (Def. 2.1.0.1).

Hyperimmunity is the key property our study introduces in denotational semantics. This property is reminiscent of the Post's notion of hyperimmune sets in recursion theory. Hyperimmunity is not only undecidable, but also surprisingly high in the hierarchy of undecidable properties (it cannot be decided by a machine with an oracle deciding the halting problem) [Nie09].

It results that fully abstract models for $\mathcal{H}^*$ form a very strange class containing models with weird behaviors. In fact the model is allowed to behave erratically as long as its misbehaviors cannot be caught by the calculus. In this condition, one can question the validity of using full abstraction as an absolute argument for the correspondence between the calculus and the model.

The other point of my study is to introduce a new tool: the *calculi with tests* (Def. 2.3.1.1). These are syntactic extensions of the λ-calculus with operators defining compact elements of the given models. Since the model appears in the syntax, we are able to perform inductions (and co-inductions) directly on the reduction steps of actual terms, rather than on the construction of Böhm trees.

A calculus with tests is a sort of dual of the set of Böhm trees. While the latter constitute a syntactical model of the λ-calculus, a calculus with tests is a semantical language of a K-model. While Böhm trees are built upon the λ-calculus and reduce the problem of full abstraction to the semantical level; a calculus with tests is built upon the model and reduces this problem to the syntactical level. We claim that, regarding relations between denotational and operational semantics, Böhm trees and λ-calculi with tests are equally powerful tools, but extend differently to other frameworks.

The comparison between these two tools is particularly emphasized in [Bre] (the long version of [Bre14]) and in this thesis. Indeed, the proof of the characterization is performed twice: once in a purely semantical style, only relying on Böhm tree techniques; and once in a purely syntactical style, relying on the λ-calculus with tests.

[1] The notion has been technically expressed in [Man09].

[2] Namely Krivine models [Kri93, Ber00] that respect the approximation property

In a nutshell, **my contributions** in this area are the following:

- the characterization of full abstraction for $\mathcal{H}^*$ (Th. 2.1.0.5) by an unforseen notion of Hyperimmunity (Def. 2.1.0.1), bridging denotational semantics and the recursion theory in yet another way,
- the developement of a new tool:[3] the calculi with tests (Def.2.3.1.1), that can compete with Böhm trees in its usefulness,
- a novel view of Böhm trees that can be themselves interpreted in different ways in Λ-models (Def. 2.2.1.7 and Def. 2.2.1.16),
- a bypassing equivalence between the equational and inequational full abstractions for $\mathcal{H}^*$ in K-models (Th. 2.1.0.5),
- and, finally, a bypassing approximation theorem applying to a large class of models (Cor. 2.4.2.14).

Bounded type systems

My second area of expertise lies in a quantitative generalisation of Curry-Howard correspondence. Some languages, logics and semantics are capable of manipulating quantitative information representing a specific physical or abstract resource (execution time, re-usability, probability, size, scheduling...). The objective is then to trace, limit or modify this resource consumption via a Curry Howard correspondence.

These calculi are extensions of the typed λ-calculus that represent the quantitative information by an algebraic structure (typically a semi-ring). This structure can be modified/used explicitly, via some (co-)effects, or implicitly, via usual operators. It can be represented either through the operational semantics or through the type system.

The whole point of this domain is to link the concrete algebraic resources appearing in operational semantics and the abstract ones appearing in type systems. The relation between those is probably of the same nature than the relation between concrete and abstract domains in abstract interpretation. Indeed, quantitative type systems are supposed to statically compute quantitative information on the resource consumption via the type-checking/inference.

One can roughly distinguish two opposite poles of study for quantitative type systems. One focuses on effects [Kat14] while the other focuses on coeffects [dLG11, GHH$^+$13a]. The distinction is not formal and may not be universal, but we believe that such a duality is at stake somewhere. In this thesis, we focus on the second, which is a recent field in full effervescence.

One particular example is the bounded linear logic (for short BLL) [GSS92a, DLH09], that achieved one of the first successes of implicit complexity: a characterization of polynomial time via a type assignment system. It consists in wisely inserting re-usability bounds for arguments at type level. For example, $3.\mathit{int}{\to}\mathit{int}$ is the type of functions that are allowed up to three usages of their argument. This time, the resource is a more abstract notion of re-usability captured by a structure similar to the polynomial ring[4] with a notion of dependency.

[3]The tool already existed in a particular case [BCEM11] but I raised it in a whole new level.

[4]More exactly polynomials with reals as coefficients but natural conserving natural numbers.

In this thesis, I am presenting my advancement regarding such a refinement of linear logic where the exponential is "bounded" by a semiring $\mathcal{S}$. These logics are called $\mathcal{S}$-bounded logics (for short $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$). Appearing simultaneously in two different situations [BGMZ14, GS14], the $\mathcal{S}$-bounded logics are used as type systems recording the resource consumption. The recorded resource can be diverse and depends on the choice of the (lax-)semiring $\mathcal{S}$. Computationally, the interesting point of this logic is its capability at recording requirements raised by coeffects [BGMZ14, POM13, POM14].

A $\mathcal{S}$-bounded logic can be seen as a refinement of the linear logic where the exponential modality is parameterized by elements of a (lax-)semiring $\mathcal{S}$. The idea is similar to the approach of BLL [GSS92a] with two exceptions: the parametricity over any (lax-)semiring offers many more choices than the sole polynomial semiring used by BLL, however, the absence of dependence considerably restricts the logical power.[5]

Unfortunately, very few concrete semantics have been designed to describe these bounded logics. In fact, I know only one realisability semantics of the original BLL [HS04] and the parameterized $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ is only given a categorical axiomatisation [BGMZ14]. Our article [BP15] aims at filling this lack for $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ and the general flow of this thesis is pursuing this heading.

The $\mathcal{S}$-bounded logics appear as refinements of the usual linear logics which semantics has been intensively studied. By linking the two, we can hope to achieve two things: exporting the huge amount of technologies developed around the semantics of linear logic and factorizing/decomposing the notion of model of linear logic in order to demystify it. Following this path, I am about to present you two different constructions that can decompose models of linear logic into models of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$.

The first construction (Def. 3.1.1.13) is rather intuitive. It starts from the remark that several well known models of linear logic are already able to distinguish, at the semantic level, the resource consumption which the $\mathcal{S}$-bounded logics are based on. The idea is to transport the whole structure of the linear logic exponential throughout a natural transformation θ_I that is parameterized by elements I of the targeted semiring $\mathcal{S}$. As a result, we we get a stratification of the original exponential in more atomic components that are more refined and that can model $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ for some $\mathcal{S}$.

This construction is first applied on the most basic model of linear logic: the relational model. There, we will identify $\mathcal{S}$-bounded logics that can be modeled by some stratification of the free exponential of the relational model and see that they are (roughly) as diverse as the possible semi-rings that can be in a sense embedded to the set-theoretical lattice $\mathcal{P}(\mathbb{N})$ (Def. 3.1.2.1). After a brief aside on other well known models such as the coherent spaces, I will come back to the relational model, but using non-free exponentials [CES10]. We will see in particular that for any choice of semiring $\mathcal{S}$, the $\mathcal{S}$-bounded logic can be modeled for a coherent choice of exponential. In a more global picture, we show the importance of studying non-free exponentials in order to model $\mathcal{S}$-bounded logics.

The second construction (Th. 3.2.3.3) is more abstract and targets a different goal. The idea is to start from any model of linear logic and construct, in the most natural way, a lax-semiring that we call *internal lax-semiring*. My intuition is that a model of linear logic is synthesized by its internal lax-semiring which already contains a large part of the computational content that can be carried in the model. We will see that a model can be turned into a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ for $\mathcal{S}$

[5] *i.e.*, the set of programs that are typable

the internal lax-semiring.

Moreover, the very natural character of the construction allows for a generalization at an abstract level. Our objective, with this generalization, is to capture the the notion of dependence of the original BLL together with the genericity of the $\mathcal{S}$-bounded logics. The early results in this directions are presented as well as their syntactical counterpart.

In a nutshell, **my contributions** in this area are the following:

- the concept of *stratification* of an exponential comonad to get a model of $B_{\mathcal{S}}$LL (Def. 3.1.1.13),
- the given of the first concrete model of $B_{\mathcal{S}}$LL, with a universal way of constructing a model of $B_{\mathcal{S}}$LL for any semiring $\mathcal{S}$ (Prop. 3.1.3.13),
- the discovery of an *internal lax-semiring* lying over any linear[6] category (Def. 3.2.2.3),
- the slicing of a linear category to get a model of $B_{\mathcal{S}}$LL, for $\mathcal{S}$ the corresponding internal lax-semiring (Th. 3.2.3.3),
- the generalization of the internal lax-semiring to a dependent semiring (Def. 3.2.4.5),
- a prototype of a dependent $\mathcal{S}$-bounded logic over a dependent semiring $\mathcal{S}$ (Def. 3.2.5.1).

Contents

This thesis is divided in three chapters: a first preliminary chapter that introduces the technical objects of the two main chapters, and two chapters of contributions following the two main lines above mentioned. Aside these chapters are a few appendices that recall general notions which are not contributions.

Preliminaries. Chapter 1 presents four technical objects that are crucial in Chapters 2 and 3. Those are the untyped λ-calculus (Sec. 1.1), the Böhm trees (Def. 1.2.2.1), the K-models (Def. 1.2.4.3) and the $\mathcal{S}$-bounded logics (Def. 1.3.1.1).

In Section 1.2, a special attention is given to models of the untyped λ-calculus, which Böhm trees[7] and K-models are part of. In Section 1.2.1, we will see that Λ-models can be described algebraically, allowing to present the set of Böhm trees as a model (Sec. 2.2.1). Then, we will see that in order to get interesting and well structured models (Sec. 1.2.3), it is useful to look for more abstract and structured categorical constructions such as Cartesian closed categories, reflexive objects and Kleisli categories. Finally, in Section 1.2.4 the K-models are presented by means of these constructions.

Section 1.3 presents in detail the $\mathcal{S}$-bounded logics (for short $B_{\mathcal{S}}$LL) for $\mathcal{S}$ a semiring. The main interest of S-bounded logics arises when implemented as a type system for a programming language enriched with co-effects. In fact, the scalars in the semiring $\mathcal{S}$ allow to write parametrized formulas that can be seen as types expressing some co-effect (i.e. requirement) of the programs having those types. However, our objective in this thesis being to study

[6] and order-enriched

[7] More exactly the set of all Böhm trees

$B_{\mathcal{S}}$LL models and their relations with linear logic, we do not develop the underlying calculus. Nonetheless, Section 1.3.2 informally presents the different applications thought examples of actual instances of the semiring $\mathcal{S}$.

The characterization of $\mathcal{H}^*$. Chapter 2 is structured in four sections. Section 2.1 presents the main theorems (Theorem 2.1.0.5 and Theorem 2.1.0.5). These two theorems correspond to the very same result: the characterization of the full abstraction with respect to $\mathcal{H}^*$ by the notion of hyperimmunity (Def. 2.1.0.1). The difference is in the point of view taken on the result, Theorem 2.1.0.5 (and Section 2.2 that is dedicated to it) presents the result from a semantical point of view, while Theorem 2.1.0.5 (and Section 2.3) has a syntactical point of view. Section 2.4 completes the chapter by relating the two proofs and by showing the diversity of the considered class of models.

The semantic point of view (Sec. 2.2) is built around the intensively studied notion of Böhm trees. Following a notion of model for coinductive terms developed in Appendix A.2.1, we generalize the notion of interpretation from λ-terms to Böhm trees. There are several such interpretations, and we are studying one in particular: the quasi-finite interpretation (Def. 2.2.1.16). The key notion is the quasi-approximation (Def. 2.2.1.17) that states that the quasi-finite interpretation factors the interpretation of any λ-term. We show the equivalence between the notion of hyperimmunity, the notion of quasi-approximation and the full abstraction for $\mathcal{H}^*$ (Theorems 2.2.2.8, 2.2.2.10 and 2.2.3.1).

The syntactic point of view (Sec. 2.3) is built around the notion of λ-calculus with tests. This notion is new and quite a lot of technicalities have to be achieved, such as confluence (Theorem 2.3.1.26) or standardization (Theorem 2.3.1.29). The λ-calculus with tests is a powerful tool that we use to prove Theorem 2.1.0.6 without any call to the notion of Böhm trees. The idea is, for any model D, to extent the λ-calculus with some tests that are defining elements of D. This results in a calculus D is fully abstract for. Once noticed that the interpretation over the λ-calculus with D-tests factors the interpretation over the λ-calculus, only remains a syntactical study to determind whether the head-observational equivalence over lambda-calculus and that over the calculus with tests concide (Theorem 2.3.2.4 and Theorem 2.3.3.5).

The study over models for $B_{\mathcal{S}}$LL. Chapter 2 is structured in two sections. Each of these sections presents a general construction aiming at transforming a linear category into a model of $B_{\mathcal{S}}$LL. However, the two constructions follow different directions and different goals. The first construction (described in Section 3.1) is a kind of "external" interpretation: it starts from a semiring $\mathcal{R}$ and analyses the properties that $\mathcal{R}$ should satisfy in order to transform a model of LL into a model of $B_{\mathcal{S}}$LL. The second construction (described in Section 3.2) instead is "internal", it tries to recover from a model of LL an associated semiring which gives a model of $B_{\mathcal{S}}$LL. Moreover, the "external" construction aims for simple (but potentially *ad hoc*) models; while the "internal" one aims at a natural (but potentially complex) model from which should emerge some natural generalization of bounded logics.

Section 3.1 presents the notion of *bounded exponential situation* of Brunel *et al* [BGMZ14] (Def. 3.1.1.2) that is a categorical axiomatization for models of $B_{\mathcal{S}}$LL. Theorem 3.1.1.16 shows that a stratification over the exponential of a linear category (Def. 3.1.1.13) leads to such a bounded exponential situation. Such a theorem is similar to saying that the codomain of a group by a surjective semigroup morphism is a group: it does not provide any actual example. The remaining of the section focuses on applying the theorem over several concrete semantics to get a better understanding of the situation. Section 3.1.2 presents the stratifications of the relational model with the free exponential (Appendix A.3.3), but also sketches some stratifications over coherent and Scott's semantics. Section 3.1.3 develops the stratifications of relational models for non-free exponentials.

Section 3.2 is built around the notion of *internal semiring* (Def. 3.2.2.1). In Section 3.2.2, Theorem 3.2.2.2 and Theorem 3.2.2.4 are expliciting this internal (lax-)semiring that is hidden in the axiomatization of linear category. In Section 3.2.3, we show that by slicing a linear category, we get a bounded exponential situation parameterized by its internal lax-semiring. Finally, in Section 3.2.4 and Section 3.2.5, we present early results concerning a semantic-oriented and dependent extension of the bounded logics.

The technical appendices. Appendix A.1 introduces all the technical definitions over category theory in the straight form of a dictionary. There, we present not only the basic definitions over categories, but several more advanced definitions such as adjunctions or (co)algebras as well the notion of 2-categories.[8]

Appendix A.2 collects several syntactical notions over term constructions and term rewriting. A special attention is given to the notion of structural coinduction. It is known that coinductive proofs have to be productive. However, in the same way that structural induction does not require a certificate of well-ordering, structural coinduction does not require any certificate of productivity. This concept is not central along the contributions. However, the notion is implicitly used in the way Böhm trees are treated (Sec 2.2.1) and in some proofs of divergence (Sec 2.3.3).

Appendix A.3 is treating the linear logic and its models. We define the intuitionistic linear logic and develop the notion of linear category which axiomatizes it. We also give the definition of a recollection of well known models of linear logic: the relational category Rel, the category Coh of coherent spaces (with both set and multiset exponentials) and the linear Scott model ScottL.

Finally, Appendix A.4 gathers basic definitions as well as several examples of semirings and lax-semirings. These examples are used as references for the Section 3, but none of them amounts to much novelty (even if some proofs are given they are quite straightforward).

[8]Notice that 2-categories are not used in all their generality, but only under an order-degeneration; nonetheless, the general case is also mentioned in the thesis.

Contents

1. Preliminaries

In this preliminary chapter we introduce the fundamental and historic notions regarding the two main chapters of the thesis:

Section 1.1 and Section 1.2 introduce the objects of Chapter 2. Section 1.1 defines the untyped λ-calculus: reduction and theories. In Section 1.2, we define the notion of Λ-model, both algebraically and categorically. We illustrate these approaches by two examples of importance for Chapter 2: the Böhm trees and the K-models.

Section 1.3 introduces the logics bounded by semirings, which are the object of Chapter 3, and gives some examples. The relative shortness of the preliminaries for Chapter 3 traduces the youthfulness of the topic.

1.1. The untyped λ-calculus

A higher order grammar

The set of *λ-terms* is defined by the following grammar [Bar84] (with $x, y, z...$ in a given a set $\mathbb{Var}$ of variables):

$$(\lambda\text{-terms}) \quad \Lambda \quad M, N \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad M\,N$$

Terms are denoted "*à la* Barendregt" which means that application is left-associative (*i.e.*, $M_1\ M_2\ M_2$ denotes $(M_1\ M_2)\ M_2$), and that nested abstractions of the form $\lambda x_1 \ldots \lambda x_k.M$ are abbreviated into $\lambda x_1 \ldots x_n.M$ or into $\lambda\vec{x}.M$.

Definition 1.1.0.1. *The set* FV(M) *of* free variables *of a λ-term M is defined by induction:*

$$\mathrm{FV}(x) := \{x\}, \qquad \mathrm{FV}(\lambda x.M) := \mathrm{FV}(M) - \{x\}, \qquad \mathrm{FV}(M\ N) := \mathrm{FV}(M) \cup \mathrm{FV}(N)$$

A variable $x \in$ FV(M) is called free variable *of M. A term M with an empty set of free variables is said to be closed. The set of all closed λ-terms is denoted Λ^o.*

Definition 1.1.0.2. *Given two λ-terms M and N, we define the* capture free substitution *$M[N/x]$ by induction on M:*[1]

- $x[N/x] := N$,
- $y[N/x] := y$ *for* $y \neq x$,
- $(\lambda x.M)[N/x] := \lambda x.M$,

- $(\lambda y.M)[N/x] := \lambda y.M[N/x]$ *for* $y \neq x$ *and* $y \notin \mathrm{FV}(N)$,
- $(M_1\ M_2)[u/x] := M_1[N/x]\ M_2[N/x])$.

Definition 1.1.0.3. *The* α-equivalence *is the smallest congruence (Def. A.2.1.25) such that* $\lambda x.M \equiv_\alpha \lambda y.M[y/x]$ *for any* $x \in \mathbb{Var}$*, any* M *and any* $y \notin \mathrm{FV}(M)$*.*

The λ-terms are defined up-to α-equivalence so that we write $M = N$ when $M \equiv_\alpha N$. Remark that the substitution becomes a total operation on α-equivalence classes.

The β-reduction

The λ-terms are subject to *β-reduction* which is generated by the axiom:

$$(\beta) \qquad\qquad (\lambda x.M)\ N \xrightarrow{\beta} M[N/x]$$

A *context* C is a λ-term with possibly some occurrences of a hole, *i.e.*:

$$\text{(contexts)} \quad \Lambda^{(\!|\cdot|\!)} \quad C \quad ::= \quad (\!|.|\!) \quad | \quad x \quad | \quad \lambda x.C \quad | \quad C_1\ C_2$$

The writing $C(\!|M|\!)$ denotes the term obtained by filling the holes of C by M. Remark that the free variables of a term M can be captured by a context C in $C(\!|M|\!)$.

A *redex* in a term M is a sub-term of the form $(\lambda x.N_1)\ N_2$.

The small step reduction $\to$ is the closure of (β) by any context, *i.e.*, its contextual closure (Def. A.2.1.25). The transitive reduction $\to^*$ is the reflexive transitive closure of $\to$.

The *normal form*s are terms without redexes, *i.e.*, of the form $\lambda x_1...x_n.y\ M_1 \cdots M_k$ with $M_1, ..., M_k$ themselves in normal form.

The big step reduction, denoted $M{\Downarrow}N$, is $M \to^* N$ for N in normal form. We write $M{\Downarrow}$ for the convergence, *i.e.*, whenever there is N such that $M \Downarrow N$, and we write $M{\Uparrow}$ for the divergence (*i.e.*, the negation of convergence).

Example 1.1.0.4.
- *The* identity term $\boldsymbol{I} := \lambda x.x$ *is such that:*

$$\boldsymbol{I}\ M \quad \to \quad M.$$

- *The* n^{th} *Church numeral, denoted by* $\underline{n}$ *, and the successor function, denoted by* $\boldsymbol{S}$ *, are defined by*

$$\underline{n} := \lambda f x.\underbrace{f\ (f \cdots\ f\ (f}_{n\ times}\ \ x) \cdots), \qquad\qquad \boldsymbol{S} := \lambda u f x.u\ f\ (f\ x).$$

Together they provide a suitable encoding for natural numbers.

- *The* looping term $\boldsymbol{\Omega} := (\lambda x.xx)\ (\lambda x.xx)$ *infinitely reduces into itself. Notice that* $\boldsymbol{\Omega}$ *is an example of diverging term:*

$$\boldsymbol{\Omega} \quad \to \quad (x\ x)[\lambda x.x\ x/x] \quad = \quad \boldsymbol{\Omega} \quad \to \quad \boldsymbol{\Omega} \quad \to \quad \cdots.$$

[1]This can be done as a coinductive procedure in a generalization to coinductive representation.

- *The* Turing fixpoint combinator $\mathbf{\Theta} := (\lambda uv.v\ (u\ u\ v))\ (\lambda uv.v\ (u\ u\ v))$ *is a term that computes the least fixpoint of its argument (if it exists):*

$$\begin{aligned}\mathbf{\Theta}\ M &\to (\lambda v.v\ ((\lambda uv.v\ (u\ u\ v))\ (\lambda uv.v\ (u\ u\ v))v))\ M\\ &= (\lambda v.v\ (\mathbf{\Theta}\ v))\ M\\ &\to M\ (\mathbf{\Theta}\ M).\end{aligned}$$

*Remark that a fixpoint combinator alone (without argument) always diverges (*i.e., $\mathbf{\Theta}\Uparrow$*):*

$$\mathbf{\Theta} \quad \to^* \quad \lambda u.u\ (\mathbf{\Theta}\ u) \quad \to^* \quad \lambda u.u\ (u\ (\mathbf{\Theta}\ u)) \quad \to \quad \cdots$$

In the last two items of Example 1.1.0.4, the combination of β-reduction on self-application gives rise to fancy computations.

The following proposition, relating recursiveness and λ-definability, provides further evidences on the computational power of the λ-calculus.

Proposition 1.1.0.5 ([Bar84, Proposition 8.2.2] [2]**).**
Let $(M_n)_{n\in\mathbb{N}}$ *be a sequence of terms such that:*

- $\forall n \in \mathbb{N}, M_n \in \Lambda^0$,
- $(n \mapsto M_n)$ *is recursive,*

then there exists F *such that:*

$$\forall n, F\ \underline{n}\ \to^* M_n.$$

Head reduction

The *head reduction* $\to_h$ is the closure of (β) by the rules:

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'} \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\ N \to_h M'\ N}$$

The transitive reduction $\to_h^*$ is the reflexive transitive closure of $\to_h$.

The *head-normal forms* are the terms where no head reduction can apply. They correspond to the terms of the form $\lambda x_1...x_n.y\ M_1 \cdots M_k$, for $M_1, ..., M_k$ any terms.

The big step head reduction, denoted $M{\Downarrow^{\mathrm{h}}}N$, is $M \to_h^* N$ for N in head-normal form. We write $M{\Downarrow^{\mathrm{h}}}$ for the *head convergence*, *i.e.*, whenever there is N such that $M{\Downarrow^{\mathrm{h}}}N$, and we write $M{\Uparrow^{\mathrm{h}}}$ for the divergence.

Occasionally, we will also use the notation $\to_h^+ := \to_h \to_h^*$ as well as $\to_{\not h} := (\to) - (\to_h)$ and its reflexive transitive closure $\to_{\not h}^*$.

Example 1.1.0.6. *The fixpoint* $\mathbf{\Theta}$ *is head converging (it is a head-normal form after reducing the single redex), and this despite the fact that* $\mathbf{\Theta} \Uparrow$.

Henceforth, convergence of a λ-term means head convergence (except when specified otherwise).

[2] This is not the original statement. We remove the dependence on $\vec{x}$ that is empty in our case and we replace the β-equivalence by a reduction since the proof of Barendregt [Bar84] works as well with this refinement.

Theories

Denotational semantics usually equates not only convertible terms, but also terms having the same "behavior" (for example, the same input-output graph or even the same internal representation after compiling). In order to consider all the possible "behaviors", we use the notion of theory that subsumes all possible ways of equating terms.

We consider also inequational theories: one program may be strictly "more defined" or "more interesting" than another one. For example, the program that computes the division of two numbers and diverges if the divisor is 0 may be considered less defined than a program that raises an error on 0.

Remark 1.1.0.7. *In the following, we introduce the notion of C-theory generalizing that of λ-theory. Indeed, some important notions like the one of sensibility or of observational equivalence, generalize over other kind of "calculi" (or any abstract rewriting system Def. A.2.1.29). In particular, we will use these notions for the calculi with tests in Section 2.3.*

Definition 1.1.0.8. *A* C-theory $\mathcal{T}$ *for a calculus* $(C, \to)$ *is a congruence* $\equiv_{\mathcal{T}}$ *over* C *(Def. A.2.1.25) that contains* $\to$*:*

$$M \to N \qquad \Rightarrow \qquad M \equiv_{\mathcal{T}} N.$$

In particular, the smallest theory is the symmetric, reflexive, transitive and contextual closure of $\to$*.*

An inequational C-theory $\mathcal{T}$ *for a calculus* $(C, \to)$ *is an inequational congruence* $\sqsubseteq_{\mathcal{T}}$ *over* C *(Def. A.2.1.25) that contains the symmetric closure of* $\to$*:*

$$M \to N \qquad \Rightarrow \qquad M \sqsubseteq_{\mathcal{T}} N \quad \text{and} \quad M \sqsupseteq_{\mathcal{T}} N.$$

In particular, the congruence induced by an inequational theory $\sqsubseteq_{\mathcal{T}}$ *is a theory denoted* $\equiv_{\mathcal{T}}$*.*

Equational and inequational theories are ordered by inclusion (of their graphs).

Example 1.1.0.9. *A* λ-theory *is theory for the* λ*-calculus.*

The set of all λ*-theories forms a complete lattice which is* 2^{ω}*-wide and* ω*-high, meaning that there is a continuum of incomparable* λ*-theory and a denumerable infinity of theory forming a strict chain.*

Here are a few λ*-theories:*

- β *: The symmetric closure of the transitive relation* $\to^*$ *is called the* β-equivalence *and is denoted* $\equiv_\beta$ *. It corresponds to the smallest* λ*-theory denoted* β*. Remarks that, due to the confluence, we have* $M \equiv_\beta N$ *iff they have a common redex* $M \to^* L^* \leftarrow N$*.*
- $\top$ *: The relation* $\equiv_\top$ *equating every* λ*-term is the biggest* λ*-theory, denoted* $\top$*. This theory is definitely non interesting, and we call* coherent *any* λ*-theory that is different from* $\top$*.*
- $\mathbf{\Omega}$ *: For any term* $M \in \Lambda$*, the minimal* λ*-theory equating* $\mathbf{\Omega}$ *(Ex. 1.1.0.4) with* M *is coherent.*

"Reasonable" theories should take into account computationally relevant features like divergence:

Definition 1.1.0.10. *A C-theory is sensible if all the diverging terms form an equivalent class:*

$$\forall M{\Uparrow}, \forall N{\Uparrow}, M \equiv N \qquad \textit{and} \qquad \forall M{\Uparrow}, \forall N{\Downarrow}, M \not\equiv N.$$

An inequational C-theory is sensible if all the diverging terms are minimal:

$$\forall M{\Uparrow}, \forall N, M \sqsubseteq N.$$

Example 1.1.0.11. *We have seen two different notions of convergence for the λ-calculus, leading to two different notions of sensibility. However, the sensibility for the full convergence $\Downarrow$ is impossible. Indeed, in a theory sensible for $\Downarrow$, we have $\mathbf{\Theta} \equiv \mathbf{\Omega}$ and, by contextual closure, $\mathbf{\Theta}\ (\lambda xy.y) \equiv \mathbf{\Omega}\ (\lambda xy.y)$, but the left term is converging and the right one is not (contradicting the sensibility).*

Thus, by sensibility for the λ-calculus, we now refer exclusively to the sensibility for the head reduction. Here are some examples of sensible λ-theories:

- $\mathcal{H}$: *The least sensible λ-theory is called $\mathcal{H}$. A remarkable property is that any coherent λ-theory above $\mathcal{H}$ is sensible,* i.e., *if all diverging terms are in the same equivalence class, then either they are alone or there is a single equivalence class. Indeed, if a converging term $M{\Downarrow}^{\mathrm{h}}$ is equivalent to $\mathbf{\Omega}$, then its head normal form $\lambda x_1...x_k.x_i\ M_1 \cdots M_n$ (with $i \le k$) is also equivalent to $\mathbf{\Omega}$ and for any N we have:*

$$\begin{aligned} N &\equiv_\beta (\lambda x_1...x_k.x_i\ M_1 \cdots M_n)\ \underbrace{(\lambda y_1...y_n.N) \cdots (\lambda y_1...y_n.N)}_{k \text{ times}} && \textit{by right-to-left } \beta\textit{-reduction} \\ &\equiv\ \Omega\ (\lambda y_1...y_n.N) \cdots (\lambda y_1...y_n.N) && \textit{by contextual closure} \\ &\equiv_{\mathcal{H}} \Omega. \end{aligned}$$

- ***BT*** : *We will see in Section 2.2.1 a λ-theory **BT** obtained by comparing Böhm trees, this theory is sensible by construction.*

Besides sensibility, another significant property of λ-theories is extensionality, stipulating that two terms having the same applicative behavior are equivalent.

Definition 1.1.0.12. *A λ-theory is* extensional *if:*

$$(\forall L \in \Lambda^o, M\ L \equiv N\ L) \quad \Rightarrow \quad M \equiv N$$

for all terms M and N. The extensional closure *$\mathcal{T}\omega$ of a λ-theory $\mathcal{T}$ is the smallest extensional λ-theory that contains it.*

Similarly, an inequational λ-theory is extensional *if:*

$$(\forall L \in \Lambda^o, M\ L \sqsubseteq N\ L) \quad \Rightarrow \quad M \sqsubseteq N$$

for all terms M and N.

Example 1.1.0.13.

ω : The λ-theory ω is the extensional completion of β (and thus the smallest extensional λ-theory).

βη : We call η-reduction the reflexive transitive contextual closure of the relation:

$$\lambda x.M\ x \geq_\eta M \qquad\qquad \text{for } x \notin \mathrm{FV}(M)$$

The λ-theory βη is the smallest congruence $\equiv_{\eta\beta}$ containing both $\equiv_\beta$ and $\geq_\eta$. If $(M\ x) \equiv_{\beta\eta} (N\ x)$ for some fresh variable x, then $M \equiv_{\beta\eta} (\lambda x.M\ x) \equiv_{\beta\eta} (\lambda x.N\ x) \equiv_{\beta\eta} N$. However, although ω includes the βη λ-theory ($(\lambda x.M\ x) \equiv_\omega M$ for x fresh), the converse fails: $M\ L \equiv_{\beta\eta} N\ L$ for any closed L does not imply $(M\ x) \equiv_{\beta\eta} (N\ x)$.

More generally we denote $\mathcal{T}\eta$ the smallest λ-theory containing both $\mathcal{T}$ and $\geq_\eta$.

Among the C-theories, the observational equivalences are particularly compelling. They stipulate that two terms are equivalents if, when embedded in any context, they provide the same observations.

Definition 1.1.0.14. *The* observational equivalence *is given by:*

$$M \equiv_o N \qquad \text{iff} \qquad \forall C \in C^{(\!|\cdot|\!)}, C(\!|M|\!)\Downarrow \;\Leftrightarrow\; C(\!|M|\!)\Downarrow.$$

The observational equivalence *is contextually closed (and is thus a C-theory) whenever*

$$\forall C \in C^{(\!|\cdot|\!)}, \quad M \to N \quad \Rightarrow \quad (C(\!|M|\!)\Downarrow \Leftrightarrow\; C(\!|N|\!)\Downarrow). \tag{1.1}$$

Under the same condition (Eq. 1.1), the observational preorder *is the inequational theory defined by:*

$$M \sqsubseteq_o N \qquad \text{iff} \qquad \forall C \in C^{(\!|\cdot|\!)}, C(\!|M|\!)\Downarrow \;\Rightarrow\; C(\!|M|\!)\Downarrow.$$

Remark 1.1.0.15. *The observational equivalence is the contextual coclosure of the following relation (called* observation*):*

$$M \simeq_o N \qquad \text{iff} \qquad M\Downarrow \;\Leftrightarrow\; N\Downarrow.$$

This means that it is the largest contextually closed equivalence that is contained in $\simeq_o$.

Example 1.1.0.16. *We have seen that sensibility only makes sense with respect to the head reduction. This time, the two different reductions for the λ-calculus define two different observational equivalences:*

$\mathcal{H}^*$: The observational equivalence for $\Downarrow^{\mathrm{h}}$ *defined by $M \equiv_{\mathcal{H}^*} N$ iff:*

$$\forall C \in \Lambda^{(\!|\cdot|\!)}, \quad C(\!|M|\!)\Downarrow^{\mathrm{h}} \quad \Leftrightarrow \quad C(\!|N|\!)\Downarrow^{\mathrm{h}}$$

$\mathcal{T}_{NF}$: The observational equivalence for $\Downarrow$ *defined by $M \equiv_{\mathcal{T}_{NF}} N$ iff:*

$$\forall C \in \Lambda^{(\!|\cdot|\!)}, \quad C(\!|M|\!)\Downarrow \quad \Leftrightarrow \quad C(\!|N|\!)\Downarrow$$

The theory $\mathcal{H}^*$ is coarser than $\mathcal{T}_{NF}$. Indeed, if $M \equiv_{\mathcal{T}_{NF}} N$ and if $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ (we can assume that $C(\!|M|\!)$ is closed) then there is a normal form $C(\!|M|\!) \to_h^* \lambda \vec{x}^n.x_i\ M_1 \cdots M_k$ so that the following converges for $\to$:

$$C[\![M]\!]\ \underbrace{(\lambda \vec{y}^{k+1}.y_{k+1}) \cdots (\lambda \vec{y}^{k+1}.y_{k+1})}_{n \text{ times}} \to^* y_{k+1},$$

however, $C(\!|N|\!)\Uparrow^{h}$ *so that*

$$C[\![N]\!]\ \underbrace{(\lambda\vec{y}^{k+1}.y_{k+1})\cdots(\lambda\vec{y}^{k+1}.y_{k+1})}_{n\ times}$$

head diverges and, in particular, also diverges for $\rightarrow$.

1.2. Λ-models

λ-theories may be extremely complex, for instance, the observational equivalence is defined by a quantification over all contexts. This is why λ-theories are generally studied via *models*, which have a natural mathematical status.

1.2.1. Definition of Λ-models

The historical algebraic models of the λ-calculus are the Λ-algebras [Bar84, Definition 5.2.2]. They were promarily designed as models of Curry's combinatory logic, and provide models of the λ-calculus via a translation of the latter into the former.

An alternative, more direct, approach is due to Salibra and Pigozzi [PS95, MS10], who defined the notion of λ-abstraction algebra. Salibra and Goldblatt [SG99] showed that Barendregt's Λ-algebras can be seen as functional λ-abstraction algebras and that any λ-abstraction algebras is the sub-algebra of a functional λ-abstraction algebra.

Definition 1.2.1.1. *A λ-abstraction algebra is given by a set $\mathcal{M}$, by an element $\tilde{x} \in \mathcal{M}$ and a function $\lambda\tilde{x} : \mathcal{M} \to \mathcal{M}$ for each $x \in \mathbb{Var}$ and by an applicative function $\bullet : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ such that:*

$$(\beta_1)\quad (\lambda\tilde{x}.\tilde{x}) \bullet M = M \qquad (\beta_2)\quad (\lambda\tilde{x}.\tilde{y}) \bullet M = \tilde{y}$$

$$(\beta_3)\quad (\lambda\tilde{x}.M) \bullet \tilde{x} = M \qquad (\beta_4)\quad (\lambda\tilde{x}\tilde{x}.M) \bullet N = \lambda\tilde{x}.M$$

$$(\beta_5)\quad (\lambda\tilde{x}.M \bullet N) \bullet L = (\lambda\tilde{x}.M) \bullet L \bullet ((\lambda\tilde{x}.N) \bullet L)$$

$$(\beta_6)\quad (\lambda\tilde{x}\tilde{y}.M) \bullet ((\lambda\tilde{x}.N) \bullet L) = \lambda\tilde{y}.(\lambda\tilde{x}.M) \bullet ((\lambda\tilde{y}.N) \bullet z)$$

$$(\alpha')\quad \lambda\tilde{x}.(\lambda\tilde{y}.M) \bullet z = \lambda\tilde{y}.(\lambda\tilde{x}.M)((\lambda\tilde{y}.N) \bullet M$$

where $L, M, N \in \mathcal{M}$ and where $x, y, z \in \mathbb{Var}$ are different.

By abuse of notation, we will call Λ-model a λ-abstraction algebra.

An ordered Λ-model is a model $\mathcal{M}$ endowed with an order $\leq$ for which $\lambda\tilde{x}$ and $\bullet$ are monotone.

Definition 1.2.1.2. *The λ-calculus being an inductive object, the interpretation $[\![.]\!] : \Lambda \to \mathcal{M}$ in a Λ-model is unique (Def. A.2.1.13).[3] This interpretation is the following one:*

$$[\![x]\!] := \tilde{x} \qquad [\![\lambda x.M]\!] := \lambda\tilde{x}.[\![M]\!] \qquad [\![M\ N]\!] := [\![M]\!] \bullet [\![N]\!].$$

It is clear that any λ-theory defines a Λ-model, but the converse is also true. Notice, however, that this is not an isomorphism (models are richer than theories).

Proposition 1.2.1.3. *Given a Λ-model $\mathcal{M}$, the following defines a Λ-theory called the* induced λ-theory*:*

$$M \equiv N \qquad \textit{iff} \qquad [\![M]\!] = [\![N]\!].$$

The key notions relating Λ-models and λ-theories are adequacy and completeness:

Definition 1.2.1.4 (*Adequacy*)**.** *A Λ-model is* adequate *for a Λ-theory $\mathcal{T}$ iff for all M, N:*

$$[\![M]\!] = [\![N]\!] \quad \Rightarrow \quad M \equiv_{\mathcal{T}} N$$

An ordered Λ-model is adequate *for an inequational Λ-theory $\mathcal{T}$ iff for all M, N:*

$$[\![M]\!] \leq [\![N]\!] \quad \Rightarrow \quad M \sqsubseteq_{\mathcal{T}} N$$

Definition 1.2.1.5 (*Completeness*)**.** *A Λ-model is* complete *for a Λ-theory $\mathcal{T}$ iff:*

$$[\![M]\!] = [\![N]\!] \quad \Leftarrow \quad M \equiv_{\mathcal{T}} N$$

An ordered Λ-model is complete *for an inequational Λ-theory $\mathcal{T}$ iff:*

$$[\![M]\!] \leq [\![N]\!] \quad \Leftarrow \quad M \sqsubseteq_{\mathcal{T}} N$$

Definition 1.2.1.6 (*Full abstraction*)**.** *An (ordered) Λ-model is* fully abstract *for an (in)equational theory $\mathcal{T}$ iff it is both adequate and complete for $\mathcal{T}$.*

The adequacy is often regarded as the minimal requirement when working on a specific equational theory. In fact, one of the main interests of denotational models is to supply tools for proving equivalence between terms.

Despite the full abstraction not being strictly needed, it is often looked at. Indeed, since it is the strongest possible property, it is invoked to show the perfection of a model. One of the goals of Chapter 2 is to explain that full abstraction is not omnipotent. We will see that it can screen unnatural and unwanted behaviors in its non-recursive fragment (*i.e.* for elements of the model that play no role in the interpretation).

Finally, we define new notions of sensibility and extensionality for models:

[3]Notice that non-inductive structures such as Böhm trees may have several or none interpretations.

Definition 1.2.1.7. *A Λ-model $\mathcal{M}$ is sensible if there is an element $\bot \in \mathcal{M}$ that interpret the diverging terms:*

$$\forall M, \qquad M{\Uparrow^{\mathrm{h}}} \quad \Leftrightarrow \quad [\![M]\!] \in \bot.$$

An ordered Λ-model is sensible if moreover $\bot$ is its least element.

Remark 1.2.1.8. *A Λ-model is sensible iff its induced λ-theory is sensible.*

The sensibility of an ordered Λ-model implies the sensibility of its induced inequational λ-theory. However, the converse does not hold since arbitrary non-definable elements may exist below the interpretation of diverging terms.

Sensibility is a nice way to prove the adequation for observational equivalences:

Lemma 1.2.1.9. *If a model $\mathcal{M}$ is sensible with respect to $\Downarrow$, then $\mathcal{M}$ is adequate for the equational theory $\mathcal{T}$ defined as the corresponding observational equivalence:*

$$M \equiv_{\mathcal{T}} N \quad \stackrel{def}{\Longleftrightarrow} \quad \forall C \in \Lambda^{(\!|\cdot|\!)},\ (C(\!|M|\!){\Downarrow}\ \textit{iff}\ C(\!|N|\!){\Downarrow}).$$

Proof. Let M and N be such that $[\![M]\!] = [\![N]\!]$ then for all $C \in \Lambda^{(\!|\cdot|\!)}$, necessarily $[\![C(\!|M|\!)]\!] = [\![C(\!|N|\!)]\!]$ and then $C(\!|M|\!)$ converges iff (sensibility) $[\![C(\!|M|\!)]\!] \neq \bot$ iff (congruence) $[\![C(\!|N|\!)]\!] \neq \bot$ iff (sensibility) $C(\!|N|\!)$ converges. □

Lemma 1.2.1.10. *If an ordered model $\mathcal{M}$ is sensible with respect to $\Downarrow$, then $\mathcal{M}$ is inequationally adequate for the inequational theory $\mathcal{T}$ defined as the corresponding observational order:*

$$M \sqsubseteq_{\mathcal{T}} N \quad \stackrel{def}{\Longleftrightarrow} \quad \forall C \in \Lambda^{(\!|\cdot|\!)},\ (C(\!|M|\!){\Downarrow} \ \Rightarrow\ C(\!|N|\!){\Downarrow}).$$

Proof. Let M and N such that $[\![M]\!] \leq_{\mathcal{M}} [\![N]\!]$ then for all $C \in \Lambda^{(\!|\cdot|\!)}$, necessarily $[\![C(\!|M|\!)]\!] \leq_{\mathcal{M}} [\![C(\!|N|\!)]\!]$. Thus if $C(\!|M|\!)$ converges, the sensibility gives $[\![C(\!|M|\!)]\!] \neq \bot$, in particular, $\bot <_{\mathcal{M}} [\![C(\!|M|\!)]\!] \leq_{\mathcal{M}} [\![C(\!|N|\!)]\!]$, so that N converges by sensibility. □

Although the notion of sensibility for models roughly corresponds to the sensibility of its induced theory, this is not the case for the extensionality:

Definition 1.2.1.11. *A Λ-model $\mathcal{M}$ is* extensional *if:*

$$\forall a, b \in \mathcal{M}, \qquad (\forall c,\ a \bullet c = b \bullet c) \quad \Rightarrow \quad a = b.$$

Similarly, an ordered Λ-model $\mathcal{M}$ is extensional *if:*

$$\forall a, b \in \mathcal{M}, \qquad (\forall c,\ a \bullet c \leq b \bullet c) \quad \Rightarrow \quad a \leq b.$$

Proposition 1.2.1.12. *If an (ordered) Λ-model is extensional then it is complete for $\beta\eta$,* i.e. $[\![\lambda xy.x\,y]\!] = [\![\lambda x.x]\!]$.

Proof. If $\mathcal{M}$ is a extensional Λ^*-model then $[\![\lambda y.x\,y]\!] \bullet a = [\![x]\!] \bullet a$ thus $[\![\lambda y.x\,y]\!] = [\![x]\!]$ and, by contextual closure, $[\![\lambda xy.x\,y]\!] = [\![\lambda x.x]\!]$. □

Remark 1.2.1.13. *Remark, however, that the extensionality of $\mathcal{M}$ does not imply the extensionality of its induced theory. Indeed, if $[\![M\,L]\!] = [\![N\,L]\!]$ for all L, the model may still contain some non-definable element so that $[\![M]\!] \bullet a \neq [\![N]\!] \bullet a$.*

In fact, for all the classes of models present in this thesis, extensional Λ^*-models correspond exactly to models that are complete for $\beta\eta$. This is a general situation which explanation is still an open question.

1.2.2. Böhm trees

Basic definitions

The Böhm trees provide one of the simplest semantic for the λ-calculus:

Definition 1.2.2.1. *The set of* Böhm trees *is the co-inductive structure generated by the grammar:*

$$\textit{(Böhm trees)} \quad \boldsymbol{BT} \quad U, V \quad ::= \quad \Omega \quad | \quad \lambda x_1...x_n.y\ U_1 \cdots U_k \quad , \forall n, \forall k \geq 0$$

The Böhm tree of a λ-term *M (*i.e.*, its interpretation), is defined by co-induction:*

- *If M head diverges, then $\boldsymbol{BT}(M) = \Omega$,*
- *if $M \to_h^* \lambda x_1...x_n.y\ N_1 \cdots N_k$ then*

$$\boldsymbol{BT}(M) = \lambda x_1...x_n.y\ \boldsymbol{BT}(N_1) \cdots \boldsymbol{BT}(N_k).$$

Notice that a Böhm tree can be described as a finitely branching tree (of possibly infinite height) where nodes are labeled either by a constant Ω, or by a list of abstractions and by a head variable.

Capital final Latin letters U, V, W... will range over Böhm trees.

Example 1.2.2.2. *The Böhm trees $\boldsymbol{BT}(\lambda x.x\,(\lambda y.x\,y))$, $\boldsymbol{BT}(x\,(\boldsymbol{I}\,\boldsymbol{I})\,(y\,\boldsymbol{\Theta}\,\boldsymbol{I}))$, $\boldsymbol{\Theta}$ and $\boldsymbol{BT}(\Theta\,(\lambda uxy.y(u\,x))\,z)$ are described in Figure 1.1.*

There exist Böhm trees that do not come from terms:

Example 1.2.2.3. *A Böhm tree with infinitely many free variables (such as the first one below) cannot be obtained from λ-terms that have finitely many free variables. Worse, if $g : \mathbb{N} \to \mathbb{N}$ is non recursive, then the second Böhm tree below does not come from any term (otherwise it would be possible to compute*

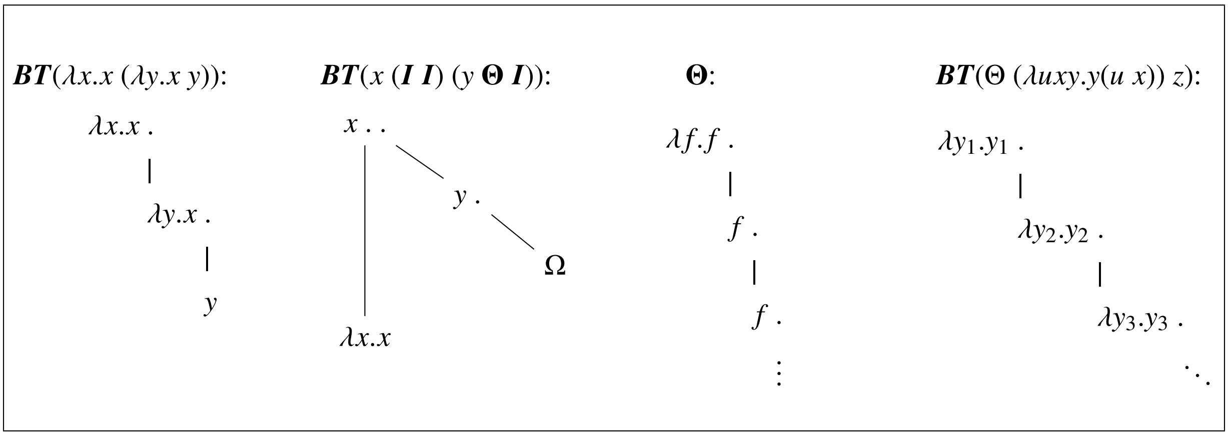
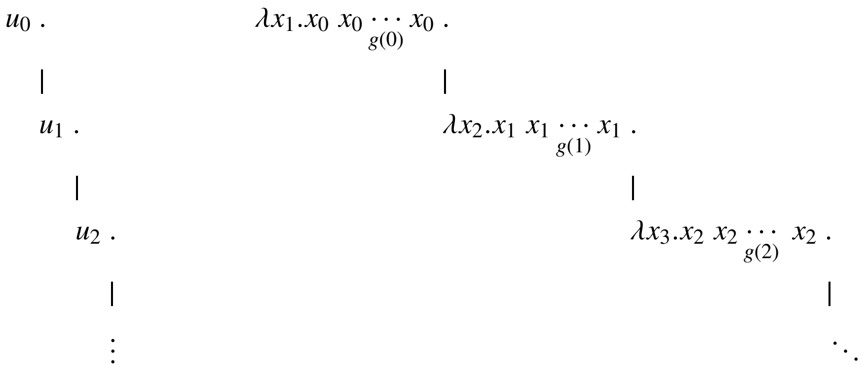


Figure 1.1.: Some examples of Böhm trees.

g from this term).



A suitable model?

A major weakness of this model is to be a "syntactical" model, in the sense that (to my knowledge) the most elegant definitions of the application of two Böhm trees is the normal-form of the reduction of $U\ V$ in an infinitary calculus (and with a possibly infinite number of steps)...

A more semantical definition would be to define $U \bullet V$ co-inductively in parallel with $U[V/x]$ by:

- For $U \bullet V$:
  - If $U = \Omega$ then $U \bullet V = \Omega$.
  - If $U = x\ U_1 \cdots U_n$ then $U \bullet V = x\ U_1 \cdots U_n\ V$.
  - If $U = \lambda x.U'$ then $U \bullet V = U'[V'/x]$.
- For $U[V/x]$:
  - If $U = \Omega$ then $U[V/x] = \Omega$.
  - If $U = \lambda \vec{y}.z\ U_1 \cdots U_n$ with $z \neq x$ then $U[V/x] = \lambda \vec{y}.z\ U_1[V/x] \cdots U_n[V/x]$.
  - If $U = \lambda \vec{y}.x\ U_1 \cdots U_n$ then $U[V/x] = \lambda \vec{y}.V \bullet U_1[V/x] \bullet \cdots \bullet U_n[V/x]$.

However, in both definitions the third item is non productive, for instance, the auto-application of $\boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda ux.x\ u))$ should reduce to Ω, but is undefined here.

Therefore, Böhm trees do not make an interesting model in themselves. However, they constitute a very powerful tool to link semantic and syntax. Indeed, our proof of Theorem 2.1.0.5 relies strongly on subtle properties of Böhm trees.

Properties

This model carries several interesting properties for the study of the untyped λ-calculus. By construction, it is sensible for the head reduction, and, moreover, it is adequate for $\mathcal{T}_{NF}$ and for $\mathcal{H}^*$ which is coarser.Moreover, those properties extend to inequations using the following natural notion of inclusion on Böhm trees:

Definition 1.2.2.4. *The* inclusion of Böhm trees *$U \subseteq V$ is co-inductively defined by:*

- *$\Omega \subseteq V$ for all V*
- *If for all $i \leq k$, $U_i \subseteq V_i$, then*

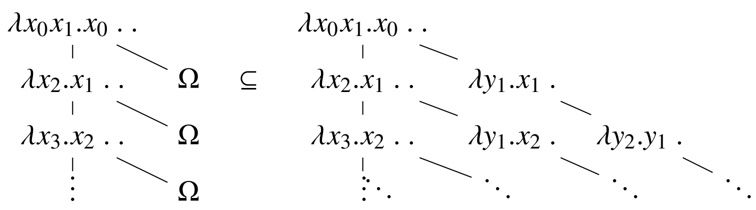
$$(\lambda x_1 ... x_n.y\ U_1 \cdots U_k)\ \subseteq (\lambda x_1 ... x_n.y\ V_1 \cdots V_k).$$

For readability, we denote $\subseteq_{\boldsymbol{BT}}$ the order induced by Böhm trees (defined by $M \subseteq_{\boldsymbol{BT}} N$ iff $\boldsymbol{BT}(M) \subseteq \boldsymbol{BT}(N)$).

The lower bounds of a Böhm tree U are obtained by replacing (possibly infinitely many) subtrees of U by Ω.

Example 1.2.2.5. *We have the inclusion*

$$\boldsymbol{\Theta}\ (\lambda uxy.x\ (u\ y)\ \boldsymbol{\Omega}) \quad \subseteq_{\boldsymbol{BT}} \quad \boldsymbol{\Theta}\ (\lambda uxy.x\ (u\ y)\ (\boldsymbol{J}\ x))$$



Proposition 1.2.2.6 ([Bar84, Proposition 16.4.7])**.** *Böhm trees are inequationally adequate (Def. 1.2.1.4) for $\mathcal{T}_{NF}$ and for $\mathcal{H}^*$:*

$$\text{if} \quad M \subseteq_{\boldsymbol{BT}} N \qquad \text{then} \quad M \sqsubseteq_{\mathcal{T}_{NF}} N \qquad \text{and} \quad M \sqsubseteq_{\mathcal{H}^*} N$$

The converse does not hold ($\subseteq_{\boldsymbol{BT}}$ is not extensional, Def. 1.2.1.11), so that we do not have full abstraction, but rather a new (inequational) λ-theory called $\mathcal{BT}$.

Forcefully adding the extensionality in $\mathcal{BT}$, we obtain the theory $\mathcal{BT}\eta$ witch is conjectured to be the same as $\mathcal{T}_{NF}$ and is definitely different from $\mathcal{H}^*$:

Example 1.2.2.7. *The term $\boldsymbol{J} = \Theta\ (\lambda uxy.x\ (u\ y))$ defines the following Böhm tree:*

$$\begin{array}{llll} \lambda x_0 x_1.x_0\ . & & & \\ \quad\ \ | & & & \\ & \lambda x_2.x_1\ . & & \\ & \quad\ \ | & & \\ & & \lambda x_3.x_2\ . & \\ & & & \ddots \end{array}$$

The behavior of this term is the same as the identity, so that we have $\boldsymbol{J} \equiv_{\mathcal{H}^*} \boldsymbol{I}$, but their Böhm trees are distinct and they are not η-convertible, so that $\boldsymbol{J} \not\equiv_{\mathcal{BT}\eta} \boldsymbol{I}$.

Böhm trees and full abstraction

We have seen that $\boldsymbol{BT}$ is not fully abstract for $\mathcal{H}^*$ since it is not extensional; however, there are refinements using the notion of infinite η expansion that permit to say something about the full abstraction (Proposition 1.2.2.12).

> **Definition 1.2.2.8.** *We write by $\geq_\eta$ the η*-reduction on Böhm trees, *that is $U \geq_\eta V$ if $U = V = \Omega$ or $U = \lambda x...x_{n+m}.y\ V_1 \cdots V_k\ x_{n+1} \cdots x_{n+m}$ and $V = \lambda x...x_n.y\ V_1 \cdots V_k$ (for $x_{n+1}, ..., x_{n+m} \notin$* FV$(V_1, ..., V_k)$*).*

> **Definition 1.2.2.9.** *We write by $\geq_{\eta\infty}$ the co-inductive version of $\geq_\eta$, that is the coinductive relation generated by:*
>
> $$\frac{}{\Omega \geq_\eta \Omega}\ (\eta\infty\omega) \qquad\qquad \frac{\forall i \leq k,\ U_i \geq_{\eta\infty} V_i \qquad \forall i \leq m,\ U_{k+i} \geq_{\eta\infty} x_{n+i}}{\lambda x_1...x_{n+m}.y\ U_1 \cdots U_{k+m} \geq_{\eta\infty} \lambda x_1...x_n.y\ V_1 \cdots V_k}\ (\eta\infty@)$$
>
> *By abuse of notations, given two λ-terms M and N, we say that M* infinitely η-expands *N, written $M \geq_{\eta\infty} N$, if $\boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$.*

Example 1.2.2.10. *We have the inequations:*

$$\begin{array}{ccccc} \boldsymbol{BT}(\boldsymbol{I}) & \leq_{\eta\infty} & \boldsymbol{BT}(\boldsymbol{J}) & \leq_{\eta\infty} & \boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda uxyz.x\ (u\ y)\ (u\ z))) \end{array}$$

$$\begin{array}{ccccc} \lambda x_0.x_0 & & \lambda x_0x_1.x_0\ . & & \lambda x_0x_1y_1.x_0\ .\ . \\ & & | & & |\qquad\diagdown\qquad\qquad\qquad \\ & \leq_{\eta\infty} & \lambda x_2.x_1\ . & \leq_{\eta\infty} & \lambda x_2y_2.x_1\ .\ .\quad \lambda y_2z_2.y_1\ .\ . \\ & & | & & |\qquad\diagdown\qquad\qquad\diagdown \\ & & \lambda x_3.x_2\ . & & \lambda x_3y_3.x_2\ .\ .\quad \lambda y_3z_3.y_2\ .\ .\ \ \ddots \\ & & | & & |\qquad\diagdown\qquad\qquad\diagdown \\ & & \vdots & & \ddots\qquad\quad \ddots\qquad\quad \ddots \end{array}$$

Remark 1.2.2.11. *The η-reduction on Böhm trees is not directly related to the η-reduction on λ-terms. For example $\mathbf{\Theta}\ (\lambda uzx.x\ (yz)) \not\leq_\eta \lambda x.\mathbf{\Theta}\ (\lambda uzx.x\ (yz))\ x$ (since the x was not free, however this reduction hold at level of Böhm trees. Conversely, $\mathbf{\Theta}\ (\lambda uz.z\ (u\ z)) \leq_\eta \mathbf{\Theta}\ (\lambda uzx.z\ (u\ z)\ x)$ while the Böhm trees are fairly different.*

However, the η-reduction on λ-terms is directly implied by the infinite η reduction.

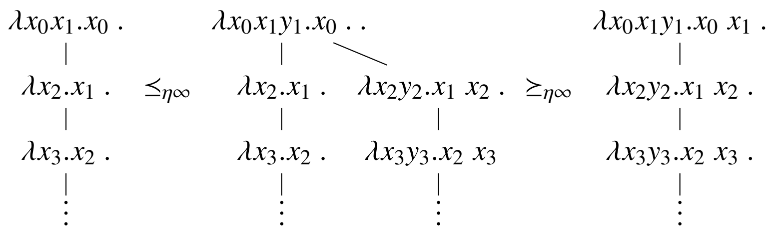
Using this notation we can characterize the notion of observational equivalence (*i.e.*, $\mathcal{H}^*$)

Proposition 1.2.2.12 ([Bar84, Theorem 19.2.9])**.** *For any terms $M, N \in \Lambda$, $M \sqsubseteq_{\mathcal{H}^*} N$ iff there exist two Böhm trees U, V such that:*

$$\boldsymbol{BT}(M) \leq_{\eta_\infty} U \subseteq V \geq_{\eta\infty} \boldsymbol{BT}(N).$$

Example 1.2.2.13. *In $\mathcal{H}^*$, we have the equivalence:*

$$\boldsymbol{J} \qquad \equiv_{\mathcal{H}^*} \qquad \mathbf{\Theta}\ (\lambda uxyz.x\ y\ (u\ z))$$



The following trivial corollary will be rather useful for proving observational equivalences:

Corollary 1.2.2.14. *For all $M, N \in \Lambda$,*

$$M \geq_{\eta\infty} N \;\Rightarrow\; M \equiv_{\mathcal{H}^*} N.$$

Proof. By Proposition 1.2.2.12 and since $\boldsymbol{BT}(M) \leq_{\eta\infty} \boldsymbol{BT}(M) \subseteq \boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$. □

1.2.3. How to get a Λ-model?

Now that we have a defined notion of Λ-model, we should be able to find models of interest. However, as it appers, the whole world of Λ-models is far too vast and contains many models which computational meaning is inexistent or inintelligible.

Resonable instances of Λ-models are to be found inside cartesian closed categories.

From CCCs to Λ-models

The Curry-Howard correspondence between LJ and the simply typed λ-calculus transports any model of LJ into a model of the simply typed λ-calculus. This means that Cartesian closed categories (Def. A.1.0.19) are categorical models of the simply typed λ-calculus. However, this does not extend directly to the untyped λ-calculus that is much larger.

In order to get a Λ-model, we need a *reflexive object* in such a CCC.

Definition 1.2.3.1. *Let C be a CCC (Def. A.1.0.19). A reflexive object is an object D of C endowed with two morphisms:*

$$\mathrm{app}: D \to (D \Rightarrow D) \qquad\qquad \mathrm{abs}: (D \Rightarrow D) \to D$$

such that $\mathrm{abs};\mathrm{app} = id_{D\Rightarrow D}$.

The idea of modelling the λ-calculus by a reflexive object is very similar to the idea of typing the λ-calculus with a unique type $* = * \Rightarrow *$ so that any term $M \in \Lambda$ defines a single proof derivation of $*, ..., * \vdash *$ in LJ.

Theorem 1.2.3.2 ([BEM07]). *Any CCC C with a reflexive object D defines a λ-abstraction algebra (Def. 1.2.1.1) inside the disjoint union of hom-set $C[D^V, D]$ for $V \subseteq_f \mathbb{Var}$:*

$$\biguplus_{V\subseteq_f \mathbb{Var}} C[D^V, D] := \{(V,\phi) \mid V \subseteq_f \mathbb{Var},\ \phi \in C[D^V, D]\}$$

quotiented by the following equivalence that morally equilizes modulo weakening of unused variables:

$$(U,\phi) \equiv (V,\psi) \qquad \textit{iff} \qquad \pi_U^{U\cup V};\phi = \pi_V^{U\cup V};\psi.$$

Due to this quotient, the interpretation $[\![M]\!]^V$ of a term M is defined modulo a set of variables V containing $\mathrm{FV}(M)$.[4]

The interpretation of a term M is a morphism $[\![M]\!] \in C[D^{\mathrm{FV}(M)}, D]$ defined by:

$$\begin{array}{lcl}
[\![x]\!]^V & := & D^V \xrightarrow{\quad\pi_x\quad} D \\
[\![\lambda x.M]\!]^V & := & D^V \xrightarrow{\pi_{V-x}} D^{V-x} \xrightarrow{\Lambda_x([\![M]\!]^{V\cup\{x\}})} D \Rightarrow D \xrightarrow{\mathrm{abs}} D \\
[\![M\,N]\!]^V & := & D^V \xrightarrow{\langle [\![N]\!]^V, [\![M]\!]^V\rangle} D\times D \xrightarrow{id_D\times \mathrm{app}} D\times(D\Rightarrow D) \xrightarrow{eval} D
\end{array}$$

From linear categoriess to CCCs

The linear logic (Def. A.3.1.1) is known (in particular) to be a rafinment of the intuisionistic lojec LJ. The most studied encoding is transporting the intuisionistic implication into a

[4]In fact, the definition is automatically extended to sets V of variables that may not contain some "unused" variables of $\mathrm{FV}(M)$.

composition of the exponential and the linear application:[5]

$$A \Rightarrow B \quad := \quad !A \multimap B$$

The translation applies also to models via the notion of Kleisli categories.

Proposition 1.2.3.3. *For any linear category $\mathcal{L}$ (Sec. A.3.2) with Cartesian products and coproducts, the Kleisli category $\mathcal{L}_!$ over the exponential comonad is a Cartesian closed category:*

- *objects are the objects of $\mathcal{L}$,*
- *morphisms are defined by $\mathcal{L}_![a,b] := \mathcal{L}[!a,b]$,*
- *composition and identity are defined by:*

$$\phi \;;^! \; \psi \;:=\; \mathsf{p}_{\mathrm{dom}(\phi)} \;;\; !\phi \;;\; \psi \qquad\qquad id^!_a := \mathsf{d}_a$$

- *Cartesian product is the Cartesian product of $\mathcal{L}$,*
- *the exponential object is defined by: $a \Rightarrow b \;:= !a \multimap b$*

In particular, any reflexive object in $\mathcal{L}_!$ is a model of the pure λ-calculus.

1.2.4. K-models

We introduce here the main object of Chapter 2: extensional K-models [Kri93][Ber00]. This class of models of the untyped λ-calculus is a subclass of filter models [CDHL84] containing many extensional models from the continuous semantics, like Scott's D_∞ [Sco72].

The category ScottL$_!$

Extensional K-models correspond to the extensional reflexive Scott domains that are prime algebraic complete lattices and whose application embeds prime elements into prime elements [Hut94, Win99]. However we prefer to exhibit K-models as the extensional reflexive objects of the category ScottL$_!$ (Prop. 1.2.4.4) which is itself the Kleisli category over the linear category ScottL (Prop. A.3.4.6).

In the following we use notations and definitions from Section A.3.4.

Definition 1.2.4.1. *We define the Cartesian closed category* ScottL$_!$ *[Hut94, Win99, Ehr12]:*

- objects *are partially ordered sets.*
- morphism *from D to E are a Scott-continuous function between the complete lattices $\mathcal{I}(D)$ and $\mathcal{I}(E)$.*

[5] notice that this is not the only encoding. This particular choice is most suited to encode call-by name implementation while other encoding can be used for call-by-value, call-by-need...

The Cartesian product *is the disjoint sum of posets. The* terminal object $\top$ *is the empty poset. The* exponential object $D{\Rightarrow}E$ *is* $\mathcal{A}_f(D)^{op}{\times}E$*. Notice that an element of* $\mathcal{I}(D{\Rightarrow}E)$ *is the graph of a morphism from D to E (see Equation* (A.3)*). This construction provides a natural isomorphism between* $\mathcal{I}(D{\Rightarrow}E)$ *and the corresponding homset. Notice that if* $\simeq$ *denotes the isomorphism in* ScottL$_!$*, then:*

$$D \Rightarrow D \Rightarrow \cdots \Rightarrow D \simeq (\mathcal{A}_f(D)^{op})^n \times D. \tag{1.2}$$

For example $D \Rightarrow (D \Rightarrow D) \simeq \mathcal{A}_f(D)^{op} \times (\mathcal{A}_f(D)^{op} \times D) = (\mathcal{A}_f(D)^{op})^2 \times D$*.*

Remark 1.2.4.2. *In the literature (*e.g. *[Hut94, Win99, Ehr12]), objects are preodered sets and the exponential object* $D \Rightarrow D$ *is defined by using finite subsets (or multisets) instead of the finite antichains. Our presentation is the quotient of the usual one by the equivalence relation induced by the preorder. The two presentations are equivalent (in terms of equivalence of category) but our choice simplifies the definition of hyperimmunity (Definition 2.1.0.1).*

An algebraic presentation of K-models

Definition 1.2.4.3 ([Kri93])**.** *An* extensional K-model *is a pair* (D, i_D) *where:*

- *D is a poset.*
- i_D *is an order isomorphism between* $D{\Rightarrow}D$ *and D.*

By abuse of notation we may denote the pair (D, i_D) simply by D when it is clear from the context we are referring to an extensional K-model.

Proposition 1.2.4.4. *Extensional K-models correspond exactly to extensional reflexive objects of* ScottL$_!$, i.e.*, an object D endowed with an isomorphism* $abs_D : (D \Rightarrow D) \to D$ *(and* $app_D := abs_D^{-1}$*).*

Proof. Given a K-model (D, i_D), the isomorphism between $D{\Rightarrow}D$ and D is given by:

$$\begin{aligned} &\forall A \in \mathcal{I}(D{\Rightarrow}D), && \mathrm{app}_D(A) = \{i_D(a,\alpha) \mid (a,\alpha) \in A\},\\ &\forall B \in \mathcal{I}(D), && \mathrm{abs}_D(B) = \{(a,\alpha) \mid i_D(a,\alpha) \in B\}. \end{aligned}$$

Conversely given an extensional reflexive object (D, app_D, abs_D) of ScottL$_!$:
First, remark that since abs_D is a monotone bijection with a monotone inverse, it is linear (preserves all sups). For all $(a,\alpha) \in D{\Rightarrow}D$, we have $\downarrow(a,\alpha) = abs(app(\downarrow(a,\alpha))) = \bigcup_{\beta \in app(\downarrow(a,\alpha))} abs(\downarrow\beta)$. Thus there is $\beta \in app(\downarrow(a,\alpha)$ such that $(a,\alpha) \in abs(\downarrow\beta)$, and since $abs(\downarrow\beta) \subseteq \downarrow(a,\alpha)$, this is an equality. Thus there is a unique β such that $app_D(a,\alpha) = \downarrow\beta$, this is $i_D(a,\alpha)$. □

In the following we will not distinguish between a K-model and its associated reflexive object, which is a model of the pure λ-calculus.

Definition 1.2.4.5. *An* extensional partial K-model *is a pair* (E, j_E) *where* E *is an object of* $\textsc{ScottL}_!$ *and* j_E *is a partial function from* $E \Rightarrow E$ *to* E *that is an order isomorphism between* $\mathrm{Dom}(j_E)$ *and* E.

$$E \quad \overset{j_E}{\longleftrightarrow} \quad \mathrm{Dom}(j_E) \quad \subseteq \quad (E \Rightarrow E)$$

Definition 1.2.4.6. *The* completion of a partial K-model (E, j_E) *is the union* $(\bar{E}, j_{\bar{E}}) = (\bigcup_{n\in\mathbb{N}} E_n, \bigcup_{n\in\mathbb{N}} j_{E_n})$ *of partial completions* (E_n, j_{E_n}) *that are extensional partial K-model defined by induction on n.* $(E_0, j_{E_0}) = (E, j_E)$ *and:*

- $|E_{n+1}| \ = \ |E_n| \cup (|E_n \Rightarrow E_n| - Dom(j_{E_n}))$
- $j_{E_{n+1}} \ = j_{E_n} \cup id_{|E_n \Rightarrow E_n| - Dom(j_{E_n})}$
- $\leq_{E_{n+1}}$ *is given by* $j_{E_{n+1}}(a, \alpha) \leq_{E_{n+1}} (b, \beta)$ *iff* $a \geq_{\mathcal{A}_f(E_n)} b$ *and* $\alpha \leq_{E_n} \beta$.

Remark that E_{n+1} corresponds to $E_n \ \Rightarrow \ E_n$ up to isomorphism, what leads to the equivalent definition:

Remark 1.2.4.7. *The completion of an extensional partial K-model* (E, j_E) *is the smallest extensional K-model* $\bar{E}$ *containing* E.

Remark 1.2.4.8. *Any extensional K-model* D *is the extensional completion of itself:* $D = \bar{D}$.

Example 1.2.4.9.

1. Scott's D_∞ *[Sco72] is the extensional completion of*

$$|D| = \{*\}, \qquad\qquad \leq_D = id, \qquad\qquad j_D = \{(\emptyset, *) \mapsto *\}.$$

   The completion is a triple $(|D_\infty|, \leq_{D_\infty}, j_{D_\infty})$ *where* $|D_\infty|$ *is generated by:*

$$\begin{array}{lllll} |D_\infty| & \alpha, \beta & ::= & * \ \mid & a {\to} \alpha \\ |!D_\infty| & a, b & \in & & \mathcal{A}_f(|D_\infty|) \end{array}$$

   except that $\emptyset{\to}* \notin |D_\infty|$*;* j_{D_∞} *is defined by* $j_{D_\infty}(\emptyset, *) = *$ *and* $j_{D_\infty}(a, \alpha) = a{\to}\alpha$ *for* $(a, \alpha) \neq (\emptyset, *)$.

2. Park's P_∞ *[Par76] is the extensional completion of*

$$|P| = \{*\}, \qquad\qquad \leq_P = id, \qquad\qquad j_P = \{(\{*\}, *) \mapsto *\};$$

   i.e., $|P_\infty|$ *is defined by the previous grammar except that* $(\{*\}{\to}*) \notin |P_\infty|$ *while* $\emptyset{\to}* \in |P_\infty|$.

3. *Norm or* D^*_∞ *[CDCZ87] is the extensional completion of*

$$|E| = \{p, q\}, \qquad\qquad \leq_E = id \cup \{p < q\},$$

$$j_E = \{(\{p\}, q){\mapsto}q, (\{q\}, p){\mapsto}p\}.$$

$$[\![x_i]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \alpha \leq \beta \in a_i\} \qquad [\![\lambda y.M]\!]_D^{\vec{x}} = \{(\vec{a},(b{\to}\alpha)) \mid (\vec{a}b,\alpha) \in [\![M]\!]_D^{\vec{x}y}\}$$

$$[\![M\,N]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \exists b, (\vec{a},(b{\to}\alpha)) \in [\![M]\!]_D^{\vec{x}} \ \wedge \forall\beta{\in}b, (\vec{a},\beta) \in [\![N]\!]_D^{\vec{x}}\}$$

Figure 1.2.: Direct interpretation of Λ in D

4. Well-stratified K-models *[Man09] are the extensional completions of some E respecting*

$$\forall(a,\alpha){\in}\mathrm{Dom}(j_E), a{=}\emptyset.$$

5. *The* inductive $\overline{\omega}$ *is the extensional completion of*

$$|E| = \mathbb{N}, \qquad \leq_E = id, \qquad j_E = \{(\{k \mid k < n\}, n){\mapsto}n \mid n \in \mathbb{N}\}.$$

6. *The* co-inductive $\overline{\mathbb{Z}}$ *is the extensional completion of*

$$|E| = \mathbb{Z}, \qquad \leq_E = id \qquad , j_E = \{(\{n\}, n+1){\mapsto}n{+}1 \mid n \in \mathbb{Z}\}.$$

7. Functionals H^f *(given* $f : \mathbb{N} \to \mathbb{N}$*) are the extensional completions of:*

$$|E| = \{*\} \cup \{\alpha_j^n \mid n \geq 0,\ 1 \leq j \leq f(n)\}, \qquad \leq_E = id,$$

$$j_E \;=\; \Big\{(\emptyset,*) \mapsto *\Big\} \,\cup\, \Big\{(\emptyset,\alpha_{j+1}^n) \mapsto \alpha_j^n \mid 1 \leq j < f(n)\Big\} \,\cup\, \Big\{(\{\alpha_1^{n+1}\},*) \mapsto \alpha_{f(n)}^n \mid n \in \mathbb{N}\Big\},$$

where $(\alpha_j^n)_{n,j}$ *is a family of atoms different from* $*$.

For the sake of simplicity, from now on we will work with a fixed extensional K-model D. Moreover, we will use the notation $a{\to}\alpha := i_D(a,\alpha)$. Notice that, due to the injectivity of i_D, any $\alpha \in D$ can be uniquely rewritten into $a{\to}\alpha'$, and more generally into $a_1{\to}\cdots{\to}a_n{\to}\alpha_n$ for any n.

Remark 1.2.4.10. *Using this notations, the model* H^f *can be summarized by writing, for each n:*

$$\alpha_1^n \;=\; \underbrace{\emptyset{\to}\cdots{\to}\emptyset}_{f(n)}{\to}\{\alpha_0^{n+1}\}{\to}*$$

Interpretation of the λ-calculus

The Cartesian closed structure of ScottL$_!$ endowed with the isomorphism app_D and abs_D of the reflexive object induced by D (Proposition 1.2.4.4) defines a standard model of the λ-calculus.

A term M with at most n free variables $x_1,\ldots,x_n$ is interpreted as the graph of a morphism $[\![M]\!]_D^{x_1\ldots x_n}$ from D^n to D (when D is obvious, we can use $[\![.]\!]^{\vec{x}}$). By Equations (A.3) and (1.2) we have:

$$[\![M]\!]_D^{x_1\ldots x_n} \subseteq (D \Rightarrow \cdots \Rightarrow D \Rightarrow D) \;\simeq\; (\mathcal{A}_f(D)^{op})^n \times D.$$

In Figure 1.2, we explicit the interpretation $[\![M]\!]_D^{x_1\ldots x_n}$ by structural induction on M.

$$\frac{\alpha \in a}{x : a \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : a \vdash M : \alpha} \qquad \frac{\Gamma \vdash M : \beta \qquad \alpha \leq \beta}{\Gamma \vdash M : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a \to \alpha} \qquad \frac{\Gamma \vdash M : a \to \alpha \qquad \forall \beta \in a,\ \Gamma \vdash N : \beta}{\Gamma \vdash M\,N : \alpha}$$

Figure 1.3.: Intersection type system computing the interpretation in D

Example 1.2.4.11.

$$\begin{aligned}
[\![\lambda x.y]\!]_D^y &= \{((a), b \to \alpha) \mid \alpha \leq_D \beta \in a\},\\
[\![\lambda x.x]\!]_D^y &= \{((a), b \to \alpha) \mid \alpha \leq_D \beta \in b\},\\
[\![\boldsymbol{I}]\!]_D &= \{a \to \alpha \mid \alpha \leq_D \beta \in a\},\\
[\![\underline{1}]\!]_D &= \{a \to b \to \alpha \mid \exists c, c \to \alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b\}.
\end{aligned}$$

In the last two cases, terms are interpreted in an empty environment. We then omit the empty sequence associated with the empty environment, e.g., *$a \to b \to \alpha$ stands for $((), a \to b \to \alpha)$.*
We can verify that extensionality holds, indeed $[\![\underline{1}]\!]_D = [\![\boldsymbol{I}]\!]_D$, since $c \to \alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b$ means that $b \to \alpha \leq_D \beta \in a$, and since any element of D is of the form $\alpha \to \beta$.

Intersection types

It is well known that the interpretation of the λ-calculus into a given K-model D is characterized by a specific *intersection type system*. In fact any element $\alpha \in D$ can be seen as an intersection type

$$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta \qquad\qquad \text{given by} \quad \alpha = \{\alpha_1, \ldots, \alpha_n\} \to \beta.$$

In Figure 1.3, we give the intersection-type assignment corresponding to the K-model induced by D.

> **Proposition 1.2.4.12.** *Let M be a term of Λ, the following statements are equivalent:*
>
> - $(\vec{a}, \alpha) \in [\![M]\!]_D^{\vec{x}}$,
> - *the type judgment $\vec{x} : \vec{a} \vdash M : \alpha$ is derivable by the rules of Figure 1.3.*

Proof. By structural induction on the grammar of Λ. □

1.3. $\mathcal{S}$-Bounded logics $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

Various systems have been recently proposed based on a notion of parameterized exponential comonad [BGMZ14, GS14] in linear logic. The idea is to parameterize the of-course modality ! with elements taken from a semiring $\mathcal{S}$. The multiplicative monoid of $\mathcal{S}$ describes how the parameters interact under the comonad structure of ! (i.e. dereliction and digging) while the additive monoid of $\mathcal{S}$ gives the interaction under the monoidal structure of ! (i.e. weakening and contraction). The axioms of the semiring allow to define a parameterized version of the usual rules of cut-elimination, preserving the confluence property (see Figure 3.2).

This approach is related to Girard, Scedrov and Scott's *bounded linear logic* (BLL) [GSS92a], and thus we refer to it as $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$. It is in some sense both a generalization and a restriction of BLL. It is a generalization because it allows to choose any semiring, as a parameter, while BLL is given with respect to a fixed notion of parameters. On the other hand, $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ is a significant restriction because its parameters are just elements of the semiring $\mathcal{S}$ while BLL deals with first-class terms extending polynomials and allowing dependencies.

The interest of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ is to offer a logical ground to the design of type systems allowing to express various co-effects, that is requirements of a program with respect to the environment. For example, in [GS14] a semiring based on contractive affine transformations has been used to design a type system with annotations on the scheduling of processes; in [BGMZ14], the semiring of non-negative real numbers is used to express the expected value of the number of times a probabilistic program calls its input during the evaluation. We briefly recall these examples in Section 1.3.2. The interesting point is that although these type systems model quite different co-effects, their soundness is rooted in the same logical framework, that is $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$.

Notice that Appendix A.4 is a collection of semiring-related definitions to which we will refer freely in this section.

1.3.1. The logic $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

Definition 1.3.1.1. *Given an ordered lax-semiring*[6] $\mathcal{S}$ *(called* bounding semiring*), we call* linear logic bounded by $\mathcal{S}$-exponentials, $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ *, the logic given by:*

- *the* formulas *defined by the grammar:*
  (formulas) $A, B, C := \alpha \mid A \otimes B \mid A \multimap B \mid A^J$ *where* $J \in \mathcal{S}$,
- *the* sequent calculus *given in Figure 1.4 modulo the equations of Figure 1.5,*
- *and the* cut-elimination procedure *defined by the usual rules of multiplicative linear logic of Figure 1.6 plus the rules of Figure 1.8 (and the minor rules of Figure 1.7).*

Remark 1.3.1.2. *One can add the additive connectives without any effort. Nonetheless, we prefer to omit their account because irrelevant for our results. In [BGMZ14], the authors use a term calculus instead of logical sequent system: the two presentations are clearly interchangeable via a Curry-Howard correspondence.*

Remark 1.3.1.3. *The derivation rules contain two different notions of weakening,* `Weak` *and* $\mathcal{S}$`weak`. *The first, called structural weakening, requires a null quantity of a fresh resource. The second, called resource weakening, requires to increase the amount of a resource already present in the context.*

[6] See Definition A.4.1.1

$$
\frac{}{A \vdash A}\,\texttt{Ax} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\mathrm{L} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R}
$$

$$
\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\texttt{Cut} \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap \mathrm{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap \mathrm{R}
$$

$$
\frac{\Gamma \vdash B \quad I \geq 0}{\Gamma, A^I \vdash B}\,\texttt{Weak} \qquad \frac{\Gamma, A \vdash B \quad I \geq 1}{\Gamma, A^I \vdash B}\,\texttt{Der} \qquad \frac{\Gamma, A^I, A^J \vdash B \quad K \geq I{+}J}{\Gamma, A^K \vdash B}\,\texttt{Contr}
$$

$$
\frac{{A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash B \quad (K_i \geq I_i{\cdot}J)}{{A_1}^{K_1}, \cdots {A_n}^{K_n} \vdash B^J}\,\texttt{Prom} \qquad \frac{\Gamma, A^I \vdash B \quad J \geq I}{\Gamma, A^J \vdash B}\,\mathcal{S}\texttt{weak}
$$

Figure 1.4.: The sequent calculus of $B_{\mathcal{S}}LL$. In a sequent $\Gamma \vdash A$, Γ is supposed to be a multiset of formulas (no implicit contraction rule is admitted).

$$
\frac{\dfrac{\dfrac{\Pi}{\Gamma \vdash C} \quad I \geq 0}{\Gamma, A^I \vdash C}\,\texttt{Weak} \quad J \geq 0}{\Gamma, A^I, B^J \vdash C}\,\texttt{Weak} \quad \equiv \quad \frac{\dfrac{\dfrac{\Pi}{\Gamma \vdash C} \quad J \geq 0}{\Gamma, B^J \vdash C}\,\texttt{Weak} \quad I \geq 0}{\Gamma, A^I, B^J \vdash C}\,\texttt{Weak}
$$

$$
\frac{\dfrac{\dfrac{\Pi}{\Gamma, A^I, A^J, B^{I'}, B^{J'} \vdash C} \quad K \geq I + J}{\Gamma, A^K, B^{I'}, B^{J'} \vdash C}\,\texttt{Contr} \quad K' \geq I' + J'}{\Gamma, A^{I+J}, B^{K'} \vdash C}\,\texttt{Contr} \quad \equiv \quad \frac{\dfrac{\dfrac{\Pi}{\Gamma, A^I, A^J, B^{I'}, B^{J'} \vdash C} \quad K' \geq I' + J'}{\Gamma, A^I, A^J, B^{K'} \vdash C}\,\texttt{Contr} \quad K \geq I + J}{\Gamma, A^K, B^{K'} \vdash C}\,\texttt{Contr}
$$

Figure 1.5.: Equations-modulo between proofs of LL.

$$
\frac{\dfrac{\dfrac{\Pi_1}{\Gamma, A \vdash B}}{\Gamma \vdash A \multimap B}\,\multimap \mathrm{R} \qquad \dfrac{\dfrac{\Pi_2}{\Sigma \vdash A} \quad \dfrac{\Pi_3}{\Delta, B \vdash C}}{\Sigma, \Delta, A \multimap B \vdash C}\,\multimap \mathrm{L}}{\Gamma, \Sigma, \Delta \vdash C}\,\texttt{Cut} \quad \longrightarrow \quad \frac{\dfrac{\dfrac{\Pi_2}{\Sigma \vdash A} \quad \dfrac{\Pi_1}{\Gamma, A \vdash B}}{\Gamma, \Sigma \vdash B}\,\texttt{Cut} \qquad \dfrac{\Pi_3}{\Delta, B \vdash C}}{\Gamma, \Sigma, \Delta \vdash C}\,\texttt{Cut}
$$

$$
\frac{\dfrac{\dfrac{\Pi_1}{\Gamma \vdash A} \quad \dfrac{\Pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R} \qquad \dfrac{\dfrac{\Pi_3}{\Sigma, A, B \vdash C}}{\Sigma, A \otimes B \vdash C}\,\otimes\mathrm{L}}{\Gamma, \Delta, \Sigma \vdash C}\,\texttt{Cut} \quad \longrightarrow \quad \frac{\dfrac{\Pi_1}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\Pi_2}{\Delta \vdash B} \quad \dfrac{\Pi_3}{\Sigma, A, B \vdash C}}{\Delta, A, \Sigma \vdash C}\,\texttt{Cut}}{\Gamma, \Delta, \Sigma \vdash C}\,\texttt{Cut}
$$

Figure 1.6.: Cut-elimination rules (multiplicative only).

$$\dfrac{\dfrac{}{A \vdash A} \qquad \dfrac{\Pi}{\Gamma, A \vdash B}}{\Gamma, A \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\Pi}{\Gamma, A \vdash B}$$

$$\dfrac{\dfrac{\Pi}{\Gamma, A \vdash B} \qquad \dfrac{}{B \vdash B}}{\Gamma, A \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\Pi}{\Gamma, A \vdash B}$$

$$\dfrac{\dfrac{\Pi_1}{\Delta \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, A', A, B \vdash C}}{\Gamma, A', A \otimes B \vdash C}\,\otimes\mathrm{L}}{\Gamma, \Delta, A \otimes B \vdash C}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash A'} \qquad \dfrac{\Pi_2}{\Gamma, A', A, B \vdash C}}{\Gamma, \Delta, A, B \vdash C}\,\texttt{Cut}}{\Gamma, \Delta, A \otimes B \vdash C}\,\otimes\mathrm{L}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma \vdash A} \qquad \dfrac{\Pi_3}{\Delta \vdash B}}{A', \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R}}{\Sigma, \Gamma, \Delta \vdash A \otimes B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma \vdash A}}{\Sigma, \Gamma \vdash A}\,\texttt{Cut} \qquad \dfrac{\Pi_3}{\Delta \vdash B}}{\Sigma, \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash A} \qquad \dfrac{\Pi_3}{A', \Delta \vdash B}}{A', \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R}}{\Sigma, \Gamma, \Delta \vdash A \otimes B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_3}{A', \Delta \vdash B}}{\Sigma, \Delta \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathrm{R}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma \vdash A} \qquad \dfrac{\Pi_3}{\Delta, B \vdash C}}{A', \Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}}{\Sigma, \Gamma, \Delta, A \multimap B \vdash C}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma \vdash A}}{\Sigma, \Gamma \vdash A}\,\texttt{Cut} \qquad \dfrac{\Pi_3}{\Delta, B \vdash C}}{\Sigma, \Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash A} \qquad \dfrac{\Pi_3}{A', \Delta, B \vdash C}}{A', \Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}}{\Sigma, \Gamma, \Delta, A \multimap B \vdash C}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_3}{A', \Delta, B \vdash C}}{\Sigma, \Delta, B \vdash C}\,\texttt{Cut}}{\Sigma, \Gamma, \Delta, A \multimap B \vdash C}\,\multimap\mathrm{L}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma, A \vdash B}}{A', \Gamma \vdash A \multimap B}\,\multimap\mathrm{R}}{\Sigma, \Gamma \vdash A \multimap B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma, A \vdash B}}{\Sigma, \Gamma, A \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma \vdash A \multimap B}\,\multimap\mathrm{R}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma \vdash B}}{A', \Gamma, A^I \vdash B}\,{\scriptstyle I \geq 0}\;\texttt{Weak}}{\Sigma, \Gamma, A^I \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma \vdash B}}{\Sigma, \Gamma \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma, A^I \vdash B}\,{\scriptstyle I \geq 0}\;\texttt{Weak}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma, A \vdash B}}{A', \Gamma, A^I \vdash B}\,{\scriptstyle I \geq 1}\;\texttt{Der}}{\Sigma, \Gamma, A \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma, A \vdash B}}{\Sigma, \Gamma, A \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma, A \vdash B}\,{\scriptstyle I \geq 1}\;\texttt{Der}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma, A^I, A^J \vdash B}}{A', \Gamma, A^K \vdash B}\,{\scriptstyle K \geq I+J}\;\texttt{Contr}}{\Sigma, \Gamma, A^K \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma, A^I, A^J \vdash B}}{\Sigma, \Gamma, A^I, A^J \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma, A^K \vdash B}\,{\scriptstyle K \geq I+J}\;\texttt{Contr}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', {A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash B}}{A', {A_1}^{K_1}, \cdots {A_n}^{K_n} \vdash B^J}\,{\scriptstyle (K_i \geq I_i \cdot J)}\;\texttt{Prom}}{\Sigma, {A_1}^{K_1}, \cdots {A_n}^{K_n} \vdash B^J}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', {A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash B}}{\Sigma, {A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash B}\,\texttt{Cut}}{\Sigma, {A_1}^{K_1}, \cdots {A_n}^{K_n} \vdash B^J}\,{\scriptstyle (K_i \geq I_i \cdot J)}\;\texttt{Prom}$$

$$\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\dfrac{\Pi_2}{A', \Gamma, A^I \vdash B}}{A', \Gamma, A^J \vdash B}\,{\scriptstyle J \geq I}\;\mathcal{S}\texttt{weak}}{\Sigma, \Gamma, A^J \vdash B}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_1}{\Sigma \vdash A'} \qquad \dfrac{\Pi_2}{A', \Gamma, A^I \vdash B}}{\Sigma, \Gamma, A^I \vdash B}\,\texttt{Cut}}{\Sigma, \Gamma, A^J \vdash B}\,{\scriptstyle J \geq I}\;\mathcal{S}\texttt{weak}$$

Figure 1.7.: Cut-elimination rules (minor rules).

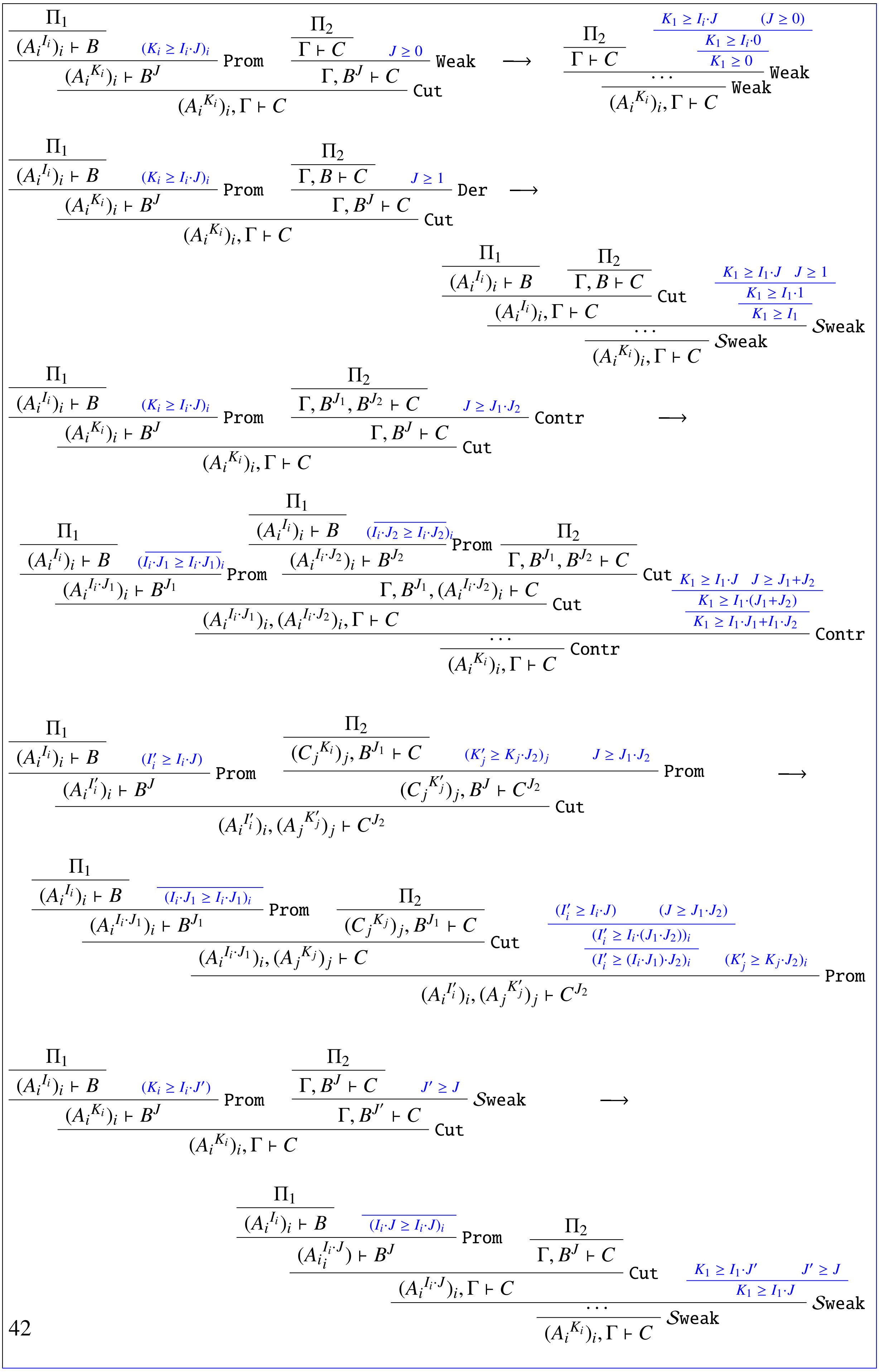


Figure 1.8.: Cut-elimination rules (exponentials only).

1.3.2. Examples

Trivial semiring: the multiplicative exponential fragment of intuitionistic linear logic is recovered from $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ by taking $\mathcal{S}$ as the one element semiring.

Boolean semiring: the Boolean semiring $\mathbb{B} = (\{tt, ff\}, \wedge, tt, \vee, ff)$ allows finer types that the trivial one, distinguishing between data that can be weakened (of type A^{ff}) from data that can be duplicated (A^{tt}). The order over $\mathbb{B}$ plays a role, also: the discrete order will make the two types disjoint, while $tt \geq ff$ will make A^{ff} a subtype of A^{tt}, so that the $(\_)^{tt}$ modality behaves as the usual of-course modality ! of linear logic.

Natural numbers: the natural number semiring $(\mathbb{N}, \times, 1, +, 0)$ yields modalities expressing the number of times a resource is to be used. The order relation then allows some flexibility: for example, the natural order $0 < 1 < 2 < \dots$ makes A^n to be the type of data that can be used up-to n times. Notice that in this case there is no modality allowing a resource to be used an indefinite number of times, so the system is not an extension of linear logic. In order to recover the usual of-course modality ! one should take the order completion $\bar{\mathbb{N}}$, adding a top-element ω.

Polynomial semiring: by taking the semiring $(\mathbb{N}[X_i]_{i\in\mathbb{N}}, \times, 1, +, 0)$ of polynomials with natural numbers as coefficients, one can express a basic form of resource dependency. One can write formulas like $A^{p(\vec{x})} \multimap B^{q(\vec{x})}$ where p, q are polynomials in the unknowns $\vec{x}$. Roughly speaking, this is the type of a function giving a result reusable $q(\vec{n})$ number of times as soon as its input can be used $p(\vec{n})$ number of times, for any sequence of natural numbers $\vec{n}$. This system has been discussed in [GSS92a] as an introduction to bounded linear logic (BLL). What is lacking with respect to the whole BLL is the possibility to bound first-order variables, so writing types of the form $A^{y\leq p(x)}$, where y is an unknown of a polynomial occurring inside A.

Affine contractive transformations: the one-dimensional contractive affine transformations can be represented by real-valued matrices $x_{s,p} = \left(\begin{smallmatrix} s & p \\ 0 & 1 \end{smallmatrix}\right)$ with $0 \leq s \leq 1$ and $0 \leq s + p \leq 1$. The value s is a scaling factor relative to the unit interval, and p is a delay from the time origin. The set of such transformations forms a monoid Aff^c_1 with composition given by matrix product.[7] By Proposition A.4.2.2, $\mathbb{N}_f\langle\mathrm{Aff}^c_1\rangle$ is a semiring so it defines the logic $\mathrm{B}_{\mathbb{N}_f\langle\mathrm{Aff}^c{}_1\rangle}\mathrm{LL}$. This system has been introduced by Ghica and Smith [GS14] in order to express at the level of types a scheduling on the execution of certain resources. For example, a formula $A^{\left[\left(\begin{smallmatrix} .5 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} .5 & .25 \\ 0 & 1 \end{smallmatrix}\right)\right]}$ represents a resource of type A that can be called twice, both calls will last $\frac{1}{2}$ the duration relative to which we are measuring, but one call starts at the beginning of the available time interval, while the other call starts when $\frac{1}{4}$ of the time has elapsed. Of course, such annotations have a meaning when the language has primitives describing processes to be scheduled. See [GS14] for more details.

[7]Notice, however, that the semiring product $I{\cdot}J$ denotes the reverse matrix product $J{\cdot}I$, this is due to a change in notation between us and [GS14].

Positive real numbers: in presence of random primitives, one can associate any resource with a discrete random variable quantifying on the number of times this resource is used during the evaluation. In [BGMZ14], $\mathrm{B}_{\mathcal{S}}$LL has been parametrised with the ordered semiring $\mathbb{R}^+ = (\mathbb{R}^+, \times, 1, +, 0, \leq)$ of the non-negative real numbers endowed with the natural order, the parameters expressing the expected values of these random variables. So for example, the type $A^{\frac{3}{2}}$ expresses a resource that, whenever it is called a number $n > 0$ of times, denoting by $p_i \in [0, 1]$ the probability of converging to a value in the i-th call, we have $\sum_{i=1}^{n} p_i = \frac{3}{2}$.

This system can be extended (syntactically) with true dependent types and be able to catch finer properties, like differential privacy [GHH+13b].

2. Characterization of full abstraction of $\mathcal{H}^*$

The histories of full abstraction and denotational semantics of λ-calculi are both rooted in four fundamental articles published in the course of a year.

In 1976, Hyland [Hyl76] and Wadsworth [Wad76] independently[1] proved the first full abstraction result of Scott's D_∞ (Ex. 1.2.4.9) for $\mathcal{H}^*$ (the theory observing the convergence of head-normalization, see Example 1.1.0.16). The following year, Milner [Mil77] and Plotkin [Plo77] showed respectively that PCF (a Turing-complete extension of the simply typed λ-calculus) has a unique fully abstract model up to isomorphism and that this model is not in the category of Scott domains and continuous functions.

Later, various articles focused on circumventing Plotkin counterexample [AMJ94, HO00] or investigating full abstraction results for other calculi [AM96, Lai97, Pao06]. However, hardly anyone pointed out the fact that Milner's uniqueness theorem is specific to PCF, while $\mathcal{H}^*$ has various models that are fully abstract but not isomorphic.

The quest for a general characterization of the fully abstract models of head normalization started by successive refinements of a sufficient, but unnecessary condition [DGFH99, X.G95, Man09], improving the proof techniques from 1976 [Hyl76, Wad76]. While these results shed some light on various fully abstract semantics for $\mathcal{H}^*$, none of them could reach full characterization.

In this chapter, we give the first full characterization of the full abstraction of $\mathcal{H}^*$ for a specific (but large) class of models. The class we choose is that of Krivine-models, or K-models [Kri93, Ber00] (Def. 1.2.4.3). This class, described in Section 1.2.4, is essentially the subclass of Scott complete lattices (or filter models [CDHL84]) which are prime algebraic. We add two further conditions: extensionality and approximability of Definition 2.2.1.12 (or equivalently the sensibility for $\Lambda_{\tau(D)}$ of Definition 2.3.1.19). The extensional and approximable K-models are the objects of our characterization and can be seen as a natural class of models obtained from models of linear logic [Gir87]. Indeed, the extensional K-models correspond to the extensional and approximable reflexive objects of the co-Kleisli category associated with the exponential comonad of Ehrhard's ScottL category [Ehr12] (Prop. 1.2.4.4).

We achieve the characterization of full abstraction for $\mathcal{H}^*$ in Theorem 2.1.0.5: a model D is fully abstract for $\mathcal{H}^*$ iff D is *hyperimmune* (Def. 2.1.0.1). Hyperimmunity is the key property our study introduces in denotational semantics. This property is reminiscent of the Post's notion of hyperimmune sets in recursion theory. Hyperimmunity is not only undecidable, but also surprisingly high in the hierarchy of undecidable properties (it cannot be decided by a machine with an oracle deciding the halting problem) [Nie09].

Roughly speaking, a model D is hyperimmune whenever the λ-terms can have access to only

[1] Notice, however, that the idea already appears in Wadsworth thesis 3 years earlier.

well-founded chains of elements of D. In other words, D might have non-well-founded chains $d_0 \geq d_1 \geq \cdots$, but these chains "grow" so fast (for a suitable notion of growth), that they cannot be contained in the interpretation of any λ-term.

The intuition that full abstraction of $\mathcal{H}^*$ is related with a kind of well-foundation can be found in the literature (*e.g.*, Hyland's [Hyl76], Gouy's [X.G95] or Manzonetto's [Man09]). Our contribution is to give, with hyperimmunity, a precise definition of this intuition, at least in the setting of K-models.

A finer intuition can be described in terms of game semantics. Informally, a game semantic for the untyped λ-calculus takes place in the arena interpreting the recursive type $o = o \to o$. This arena is infinitely wide (by developing the antecedent of the implication $o \to o$) and infinitely deep (by developing the consequent of the implication $o \to o$). Moves therein can thus be characterized by their nature (question or answer) and by a word over natural numbers. For example, $q(2.3.1)$ represents a question in the underlined "o" in $o = o \to (o \to o \to (\underline{o} \to o) \to o) \to o$. Plays in this game are potentially infinite sequences of moves, where a question of the form $q(w)$ is followed by any number of deeper questions/answers, before an answer $a(w)$ is eventually provided, if any.

A play like $q(\epsilon), q(1)...a(1), q(2)...a(2), q(3)...$ is admissible: one player keeps asking questions and is infinitely delaying the answer to the initial question, but some answers are given so that the stream is productive. However, the full abstraction for $\mathcal{H}^*$ forbids non-productive infinite questioning like in $q(\epsilon), q(1), q(1.1), q(1.1.1)...$, in general. Nevertheless, disallowing *all* such strategies is sufficient, but not necessary to get full abstraction. The hyperimmunity condition is finer: non productive infinite questioning is allowed *as long as* the function that choose the next question grows faster than any recursive function (notice that in the example above that choice is performed by the constant $(n \mapsto 1)$ function). For example, if $(u_i)_{i \geq 0}$ grows faster than any recursive function, the play $q(\epsilon), q(u_1), q(u_1.u_2), q(u_1.u_2.u_3)...$ is perfectly allowed.

Incidentally, we obtain a significant corollary (also expressed in Theorem 2.1.0.5) stating that full abstraction coincides with inequational full abstraction for $\mathcal{H}^*$ (equivalence between observational and denotational orders). This is in contrast to what happens to other calculi [Sto90, EPT14].

In the literature, most of the proofs of full abstraction for $\mathcal{H}^*$ are based on Nakajima trees [Nak75] or some other notion of quotient of the space of Böhm trees, using the characterization of the observational equivalence (see Proposition 1.2.2.12). The usual approach is too coarse because it considers arbitrary Böhm trees which are not necessarily images of actual λ-terms. To overcome this we propose two different techniques leading to two different proofs of the main result: one purely semantical and another one purely syntactical.

Section 2.2 deals with the semantical proof. This proof follows the line of historical ones while overcoming weaknesses of Nakajima trees with a notion of quasi-approximation property (Def. 2.2.1.17), that involves recursivity in a refined way. Quasi-approximability is a key tool in the proof, which is otherwise quite standard. However, since Böhm trees are specific to the λ-calculus and head reduction, there is not much hope to extend the proof to many other calculi/strategies (such as differential λ-calculus [ER04], or call-by-value strategies).

Section 2.3 deals with the syntactical proof. It approaches the problem from a novel angle. It consists in the use of a new tool: the *calculi with tests* (Def. 2.3.1.1). These are syntactic extensions of the λ-calculus with operators defining compact elements of the given models.

Since the model appears in the syntax, we are able to perform inductions (and coinductions) directly on the reduction steps of actual terms, rather than on the construction of Böhm trees.

The idea of test mechanisms as syntactic extensions of the λ-calculus was first used by Bucciarelli *et al.* [BCEM11]. Even though it was mixed with a resource-sensitive extension, the idea was already used to define morphisms of the model. Nonetheless, we can notice that older notions like Wadsworth's labeled $\lambda\bot$-calculus [Wad76] seem related to calculi with tests. The calculi with tests are not *ad hoc* tricks, but powerful and general tools.

2.1. The theorem

We state our main result, showing an equivalence between hyperimmunity (Def. 2.1.0.1) and full abstraction for $\mathcal{H}^*$ with two theorems (Th. 2.1.0.5 and 2.1.0.6) that differ in their premises. We will prove later (Th. 2.4.1.9) that these premises are in fact equivalent.

Recall from Section 1.2.4 and Appendix A.3.4 that an extensional K-model is a pair $(D, (\_ \to \_))$ where:

- D is a poset.
- $(\_{\to}\_)$ is an order isomorphism between $D \Rightarrow D$ and D (where $D \Rightarrow D \simeq \mathcal{A}_f(D)^{op} \times D$).

Definition 2.1.0.1 (*Hyperimmunity*)**.** *A (possibly partial) extensional K-model D is said to be* hyperimmune *if for every sequence* $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$*, there is no recursive function* $g : \mathbb{N} \to \mathbb{N}$ *satisfying, for all* $n \geq 0$*:*

$$\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n \qquad \text{and} \qquad \alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}. \tag{2.1}$$

Notice, in the above definition, that each antichain $a_{n,i}$ always exist and are uniquely determined by the isomorphism between D and $D \Rightarrow D$ that allow us to unfold any element α_i as an arrow (of any length).

The idea is the following. The sequence $(\alpha_n)_{n\geq 0}$ is morally describing a non well-founded chain of elements of D, through the isomorphism $D \simeq D \Rightarrow D$, allowing us to see any element α_i as an arrow (of any length):

$$\begin{array}{lllll}
\alpha_1 = a_{1,1} \to \cdots\ a_{1,i_1} \cdots \to a_{1,g(1)} \to \alpha'_1 & & & \\
\qquad\qquad\quad \Cup & & & \\
\qquad\qquad\quad \alpha_2 = a_{2,1} \to \cdots\ a_{2,i_2} \cdots \to a_{2,g(2)} \to \alpha'_2 & & \\
\qquad\qquad\qquad\qquad\qquad\quad \Cup & \\
\qquad\qquad\qquad\qquad\qquad\quad \alpha_3 = a_{3,1} \to \cdots\ a_{3,i_3} \cdots \to a_{3,g(3)} \to \alpha'_3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{array}$$

The growth rate $(i_n)_n$ of the chain $(\alpha_n)_n$ depends on how many arrows must be displayed in α_i in order to see α_{i+1} as an element of the antecedent of one of them. Now, the hyperimmunity means that if any such non-well founded chain $(\alpha_n)_n$ exists, then its growth rate $(i_n)_n$ cannot be bounded by any recursive function g.

Remark 2.1.0.2. *It would not be sufficient to simply consider the function* $n \mapsto i_n$ *such that* $\alpha_{n+1} \in a_{n,i_n}$ *rather than the bounding function g. Indeed,* $n \mapsto i_n$ *may not be recursive even while g is.*

Proposition 2.1.0.3. *For any extensional partial K-model E (Def. 1.2.4.5), the completion $\overline{E}$ (Def. 1.2.4.6) is hyperimmune iff E is hyperimmune.*

Proof. The left-to-right implication is trivial.
The right-to-left one is obtained by contradiction:
Assume we had $(\alpha_n)_{n\geq 0} \in \bar{E}^{\mathbb{N}}$ and a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that for all $n \geq 0$:

$$\alpha_n = a_{n,1}\to\cdots\to a_{n,g(n)}\to\alpha'_n \qquad\qquad \text{and} \qquad\qquad \alpha_{n+1} \in \bigcup_{i\leq g(n)} a_{n,i}$$

Recalls that the sequence $(E_k)_{k\geq 0}$ of Definition 1.2.4.6 approximates the completion $\bar{E}$.
Then we have the following:

- There exists k such that $\alpha_0 \in E_k$: Indeed, $\alpha_0 \in \bar{E} = \bigcup_k E_k$.
- If $\alpha_n \in E_{k+1}$ then $\alpha_{n+1} \in E_k$: There is $i \leq g(n)$ such that $\alpha_{n+1} \in a_{n,i} \subseteq E_k$.
- If $\alpha_n \in E_0 = E$ then $\alpha_{n+1} \in E_0 = E$: By surjectivity of j_E.

Thus there is k such that $(\alpha_n)_{n\geq k} \in E^{\mathbb{N}}$, what would break the hyperimmunity of E. □

Example 2.1.0.4.

- *The well-stratified K-models of Example 1.2.4.9*(4) *(and in particular D_∞ of Item* (1)*) are trivially hyperimmune: already in the partial K-model, there are not even α_1, α_2 and n such that $\alpha_1 = a_1\to\cdots\to a_n\to\alpha'_1$ and $\alpha_2 \in a_n$ (since $a_n = \emptyset$). The non-hyperimmunity of the partial K-model can be extended through the completion using Proposition 2.1.0.3.*
- *The same holds for $\overline{\omega}$ (Ex. 1.2.4.9*(5)*). Indeed, any such $(\alpha_n)_n$ in the partial K-model would respect $\alpha_{n+1}<_{\mathbb{N}}\alpha_n$,hence $(\alpha_n)_n$ must be finite by well-foundation of* $\mathbb{N}$.
- *On the other hand the models P_∞, D^*_∞ and $\overline{\mathbb{Z}}$ (Examples 1.2.4.9*(2), (3) *and* (6)*) are not hyperimmune. Indeed for all of them $g = (n \mapsto 1)$ satisfies the condition of Equation* (2.1)*, the respective non-well founded chains $(\alpha_i)_i$ being $(*, *, \dots)$, $(p, q, p, q, \dots)$, and $(0, -1, -2, \dots)$:*

$$\begin{array}{ccc}
* = \{*\} \to * & p = \{q\} \to p & 0 = \{1\} \to 0 \\
\Cup & \Cup & \Cup \\
* = \{*\} \to * & q = \{p\} \to q & 1 = \{2\} \to 1 \\
\Cup & \Cup & \Cup \\
* = \{*\} \to * & p = \{q\} \to p & 2 = \{3\} \to 2 \\
\ddots & \ddots & \ddots
\end{array}$$

- *More interestingly, the model H^f (Ex. 1.2.4.9*(7)*) is hyperimmune iff f is an* hyperimmune function *[Nie09],* i.e., *iff there is no recursive $g : \mathbb{N} \to \mathbb{N}$ such that $f \leq g$ (pointwise order); otherwise* *the corresponding sequence is* $(\alpha_1^i)_i$.

$$\alpha_1^0 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(0)\ times} \to \{\alpha_1^1\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Uparrow$$

$$\alpha_1^1 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(1)\ times} \to \{\alpha_1^2\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Uparrow$$

$$\alpha_1^2 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(2)\ times} \to \{\alpha_1^3\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Uparrow$$

$$\ddots$$

The following theorems constitute the main result of this chapter. They show the equivalence between hyperimmunity and (inequational) full abstraction for $\mathcal{H}^*$ under certain conditions. These conditions, namely the approximation property and the sensibility for $\Lambda_{\tau(D)}$, are equivalent conditions (see Theorem 2.4.1.9) and are defined in more details in Definitions 2.2.1.12 and 2.3.1.19.

Theorem 2.1.0.5 (Main theorem (semantic)). *For any extensional and approximable K-model D (Def. 2.2.1.12), the following are equivalent:*

1. *D is hyperimmune,*
2. *D is inequationally fully abstract for* $\mathcal{H}^*$*,*
3. *D is fully abstract for* $\mathcal{H}^*$*.*

Theorem 2.1.0.6 (Main theorem (syntactic)). *For any extensional K-model D sensible for* $\Lambda_{\tau(D)}$ *(Def. 2.3.1.19), the following are equivalent:*

1. *D is hyperimmune,*
2. *D is inequationally fully abstract for* $\mathcal{H}^*$*,*
3. *D is fully abstract for* $\mathcal{H}^*$*.*

Example 2.1.0.7. *In Example 1.2.4.9, the models* D_∞ *and* $\overline{\omega}$ *(Items* (1) *and* (5)*), and the well-stratified K-models are (inequationally) fully abstract as well as the* H^f *which function f is hyperimmune. While the models* P_∞, D_∞^*, $\overline{\mathbb{Z}}$ *(Items* (2), (3) *and* (6)*) are not, as well as the* H^f *for f not hyperimmune.*

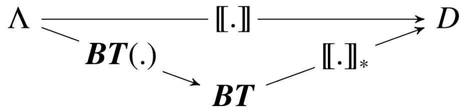
Indeed, we have seen in Example 2.1.0.4 that the first models are hyperimmune but not the second and we will see in Example 2.4.2.13 that each of them are sensible for $\Lambda_{\tau(D)}$.

Despite being based on equivalent conditions, the two theorems will be proven separately. We use two different proofs, as they give different perspectives of the same problem. The first proof is a semantical proof based on Böhm trees, it is the object of Section 2.2. The second one is a syntactical proof based on λ-calculi with tests; it is the object of Section 2.3. Section 2.4 gathers several bypassing properties and theorems regarding λ-calculi with tests, among them is the proof of equivalence between approximation property and sensibility for $\Lambda_{\tau(D)}$ (Th. 2.4.1.9).

The two proofs will split into three parts. Sections 2.2.1 and 2.3.1 will present needed definitions and theorems referring respectively to Böhm trees and λ-calculi with tests. Sections 2.2.2 and 2.3.2 prove the implication ((1)⇒(2)), showing that the (inequational) theory of D is $\mathcal{H}^*$ when assuming the hyperimmunity (Th. 2.3.2.4). Sections 2.2.3 and 2.3.3 prove the implication ((3)⇒(1)), exhibiting a counterexample to full abstraction when D is not hyperimmune (Th. 2.2.3.1). The implication ((2)⇒(3)) is trivial.

2.2. Semantical proof using Böhm trees

The main idea of this proof is not new, it consists in using Böhm trees to decompose the interpretation of the λ-calculus. In order to do so, we need to interpret them into our K-model D so that the following diagram commutes:



The approximation and quasi-approximation properties of Definitions 2.2.1.12 and 2.2.1.17 exactly state this decomposition for two specific choices of interpretation. Indeed, we will see in Definition 2.2.1.7 that there are many different possible interpretations of the Böhm trees, we will mainly focus on the inductive interpretation (Def. 2.2.1.10) and the quasi-finite interpretation (Def. 2.2.1.16).

The approximation and quasi-approximation properties will have different roles. The approximation property, *i.e.*, the decomposition via the inductive interpretation, mainly says that the interpretation of terms is approximable by finite Böhm trees. Approximation property is a hypothesis of Theorem 2.1.0.5 and it holds in all known candidates to full abstraction, *i.e.*, extensional and sensible models (Ex. 2.2.1.14). We even conjecture, in fact, that all K-models that are fully abstract for $\mathcal{H}^*$ respect the approximation property.

The quasi-approximation property is a fairly finer property[2] that is based on deep references to recursivity theory. The quasi-approximation property will be proven equivalent to both full abstraction for $\mathcal{H}^*$ and hyperimmunity in the presence of the approximation property.

Theorem 2.2.0.8 (Developed semantic theorem). *For any extensional K-model D sensible for $\Lambda_{\tau(D)}$ (Def. 2.3.1.19), the following are equivalent:*

1. *D is hyperimmune,*
2. *D respects the quasi-aproximation property,*
3. *D is inequationally fully abstract for* Λ*,*
4. *D is fully abstract for* Λ.

Proof.

- (1) $\Rightarrow$ (2): Theorem 2.2.2.8,
- (2) $\Rightarrow$ (3): inequational adequation is the object of Theorem 2.2.2.9 and inequational completeness the one of Theorem 2.2.2.10,
- (3) $\Rightarrow$ (4): trivial,

[2]Even if technically independent.

- (4) ⇒ (1): Theorem 2.2.3.1.

□

2.2.1. Böhm trees

Subclasses of Böhm trees

Before saying anything on interpretations of Böhm trees in a K-model, we need a few definitions over non conventional representations (def. A.2.1.9) of Böhm trees (def. 1.2.2.1). Indeed, it is convenient to define dense subclasses of Böhm trees (for the inclusion order) that will work as potential bases. Such bases can be used to interpret any Böhm tree in our models as the sup of the interpretations of its approximants.[3]

The only base that appears in the literature is the class $\boldsymbol{BT_f}$ of finite Böhm trees. However, we will oppose it the larger classes $\boldsymbol{BT_{\Omega f}}$ and $\boldsymbol{BT_{qf}}$ of Ω-finite and quasi-finite Böhm trees. Larger bases are generally less interesting, but here we will see that they enforce a notion of "stability" for recursive Böhm trees.

Concretely, Ω-finite Böhm trees basically constitute a base where approximants of an actual term (via its translation into a Böhm tree) are all recursive Böhm trees (Lemma. 2.2.1.3). The quasi-finite Böhm trees are then Ω-finite Böhm trees that are somehow stable by $\leq_{\eta\infty}$ and $\geq_{\eta\infty}$ (Lemma. 2.2.1.6).

Definition 2.2.1.1. *We define the following representations (Def. A.2.1.9) over Böhm trees:*

- *The set of* finite Böhm trees*, denoted $\boldsymbol{BT_f}$, is the set of Böhm trees inductively generated by their grammar (or equivalently Böhm trees of finite height). Given a term M, we denote $\boldsymbol{BT_f}(M)$ the set of finite Böhm trees U such that $U \subseteq \boldsymbol{BT}(M)$.*
- *The set of* Ω-finite Böhm trees*, denoted $\boldsymbol{BT_{\Omega f}}$, is the set of Böhm trees that contain a finite number of occurrences of* Ω.
- *The set of* quasi-finite Böhm tree*, denoted $\boldsymbol{BT_{qf}}$, is the set of those Ω-finite Böhm trees having their number of occurences of each (free and bounded) variables recursively bounded. Formally, there is a recursive function g such that variables abstracted at depth[4] n cannot occur at depth greater that $g(n)$.*

Capital final Latin letters X, Y, Z... will range over any of those classes of Böhm trees. We will use the notation $\subseteq_f$ (resp. $\subseteq_{\Omega f}$ and $\subseteq_{qf}$) for the inclusion restricted to $\boldsymbol{BT_f} \times \boldsymbol{BT}$ (resp. $\boldsymbol{BT_{\Omega f}} \times \boldsymbol{BT}$ and $\boldsymbol{BT_{qf}} \times \boldsymbol{BT}$).

In particular, to any finite Böhm tree U corresponds a term M obtained by replacing every symbol Ω by the diverging term $\boldsymbol{\Omega}$. By abuse of notation, we may use one instead of the other.

Example 2.2.1.2. *The identity $\boldsymbol{I}$ corresponds to a finite Böhm tree and thus is in all three classes. The term $\lambda z.\boldsymbol{\Theta}\ (\lambda ux.z\ u)$ has a Böhm tree that is Ω-finite but not quasi-finite. The term $\boldsymbol{\Theta}\ (\lambda ux.x\ u\ \Omega)$ has a*

[3] Recall that coinductive structures may have several interpretations into a given model (def. A.2.1.27).

[4] We consider that free variables are "abstracted" at depth 0.

Böhm tree that is neither of these classes.

$$\boldsymbol{BT}(\lambda z.\boldsymbol{\Theta}\ (\lambda ux.z\ u)) = \begin{array}{r}\lambda z x_1.z\ .\\ |\\ \lambda x_2.z\ .\\ |\\ \lambda x_3.z\ .\\ \vdots\end{array} \qquad \boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda ux.x\ u\ \Omega)) = \begin{array}{r}\lambda x_1.x_1\ .\ \Omega\\ |\\ \lambda x_2.x_2\ .\ \Omega\\ |\\ \lambda x_3.x_3\ .\ \Omega\\ \vdots\end{array}$$

Lemma 2.2.1.3. *For all terms M, if $X \in \boldsymbol{BT}_{\Omega f}$ and $X \subseteq \boldsymbol{BT}(M)$, then X is a recursive Böhm tree (def. A.2.1.9).*

Proof. First remark that only X has to be recursive, not the proof of $X \subseteq \boldsymbol{BT}(M)$. Moreover, we only have to show that there exists a recursive construction of X, we do not have to generate it constructively.

There is a finite number of Ω's in X whose positions $p \in P$ can be guessed beforehand by an oracle that is finite thus recursive. After that, it suffices to compute the Böhm tree of M except in these positions where we directly put an Ω. This way the program is always productive as any Ω of M (*i.e.*, any non terminating part of the process of computation of $\boldsymbol{BT}(M)$) will be shaded by a guessed Ω of X (potentially far above). □

Lemma 2.2.1.4. *Let $U, V \in \boldsymbol{BT}$. If $U \leq_{\eta\infty} V$ (def. 1.2.2.9), there is a bijection between the Ω's in U and those in V.*

Proof. Recall that $U \leq_{\eta\infty} V$ is the relation which proofs range over the coinductive sequents generated by

$$\frac{}{\Omega \geq_\eta \Omega}\ (\eta\infty\omega) \qquad \frac{\forall i \leq k,\ U_i \geq_{\eta\infty} V_i \qquad \forall i \leq m,\ U_{k+i} \geq_{\eta\infty} x_{n+i}}{\lambda x_1 ... x_{n+m}.y\ U_1 \cdots U_{k+m} \geq_{\eta\infty} \lambda x_1 ... x_n.y\ V_1 \cdots V_k}\ (\eta\infty @)$$

Remark that this system is deterministic so that a sequent $U \leq_{\eta\infty} V$ has at most one proof. In particular the occurrences of rule $(\eta\infty\omega)$ describe the pursued bijection. □

Lemma 2.2.1.5. *For all $U, V \in \boldsymbol{BT}$ such that $U \leq_{\eta\infty} V$, $U \in \boldsymbol{BT}_{qf}$ iff $V \in \boldsymbol{BT}_{qf}$.*

Proof. By lemma 2.2.1.4, we know that if $X \leq_{\eta\infty} V$ or $X \geq_{\eta\infty} V$ then $V \in \boldsymbol{BT}_{\Omega f}$.

It is easy to see that if variables occurrences are bounded by g in X they will be bounded by $(n \mapsto max(g(n), 1))$ in V. Indeed an $\eta\infty$-expansion/reduction will not change the depth of any variable, and will only delete/introduce abstraction whose variable will be used exactly once at depth 1. □

Lemma 2.2.1.6. *Both ordering $\leq_{\eta\infty}$ and $\geq_{\eta\infty}$ distribute over $\subseteq_{qf}$, and the ordering $\geq_{\eta\infty}$ distributes over $\subseteq_f$:*

- *For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT}_{qf}$ such that $X \subseteq_{qf} U \leq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT}_{qf}$ such that*

$$X \leq_{\eta\infty} Y \subseteq_{qf} V.$$

- *For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT_{qf}}$ such that $X \subseteq_{qf} U \geq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT_{qf}}$ such that*

$$X \geq_{\eta\infty} Y \subseteq_{qf} V.$$

- *For all $U, V \in \boldsymbol{BT}$ and $X \in \boldsymbol{BT_{f}}$ such that $X \subseteq_{f} U \geq_{\eta\infty} V$, there is $Y \in \boldsymbol{BT_{f}}$ such that*

$$X \geq_{\eta\infty} Y \subseteq_{f} V.$$

Proof.

- Distribution of $\leq_{\eta\infty}$ over $\subseteq_{qf}$:
  We create $Y \in \boldsymbol{BT}$ such that $X \leq_{\eta\infty} Y \subseteq V$ by co-induction (remark that, by lemma 2.2.1.5, we obtain $V \in \boldsymbol{BT_{qf}}$):
  - Either $X = \Omega$: then $Y = \Omega$.
  - Otherwise

    $$X = \lambda x_1...x_n.y\ X_1 \cdots X_m, \quad U = \lambda x_1...x_n.y\ U_1 \cdots U_m, \quad V = \lambda x_1...x_{n+k}.y\ V_1 \cdots V_{m+k},$$

    such that $X_i \subseteq_{qf} U_i \leq_{\eta\infty} V_i$ for $i \leq m$ and $x_{n+i} \leq_{\eta\infty} V_{m+i}$ (thus $V_{m+i} \in \boldsymbol{BT_{qf}}$) for $i \leq k$. By co-induction hypothesis we have $(Y_i)_{i\ln m}$ such that $X_i \leq_{\eta\infty} Y_i \subseteq V_i$ for $i \leq m$, we thus set

    $$Y = \lambda x_1...x_{n+k}.y\ Y_1 \cdots Y_m\ V_{m+1} \cdots V_{m+k}.$$

- Distribution of $\geq_{\eta\infty}$ over $\subseteq_{qf}$:
  We create $Y \in \boldsymbol{BT}$ such that $X \geq_{\eta\infty} Y \subseteq V$ by co-induction, by lemma 2.2.1.5 we then obtain that $V \in \boldsymbol{BT_{qf}}$:
  - Either $X = \Omega$: then $Y = \Omega$.
  - Otherwise

    $$X = \lambda x_1...x_{n+k}.y\ X_1 \cdots X_{m+k}, \quad U = \lambda x_1...x_{n+k}.y\ U_1 \cdots U_{m+k}, \quad V = \lambda x_1...x_n.y\ V_1 \cdots V_m,$$

    such that $X_i \subseteq_{qf} U_i \leq_{\eta\infty} V_i$ for $i \leq m$ and $X_{m+i} \subseteq_{qf} U_{m+i} \geq_{\eta\infty} x_{n+i}$ for $i \leq k$. By co-induction hypothesis we have $(V_i)_{i\leq m+k}$ such that $X_i \geq_{\eta\infty} Y_i \subseteq V_i$ for $i \leq m$, and $X_{m+i} \geq_{\eta\infty} Y_{m+i} \subseteq x_{n+i}$ for $i \leq k$; we thus set

    $$Y = \lambda x_1...x_{n+k}.y\ Y_1 \cdots Y_m.$$

- Distribution of $\geq_{\eta\infty}$ over $\subseteq_{f}$:
  We create $Y \in \boldsymbol{BT_f}$ similarly to the previous case except that we proceed by induction on X:
  - Either $X = \Omega$: then $Y = \Omega$.
  - Otherwise

    $$X = \lambda x_1...x_{n+k}.y\ X_1 \cdots X_{m+k}, \quad U = \lambda x_1...x_{n+k}.y\ U_1 \cdots U_{m+k}, \quad V = \lambda x_1...x_n.y\ V_1 \cdots V_m,$$

    such that $X_i \subseteq_{f} U_i \leq_{\eta\infty} V_i$ for $i \leq m$ and $X_{m+i} \subseteq_{f} U_{m+i} \geq_{\eta\infty} x_{n+i}$ for $i \leq k$. By co-induction hypothesis we have $(V_i)_{i\leq m+k}$ such that $X_i \geq_{\eta\infty} Y_i \subseteq_{f} V_i$ for $i \leq m$, and $X_{m+i} \geq_{\eta\infty} Y_{m+i} \subseteq_{f} x_{n+i}$ for $i \leq k$; we thus set

    $$Y = \lambda x_1...x_{n+k}.y\ Y_1 \cdots Y_m.$$

□

Interpretations of Böhm trees

Böhm trees can be seen as normal forms of infinite depth. As such, one can give them an interpretation extending the interpretation of the λ-calculus in our semantic through a fixpoint. However, there is no *a priori* reason to choose one specific fixpoint. We will formalize the notion of interpretations of Böhm trees in Definition 2.2.1.7, then, using their description as fixpoint, we will see in Property 2.2.1.9 that they form a complete lattice.

The minimal interpretation, called the inductive interpretation (Def. 2.2.1.10), is the canonical choice and has been used often in the literature to describe the approximation property (Def. 2.2.1.12). The approximation property informally states the coherence of the interpretation of terms and the inductive interpretation of Böhm trees.

But the complete lattice of interpretations is richer than the sole inductive one. Another canonical interpretation is the maximal one, called co-inductive interpretation (Def. 2.2.1.10). Unfortunately, no equivalent version of approximation property can be given for the co-inductive interpretation (more exactly, no K-model can verify it).

However, we can look for an interpretation that is both: as large as possible and with a useful notion of coherence with the λ-calculus. We found the quasi-finite interpretation (Def. 2.2.1.16) that is basically the minimal interpretation which restriction to quasi-finite Böhm trees corresponds to the co-inductive interpretation. The property that states the coherence of the interpretation is the quasi-approximation property (Def. 2.2.1.17). We will see later on that in the presence of the approximation property and the extensionality, the quasi-approximation property is equivalent to the hyperimmunity and the full abstraction for $\mathcal{H}^*$.

Definition 2.2.1.7. *We call* proto-interpretation of Böhm trees *any total function* $[\![-]\!]_*$ *that map elements* $U \in \boldsymbol{BT}$ *to initial segments of* $D^{\mathrm{FV}(U)} \Rightarrow D$.

An interpretation of Böhm trees *is a proto-interpretation* $[\![.]\!]_*$ *respecting the following:*

- *The interpretation of* Ω *is always empty:*

$$[\![\Omega]\!]_*^{\vec{x}} = \emptyset.$$

- *The interpretation of an abstraction* $\lambda y.U$ *satisfies:*

$$[\![\lambda y.U]\!]_*^{\vec{x}} = \{(\vec{a}, b{\to}\alpha) \mid (\vec{a}b, \alpha) \in [\![U]\!]_*^{\vec{x}y}\}.$$

- *The interpretation of a stack of applications* $x_i\ U_1 \cdots U_n$ *(for* $n \geq 0$*), has to respect:*

$$[\![x_i\ U_1 \cdots U_n]\!]_*^{\vec{x}} = \{(\vec{a}, \alpha) \mid \exists b_1{\to}\cdots{\to}b_n{\to}\alpha \leq \alpha' \in a_i, \forall i \leq n, \forall \beta \in b_i, (\vec{a}, \beta) \in [\![U_i]\!]_*^{\vec{x}}\}$$

Remark 2.2.1.8. *On finite Böhm trees, every interpretation collapses, thus we can denote* $[\![X]\!]^{\vec{x}}$ *for any* $X \in \boldsymbol{BT_f}$ *without ambiguity. Moreover, if the model is sensible,* $[\![X]\!]^{\vec{x}}$ *is the same as the interpretation of* X *considered as a* λ*-term (by replacing occurrences of* Ω *by the diverging term* $\boldsymbol{\Omega}$*).*

The interpretations, however, differ on their infinite Böhm trees. Fortunately, the set of interpretations forms a complete lattice.

$$\frac{\Gamma, x : a \vdash U : \alpha}{\Gamma \vdash \lambda x.U : a \to \alpha}\ (BT\text{-}\lambda)$$

$$\frac{b_1 \to \cdots \to b_n \to \beta \in a \qquad \alpha \leq \beta \qquad \forall i \leq n, \forall \gamma \in b_i,\ \Gamma, x : a \vdash U_i : \gamma}{\Gamma, x : a \vdash x\, U_1 \cdots U_n : \alpha}\ (BT\text{-}@)$$

Figure 2.1.: Intersection type system for Böhm trees. Notice that the intersection is hiddent in the membership condition in the first premise of $(BT\text{-}@)$.

Proposition 2.2.1.9. *The poset of interpretations (with pointwise inclusion) is a complete lattice.*

Proof. We are showing that interpretations are the fixpoints of a Scott-continuous (even linear) function ζ in the complete lattice of proto-interpretations (with pointwise order).

The function ζ , maps an interpretation $[\![.]\!]_*$ to the interpretation $[\![.]\!]_{\zeta(*)}$ defined by

- The interpretation of Ω is always empty:

$$[\![\Omega]\!]^{\vec{x}}_{\zeta(*)} = \emptyset.$$

- The interpretation of $\lambda y.U$ is the same as for λ-terms:

$$[\![\lambda y.U]\!]^{\vec{x}}_{\zeta(*)} = \{(\vec{a}, b \to \alpha) \mid (\vec{a}b, \alpha) \in [\![U]\!]^{\vec{x}y}_*\}.$$

- $x_i\ U_1 \cdots U_n$, have to respect:

$$[\![x_i\ U_1 \cdots U_n]\!]^{\vec{x}}_{\zeta(*)} = \{(\vec{a}, \alpha) \mid \exists b_1 \to \cdots \to b_n \to \alpha \leq \alpha' \in a_i, \forall i \leq n, \forall \beta \in b_i, (\vec{a}, \beta) \in [\![U_i]\!]^{\vec{x}}_*\},$$

This three equations trivially preserve the sups, so that ζ is continuous (even linear). It is well known that fixpoints of a Scott-continuous function form a complete lattice. ◻

Definition 2.2.1.10. *The minimal interpretation is the* inductive interpretation

$$[\![U]\!]^{\vec{x}}_{ind} = \bigcup_{\substack{X \subseteq U \\ X \in BT_f}} [\![X]\!]^{\vec{x}}.$$

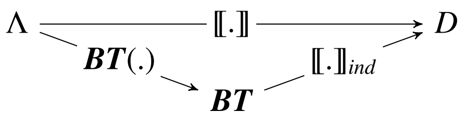
The maximal interpretation is called the co-inductive interpretation *and denoted* $[\![.]\!]^{\vec{x}}_{coind}$.

The idea of intersection types can be generalized to Böhm trees. We introduce in Figure 2.1 the corresponding intersection type system. There is no rule for Ω since it has an empty interpretation. Remark, moreover, that the rule $(BT\text{-}@)$ seems complicated, but is just the aggregation of rules $(I\text{-}id)$, $(I\text{-}w)$, $(I\text{-}\leq)$ and $(I\text{-}@)$ of Figure 1.3. The difference between the inductive and the co-inductive interpretations lies on the finiteness of the allowed derivations in this system.

Proposition 2.2.1.11. *Let U be a Böhm tree, then:*

- $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}}_{ind}$ *iff the type judgment* $\vec{x} : \vec{a} \vdash U : \alpha$ *has a finite derivation using the rules of Figure 2.1.*
- $(\vec{a}, \alpha) \in [\![U]\!]^{\vec{x}}_{coind}$ *iff the type judgment* $\vec{x} : \vec{a} \vdash U : \alpha$ *has a possibly infinite derivation using the rules of Figure 2.1.*

Definition 2.2.1.12. *We say that D respects the* approximation property*, or that D is* approximable*, if the interpretation of any term corresponds to the inductive interpretation of its Böhm tree,* i.e. *if the following diagram commutes:*



Lemma 2.2.1.13. *If D is extensional and approximable, and if M and N are two terms such that* $M \geq_{\eta\infty} N$ *(def. 1.2.2.9), then* $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$.

Proof. Let $(a, \alpha) \in [\![M]\!]^{\vec{x}}$, by the approximation property there is a finite $U \subseteq_f \boldsymbol{BT}(M)$ such that $(a, \alpha) \in [\![U]\!]^{\vec{x}}$. Since $U \subseteq_f \boldsymbol{BT}(M) \geq_{\eta\infty} \boldsymbol{BT}(N)$, we can apply Lemma 2.2.1.6 to find $V \in \boldsymbol{BT_f}$ such that $U \geq_{\eta\infty} V \subseteq_f \boldsymbol{BT}(N)$. However, between finite Böhm trees, an $\infty\eta$-expansion is a usual η-expansion, so that $U \geq_\eta V \subseteq_f \boldsymbol{BT}(N)$. We thus have (using the extensionality), $(a, \alpha) \in [\![U]\!]^{\vec{x}} \subseteq [\![V]\!]^{\vec{x}} \subseteq [\![M]\!]^{\vec{x}}$. □

The approximation property is a common condition enjoyed by all known K-models.[5]

Example 2.2.1.14. *All the K-models of Example 1.2.4.9 except* P_∞ *(that is not even sensible) are approximable, regardless of them being fully abstract or not.*

Proof. By Example 2.4.2.4 and Corollary 2.4.2.14. □

Our goal is to modify the approximation property so that we could characterize the full abstraction.

Remark 2.2.1.15. *A vain attempt would consist on replacing the inductive interpretation (in the definition of the approximation property) by the co-inductive one. In fact, the resulting property would never hold:*

For any sensible K-model and any $\alpha \in D$*, if* $M = \boldsymbol{\Theta}\ (\lambda u.z\ u)$*, then*

$$(\{\{\alpha\}{\to}\alpha\}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^{z}_{coind} \qquad\qquad (\{\{\alpha\}{\to}\alpha\}, \alpha) \notin [\![M]\!]^{z}.$$

Indeed, If we had $(\{\{\alpha\}{\to}\alpha\}, \alpha) \in [\![M]\!]^{z}$ *it would give* $\alpha \in [\![M[\boldsymbol{I}/z]]\!] = [\![\boldsymbol{\Theta}\ \boldsymbol{I}]\!] = \emptyset$*. Moreover, since* $\boldsymbol{BT}(M) = z\ \boldsymbol{BT}(M)$*, we get to co-inductively show that* $(\{\{\alpha\}{\to}\alpha\}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^{z}_{coind}$.

[5]Provided that they equalize terms with the same Böhm trees (which is a necessary condition for full abstraction).

In this example, the co-inductive interpretation of $\boldsymbol{BT}(\boldsymbol{\Theta}\ (\lambda ux.z\ u))$ is incoherent with the term interpretation because it uses the z infinitely often. In the category Rel, for example, this would not hold (even if other problems would come later). In order to get rid of these incoherence we can use a guarded fixpoint.

In order to recover a meaningful property, we will use the *quasi-finite interpretation*. This is the least interpretation which restriction to quasi-finite Böhm trees is the co-inductive interpretation.

Definition 2.2.1.16. *The* quasi-finite interpretation of Böhm trees *is defined by*

$$[\![U]\!]_{qf}^{\vec{x}} = \bigcup_{\substack{X \subseteq U \\ X \in \boldsymbol{BT}_{qf}}} [\![X]\!]_{coind}^{\vec{x}}.$$

Definition 2.2.1.17. *We say that D respects the* quasi-approximation property, *or is* quasi-approximable, *if the interpretation of any term corresponds to the quasi-finite interpretation of its Böhm tree,* i.e. *if the following diagram commutes:*

$$\begin{array}{ccccc} \Lambda & & \xrightarrow{\quad [\![.]\!] \quad} & & D \\ & \searrow^{\boldsymbol{BT}(.)} & & \nearrow_{[\![.]\!]_{qf}} & \\ & & \boldsymbol{BT} & & \end{array}$$

Example 2.2.1.18. *We will prove that the quasi-approximation property is equivalent to hyperimmunity and full abstraction for $\mathcal{H}^*$ (in presence of approximation property and extensionality). So models that are hyperimmune, like D_∞, respect it and those that are not, like D_∞^*, does not. In the case of D_∞^*, for example, the quasi-approximation property is refuted by $\boldsymbol{J}$, indeed $p \in [\![\boldsymbol{BT}(\boldsymbol{J})]\!]_{qf} - [\![\boldsymbol{J}]\!]$.*

Technical lemmas

This section shows that the relations $\subseteq$ and $\leq_{\eta\infty}$ in $\boldsymbol{BT}$ are pushed along the co-inductive interpretation into the inclusion (Lemma. 2.2.1.19) and the equality (Lemma. 2.2.1.20) at the level of the model. These properties will be useful as they generalize easily to the quasi-finite interpretation.

Lemma 2.2.1.19. *Let D be an extensional K-model.*
Let U, V be two Böhm trees such that $U \subseteq V$.
Then $[\![U]\!]_{coind}^{\vec{x}} \subseteq [\![V]\!]_{coind}^{\vec{x}}$.

Proof. We will show that the proto-interpretation $[\![V]\!]_* = \bigcup_{U \subseteq V} [\![U]\!]_{coind}$ over Böhm trees is an interpretation. This is sufficient since, $[\![\_]\!]_{coind}$ being the greatest interpretation, we will have

$$[\![V]\!]_{coind} \subseteq \bigcup_{U \subseteq V} [\![U]\!]_{coind} = [\![V]\!]_* \subseteq [\![V]\!]_{coind}$$

- Interpretation over Ω:

$$\begin{aligned}
[\![\Omega]\!]_*^{\vec{x}} &= \bigcup_{U \subseteq \Omega} [\![U]\!]_{coind}^{\vec{x}} \\
&= [\![\Omega]\!]_{coind}^{\vec{x}} \\
&= \emptyset.
\end{aligned}$$

- Interpretation over abstractions:

$$\begin{aligned}
[\![\lambda y.V]\!]_*^{\vec{x}} &= \bigcup_{U \subseteq \lambda y.V} [\![U]\!]_{coind}^{\vec{x}} \\
&= [\![\Omega]\!]_{coind} \cup \bigcup_{U' \subseteq V} [\![\lambda y.U']\!]_{coind}^{\vec{x}} \\
&= \bigcup_{U' \subseteq V} \{(\vec{a}, b \to \alpha) \mid (\vec{a}b, \alpha) \in [\![U']\!]_{coind}^{\vec{x}y}\} \\
&= \{(\vec{a}, b \to \alpha) \mid (\vec{a}b, \alpha) \in \bigcup_{U' \subseteq V} [\![U']\!]_{coind}^{\vec{x}y}\} \\
&= \{(\vec{a}, b \to \alpha) \mid (\vec{a}b, \alpha) \in [\![V]\!]_*^{\vec{x}y}\}.
\end{aligned}$$

- Interpretation over applications:

$$\begin{aligned}
[\![x_m\ V_1 \cdots V_k]\!]_* &= \bigcup_{U \subseteq x_m\ V_1 \cdots V_k} [\![U]\!]_{coind}^{\vec{x}} \\
&= [\![\Omega]\!]_{coind} \cup \bigcup_{U_j \subseteq V_j} [\![x_m\ U_1 \cdots U_k]\!]_{coind}^{\vec{x}} \\
&= \bigcup_{U_j \subseteq V_j} \{(\vec{a}, \alpha) \mid \exists b_1 \to \cdots \to b_k \to \alpha \leq \alpha' \in a, \forall j \leq k, \forall \beta \in b_j,\ (\vec{a}, \beta) \in [\![U_j]\!]_{coind}^{\vec{x}}\} \\
&= \{(\vec{a}, \alpha) \mid \exists b_1 \to \cdots \to b_k \to \alpha \leq \alpha' \in a, \forall j \leq k, \forall \beta \in b_j,\ (\vec{a}, \beta) \in \bigcup_{U_j \subseteq V_j} [\![U_j]\!]_{coind}^{\vec{x}}\} \\
&= \{(\vec{a}, \alpha) \mid \exists b_1 \to \cdots \to b_k \to \alpha \leq \alpha' \in a, \forall j \leq k, \forall \beta \in b_j,\ (\vec{a}, \beta) \in [\![V_j]\!]_*^{\vec{x}}\}
\end{aligned}$$

□

Lemma 2.2.1.20. *Let D be an extensional K-model.*
Let U, V be two Böhm trees such that $U \leq_{\eta\infty} V$.
Then $[\![U]\!]_{coind}^{\vec{x}} = [\![V]\!]_{coind}^{\vec{x}}$.

Proof. We will prove separately the two inclusions.

- We will show that the proto-interpretation $[\![V]\!]_* = \bigcup_{U \leq_{\eta\infty} V} [\![V]\!]_{coind}$ over Böhm trees is an interpretation. This is sufficient since, $[\![\_]\!]_{coind}$ being the greatest interpretation, we will have

$$[\![V]\!]_{coind} \subseteq \bigcup_{U \leq_{\eta\infty} V} [\![V]\!]_{coind}^{\vec{x}} = [\![V]\!]_* \subseteq [\![V]\!]_{coind}.$$

– Interpretation over Ω:

$$\begin{aligned}[\![\Omega]\!]^{\vec{x}}_* &= \bigcup_{U \preceq_{\eta\infty} \Omega} [\![U]\!]^{\vec{x}}_{coind} \\ &= [\![\Omega]\!]^{\vec{x}}_{coind} \\ &= \emptyset.\end{aligned}$$

– If $V_s \not\succeq_{\eta\infty} x_k$ and $V_{s+i} \succeq_{\eta\infty} x_{k+i}$ (for $1 \le i \le m$) and $j \le k$:

$$\begin{aligned}
&[\![\lambda x_{n+1}...x_{k+m}.x_j\ V_1 \cdots V_{s+m}]\!]^{\vec{x}^n}_* \\
&= \bigcup_{U \preceq_{\eta\infty} \lambda x_{n+1}...x_{k+m}.x_j\ V_1 \cdots V_{s+m}} [\![U]\!]^{\vec{x}^n}_{coind} \\
&= \bigcup_{m' \le m} \bigcup_{U_t \preceq_{\eta\infty} V_t} [\![\lambda x_{n+1}...x_{k+m'}.x_j\ U_1 \cdots U_{s+m'}, (u_i)_{i>n}]\!]^{\vec{x}^n}_{coind} \\
&= \bigcup_{m' \le m} \bigcup_{U_t \preceq_{\eta\infty} V_t} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m'} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![U_t]\!]^{\vec{x}^{k+m'}}_{coind}\} \\
&= \bigcup_{m' \le m} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m'} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in \bigcup_{U_t \preceq_{\eta\infty} V_t} [\![U_t]\!]^{\vec{x}^{k+m'}}_{coind}\} \\
&= \bigcup_{m' \le m} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m'} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]^{\vec{x}^{k+m'}}_*\} \\
&= \bigcup_{m' \le m} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to a_{k+m'+1} \to \cdots a_{n+m} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]^{\vec{x}^{k+m'}}_*\} \\
&= \bigcup_{m' \le m} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to a_{k+m'+1} \to \cdots a_{k+m} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]^{\vec{x}^{k+m'}}_* \\
&\qquad \forall m' \le t \le m, \forall \beta \in a_{k+t}, (\vec{a}, \beta) \in [\![x_{k+t}]\!]^{\vec{x}^{k+m'}}_*\} \\
&= \bigcup_{m' \le m} \{(a_i)_{i \le n}, a_{n+1} \to \cdots a_{k+m} \to \alpha) \mid \exists c_1 \to \cdots \to c_{s+m'} \to a_{k+m'+1} \to \cdots a_{k+m} \to \alpha \le \alpha' \in a_j, \\
&\qquad \forall t \le s+m', \forall \beta \in c_t,\ (\vec{a}, \beta) \in [\![V_t]\!]^{\vec{x}^{k+m'}}_* \\
&\qquad \forall m' \le t \le m, \forall \beta \in a_{k+t}, (\vec{a}, \beta) \in [\![V_{s+t}]\!]^{\vec{x}^{k+m'}}_*\}
\end{aligned}$$

This proves that if $U \preceq_{\eta\infty} V$ then $[\![U]\!]^{\vec{x}}_{coind} \subseteq [\![V]\!]^{\vec{x}}_* \subseteq [\![V]\!]^{\vec{x}}_{coind}$.

- To prove the converse, it is sufficient to show that the proto-interpretation $[\![V]\!]^{\vec{x}}_* = \bigcup_{U \succeq_{\eta\infty} V} [\![U]\!]^{\vec{x}}_{coind}$ is an interpretation:

–

$$
\begin{aligned}
[\![\Omega]\!]^{\vec{x}}_{*} &= \bigcup_{U\geq_{\eta\infty}\Omega} [\![U]\!]^{\vec{x}}_{coind}\\
&= [\![\Omega]\!]^{\vec{x}}_{coind}\\
&= \emptyset.
\end{aligned}
$$

–

$$
\begin{aligned}
&[\![\lambda x_{n+1}...x_s.x_j\ V_1\cdots V_k]\!]^{\vec{x}}_{*}\\
&= \bigcup_{U\geq_{\eta\infty}\lambda x_{n+1}...x_s.x_j\ V_1\cdots V_k} [\![U]\!]^{\vec{x}}_{coind}\\
&= \bigcup_{m}\bigcup_{U_i\geq_{\eta\infty}V_i}\bigcup_{U_{k+i}\geq_{\eta\infty}x_{s+i}} [\![\lambda x_{n+1}...x_{s+m}.x_j\ U_1\cdots U_{k+m}]\!]^{\vec{x}}_{coind}\\
&= \bigcup_{m}\bigcup_{U_i\geq_{\eta\infty}V_i}\bigcup_{U_{k+i}\geq_{\eta\infty}x_{s+i}} \{((a_i)_{i\leq n}, a_{n+1}\rightarrow\cdots a_{s+m}\rightarrow\alpha) \mid \exists c_1\rightarrow\cdots\rightarrow c_{k+m}\rightarrow\alpha\leq\alpha'\in a_j,\\
&\qquad\qquad \forall t\leq k{+}m, \forall\beta\in c_t,\ (\vec{a},\beta)\in[\![U_t]\!]^{\vec{x}^{s+m}}_{coind}\}\\
&= \bigcup_{m}\{((a_i)_{i\leq n}, a_{n+1}\rightarrow\cdots a_{s+m}\rightarrow\alpha) \mid \exists c_1\rightarrow\cdots\rightarrow c_{k+m}\rightarrow\alpha\leq\alpha'\in a_j,\\
&\qquad\qquad \forall t\leq k, \forall\beta\in c_t,\ (\vec{a},\beta)\in \bigcup_{U_t\geq_{\eta\infty}V_t}[\![U_t]\!]^{\vec{x}^{s+m}}_{coind}\\
&\qquad\qquad \forall t\leq m, \forall\beta\in c_{k+t},\ (\vec{a},\beta)\in \bigcup_{U_{k+t}\geq_{\eta\infty}x_{n+t}}[\![U_t]\!]^{\vec{x}^{s+m}}_{coind}\}\\
&= \bigcup_{m}\{((a_i)_{i\leq n}, a_{n+1}\rightarrow\cdots a_{s+m}\rightarrow\alpha) \mid \exists c_1\rightarrow\cdots\rightarrow c_{k+m}\rightarrow\alpha\leq\alpha'\in a_j,\\
&\qquad\qquad \forall t\leq k, \forall\beta\in c_t,\ (\vec{a},\beta)\in[\![V_t]\!]^{\vec{x}^{s+m}}_{*}\\
&\qquad\qquad \forall t\leq m, \forall\beta\in c_{k+t},\ (\vec{a},\beta)\in[\![x_{n+t}]\!]^{\vec{x}^{s+m}}_{*}\}\\
&= \bigcup_{m}\{((a_i)_{i\leq n}, a_{n+1}\rightarrow\cdots a_{s+m}\rightarrow\alpha) \mid \exists c_1\rightarrow\cdots\rightarrow c_{k+m}\rightarrow\alpha\leq\alpha'\in a_j,\\
&\qquad\qquad \forall t\leq k, \forall\beta\in c_t,\ (\vec{a},\beta)\in[\![V_t]\!]^{\vec{x}^{s+m}}_{*}\}
\end{aligned}
$$

This proves that if $U \geq_{\eta\infty} V$ then $[\![U]\!]^{\vec{x}}_{coind} \subseteq [\![V]\!]^{\vec{x}}_{*} \subseteq [\![V]\!]^{\vec{x}}_{coind}$.

□

2.2.2. Hyperimmunity implies full abstraction

In this section we will prove the step (1) ⇒ (2) of the main theorem (Th. 2.1.0.5). This will be done using the quasi-approximation property to decompose the proof in two steps. Indeed, we will see that in the presence of the approximation property, hyperimmunity implies the quasi-approximation property that itself implies the full abstraction for $\mathcal{H}^*$. Those two implications will be proven separately in Theorems 2.2.2.8 and 2.2.2.10.

Hyperimmunity and approximation imply quasi-approximation

Firstly, we are introducing the tree-hyperimmunity that is equivalent to hyperimmunity (Lemma 2.2.2.2).

The reason to introduce this new formalism is quite simple. For the proof of Theorem 2.2.2.8, we will have to contradict hyperimmunity starting from a term M that contradicts the quasi-approximation.

Recall that refuting the hyperimmunity amounts to exhibiting an non-hyperimmune function (*i.e.*, bounded by a recursive function g) and a sequence $(\alpha_i)_i \in D^{\mathbb{N}}$ with a non well founded chain bounded by g (see Definition 2.1.0.1).

The refutation of the quasi-approximation by M gives a recursive procedure that bounds the non-hyperimmune function g. However, the procedure does generally not directly construct the values of this function, but also performs a lot of useless computation; this is due to the refuting term M not being optimal. Thus, we will simply construct an infinite tree and use König lemma[6] to find an infinite branch that contradicts the hyperimmunity.

Generalizing hyperimmunity from sequences to trees allows us to apply a well known theorem of recursivity theory. This theorem states the equivalence between hyperimmune functions and infinite paths in recursive $\mathbb{N}$-labeled trees.[7] That is why the hyperimmune function becomes an infinite recursive $\mathbb{N}$-labeled tree. The sequence $(\alpha_i)_i \in D^{\mathbb{N}}$, similarly, becomes a partial (but infinite) labeling of the recursive tree. The sequence has to be partial in order to select a specific hyperimmune path.

Definition 2.2.2.1. *Let D be a K-model.*
A $\mathbb{N}$-labeled *tree T is a finitely branching tree where nodes are labeled by $\mathbb{N}$, we denote by $T(\mu)$ the $\mathbb{N}$-label of μ.*
*A D-*decoration *of a $\mathbb{N}$-labeled T is a partial function of infinite domain $\ell_D : T \to D$ such that for every couple of nodes ν and μ that are father and son in T, if $\mu \in dom(\ell_D)$, then $\nu \in dom(\ell_D)$ and:*

$$\ell_D(\nu) = a_1 \to \cdots \to a_{T(\mu)} \to \alpha \qquad \Rightarrow \qquad \ell_D(\mu) \in a_{T(\mu)}.$$

D is tree-hyperimmune *if any D-decoration of any T is non-recursive.*

Lemma 2.2.2.2. *A K-model D is tree-hyperimmune iff it is hyperimmune.*

Proof.

- We assume that there is a recursive g and a sequence $(\alpha_n)_n$ refuting the hyperimmunity. We define the tree T given by the set of nodes $\{\omega \in \mathbb{N}^* \mid \forall n \leq |\omega|, \omega_n \leq g(n)\}$ of finite sequences bounded by g and ordered by prefix; the $\mathbb{N}$-labeling is given by $T(\epsilon) = 0$ and $T(\omega.n) = n$. Then T is recursive and we have ℓ_D partially defined by induction:
  - $\ell_D(\epsilon) = \alpha_0$ is always defined,
  - $\ell_D(\omega.n) = \alpha_{|\omega.n|}$ is defined if $\ell_D(\omega) = \alpha_{|\omega|} = a_1 \to \cdots \to a_n \to \alpha$ and $\alpha_{|\omega.n|} = \alpha_{|\omega|+1} \in a_n$.

[6] König lemma states that any infinite tree that is finitely branching accepts an infinite branch/path.
[7] Trees with nodes indexed by natural numbers

The decoration is infinite since, for all depth d, $\alpha_{d+1} \in \bigcup_{n\leq g(d)} a_n$ for $\alpha_d = a_1 \to \cdots a_{g(d)} \to \alpha'_d$. This contradicts the tree-hyperimmunity.

- If D is not tree-hyperimmune, then there is a finitely branching, $\mathbb{N}$-labeled, and recursive tree T and an infinite decoration ℓ_D. By König lemma, the sub-tree that constitute the domain of ℓ_D (that is infinite and finitely branching) accepts an infinite branch $(\mu_n)_n$. If we denotes $\alpha_n = \ell_D(\mu_n)$, we have $\alpha_{n+1} \in a_{T(\mu_{n+1})}$ for $\alpha_n = a_1 \to \cdots \to a_{T(\mu_{n+1})} \to \alpha'$. Since the sequence $(T(\mu_{n+1}))_n$ is majored by the maximal $\mathbb{N}$-label on depth $n+1$ in T, that is recursive, we are contradicting the hyperimmunity.

□

Remark 2.2.2.3. *In the following, internal nodes of a quasi-finite Böhm tree are denoted by X, Y... as they are assimilated with the quasi-finite Böhm tree which root is the node at issue.*

Definition 2.2.2.4. *Let X be a quasi-finite Böhm tree that is recursive and close.*
The play over *X is a (non-necessarily finitely branching) tree T which nodes, denoted $P(Y)$ (or $O(Y)$) are uniquely determined by a letter P or O and a node Y over X.*

- *The nodes at even depth are called* player nodes. *They are denoted $P(Y)$.*
- *The nodes at odd depth are called* opponent nodes. *They are denoted $O(Y)$.*

The tree is given by:

- *the root is $P(X)$,*
- *the opponent node $O(\lambda x_1...x_m.z\ Y_1 \cdots Y_k)$ has k sons which are the $P(Y_i)$ for $i \leq k$,*
- *the player node $P(\lambda x_1...x_m.z\ Y_1 \cdots Y_k)$ has for sons every $O(Z)$ for Z a node over Y_1, ..., or Y_n which head variable is one of the $x_1, \ldots, x_m$.*

Proposition 2.2.2.5. *Let X be a quasi-finite Böhm tree that is recursive and close and T the play over X.*
For every node Y of X, $P(Y)$ is a node of T. For every node Y of X that is not an Ω, $O(Y)$ is a node of T.

Proof. By structural induction over the nodes Y of X:

- If Y is a node of X, then either $Y = X$ and $P(X)$ is the root of T, or Y has a father Y' in X. In the last case, $O(Y')$ is a node of T by IH and $P(Y)$ is a son of $O(Y')$.
- If $Y' = \lambda x_1...x_m.z\ Y_1 \cdots Y_k$ is a node of X, then by closeness of X, there is a forebear forbear of Y in X where z is abstracted (potentially $Y = Y'$), *i.e* $Y = \lambda y_1...y_{m'}.z'\ Y'_1 \cdots Y'_k$ with $z = y_i$. By IH, $P(Y)$ is a node of T and $O(Y)$ is its son.

□

Definition 2.2.2.6. *Let X be a quasi-finite Böhm tree that is recursive and close. The* labeled play over X *is the play over X together with the $\mathbb{N}$-labeling ℓ defined as follow:*

- *the labeling of the root is $\ell(P(X)) = 0$,*
- *for any Y, $P(Y)$ has for father $O(\lambda x_1...x_m.z\ Y_1 \cdots Y_k)$ with Y one of the Y_i, the $\mathbb{N}$-label of $P(Y)$ is the corresponding index of application i,*
- *for any $Y = \lambda x_1...x_m.z\ Y_1 \cdots Y_k$, the father of $O(Y)$ is $P(Y')$ for Y' that is the forbear of Y in X where z is abstracted (potentially $Y' = Y$),* i.e $Y' = \lambda y_1...y_{m'}.z'\ Y'_1 \cdots Y'_k$ *with $z = y_i$. The $\mathbb{N}$-label of $O(Y)$ is the corresponding index of abstraction i.*

Proposition 2.2.2.7. *For any quasi-finite $X \in \boldsymbol{BT_{qf}}$, the labeled play T over X is recursive finitely branching and $\mathbb{N}$-labeled.*

Proof.

- The tree T is finitely branching: An opponent node $P(\lambda x_1 \dots x_n.z\ Y_n \cdots Y_k)$ has exactly k sons which are the $O(Y_i)$ for $i \leq k$. A player node $O(\lambda x_1 \dots x_n.z\ Y_n \cdots Y_k)$ has one son for each occurrence of its abstracted variables, which result in a finite number by quasi-finiteness of X.
- The tree T is recursive: by recursivity and quasi-finiteness of X.

□

Theorem 2.2.2.8 (Hyperimmunity and approximation imply quasi-approximation). *Any hyperimmune approximable K-model D is also quasi-approximable.*

Proof. We will prove the contrapositive: We assume that D is approximable but not quasi-approximable, then we show that D is not tree-hyperimmune (and thus not hyperimmune by Lemma 2.2.2.2).
Since D is not quasi-approximable, there is a λ-term $M \in \Lambda$ such that $[\![M]\!]^{\vec{x}} \neq [\![\boldsymbol{BT}(M)]\!]^{\vec{x}}_{qf}$. We can assume that M is closed (otherwise we could have taken $\lambda x_1...x_m.M$).
But the approximation property gives that $[\![M]\!]^{\vec{x}} = [\![\boldsymbol{BT}(M)]\!]_{ind} \subset [\![\boldsymbol{BT}(M)]\!]_{qf}$. Thus there is α such that $\alpha \in [\![\boldsymbol{BT}(M)]\!]_{qf}$ but $\alpha \notin [\![\boldsymbol{BT}(M)]\!]_{ind}$.
By Definition 2.2.1.16 of $[\![.]\!]_{qf}$, there is a Böhm tree $X \subseteq \boldsymbol{BT}(M)$ quasi-finite such that $\alpha \in [\![X]\!]_{coind}$.
Since X is quasi-finite (and in particular Ω-finite), its tree is recursive by Lemma 2.2.1.3. And since D is approximable, for all finite Böhm tree $Z \subseteq X$, $\alpha \notin [\![Z]\!] = [\![Z]\!]_{coind}$; in particular X is infinite.
Let T be the labeled play over X.
It remains to partially (but infinitely) label the tree T with elements of D breaking the conditions of tree-hyperimmunity.
We will give, inductively, two infinite sequences $(Y_n)_n$ and $(Z_n)_n$ of nodes of X, two infinite sequences $(\alpha^n)_n$ and $(\beta^n)_n$ of elements of D, and for all $n \in \mathbb{N}$:

$$a^n_1, ..., a^n_{\#FV(Y_n)} \in \mathcal{A}_f(D)^{\#FV(Y_n)} \qquad l^n_1, ..., l^n_{\#FV(Y_n)} \in \mathbb{N}^{\#FV(Y_n)} \qquad k^n_1, ..., k^n_{\#FV(Y_n)} \in \mathbb{N}^{\#FV(Y_n)}$$

such that there is $(a_1^n...a_m^n, \alpha^n) \in [\![Y_n]\!]_{qf}^{\vec{x}} - [\![Y_n]\!]_{inf}^{\vec{x}}$ and so that x_i is the $(l_1^n)^{th}$ variable of Y_{k^n},

- $Y_0 = X$ and $\alpha^0 = \alpha$.
- $Y_n \neq \Omega$ by non emptiness of $[\![Y_n]\!]_{qf}^{\vec{x}}$.
- If $Y^n = \lambda x_{m+1}...x_{m'}.x_i\, X_1 \cdots X_k$ has $x_1...x_m$ as free variables:
  If we unfold $\alpha^n = a_{m+1}^n \to \cdots a_{m'}^n \to \alpha'$ then there exist $\beta = b_1 \to \cdots b_k \to \alpha'_0 \in a_i^n$ such that for all j and all $\gamma \in b_j$, we have $(a_1^n...a_{m'}^n, \gamma) \in [\![X_j]\!]_{qf}^{\vec{x}}$. In particular there is $j \leq k$ and $\gamma \in b_j$ such that $(a_1^n...a_{m'}^n, \gamma) \in [\![X_j]\!]_{qf}^{\vec{x}} - [\![X_j]\!]_{ind}^{\vec{x}}$. We have then

$$\begin{array}{lll} Z_n = Y_n & & \beta^n = \beta \\ Y_{n+1} = X_j & \forall i,\ a_i^{n+1} = a_i^n & \alpha^{n+1} = \gamma \end{array}$$

  Remark that we have $P(Y_{k^n})$ that is the father of $O(Z_n) = (Z_n, l_i^n)$ that, itself, is the father of $P(Y_{n+1}) = (Y_{n+1}, j)$ and that $\beta \in a_i^n$ and $\alpha^{n+1} = \gamma \in b_j$.

As a result, we can D-label $\ell_D(P(Y_n)) = \alpha^n$ and $\ell_D(O(Z_n)) = \beta^n$. □

Quasi-approximation and extensionality imply full abstraction

Theorem 2.2.2.9 (Adequacy). *Let D be a K-model respecting the quasi-approximation property. Then it is inequationally adequate,* i.e. *for all M and N such that $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$, $M \sqsubseteq_{\mathcal{H}^*} N$ (*c.f. *Definition 1.2.1.4).*

Proof. D is sensible (diverging terms have empty interpretations). Indeed, for any head-diverging term M, $\boldsymbol{BT}(M) = \Omega$ and thus

$$[\![M]\!]^{\vec{x}} = [\![\boldsymbol{BT}(M)]\!]_{qf}^{\vec{x}} = [\![\Omega]\!]_{qf}^{\vec{x}} = \emptyset.$$

We conclude by Lemma 1.2.1.10. □

Theorem 2.2.2.10 (Completion). *Let D be a quasi-approximable extensional K-model. D is inequationally complete,* i.e. *for all M and N such that $M \sqsubseteq_{\mathcal{H}^*} N$, there is $[\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}$ (*c.f. *Definition 1.2.1.5).*

Proof. Let $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$.
By the quasi-approximation property, there is $W \subseteq \boldsymbol{BT_{qf}}(M)$ such that $(\vec{a}, \alpha) \in [\![W]\!]_{coind}^{\vec{x}}$.
Using Proposition 1.2.2.12, there is U and V such that $\boldsymbol{BT}(M) \leq_{\eta\infty} U \subseteq V \geq_{\eta\infty} \boldsymbol{BT}(N)$. By Lemma 2.2.1.6, there is $X, Y, Z \in \boldsymbol{BT_{qf}}$ such that:

$$\begin{array}{ccccccc} \boldsymbol{BT}(M) & \leq_{\eta\infty} U & \subseteq & V & \geq_{\eta\infty} & \boldsymbol{BT}(N) \\ \Cup_{qf} & \Cup_{qf} & & \Cup_{qf} & & \Cup_{qf} \\ W & \leq_{\eta\infty} X & \subseteq & Y & \geq_{\eta\infty} & Z \end{array}$$

Thus:

$$\begin{array}{rcll} (\vec{a}, \alpha) \in [\![W]\!]_{coind}^{\vec{x}} & = & [\![X]\!]_{coind}^{\vec{x}} & \text{Lemma 2.2.1.19} \\ & \subseteq & [\![Y]\!]_{coind}^{\vec{x}} & \text{Lemma 2.2.1.20} \\ & = & [\![Z]\!]_{coind}^{\vec{x}} & \text{Lemma 2.2.1.19} \\ & \subseteq & [\![N]\!]^{\vec{x}} & \text{quasi-approximation} \end{array}$$

□

2.2.3. Full abstraction implies hyperimmunity

The counterexample

Suppose that D is approximable but not hyperimmune. By Definition 2.1.0.1 of hyperimmunity, there exists a recursive $g : (\mathbb{N} \to \mathbb{N})$ and a sequence $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$ such that

$$\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n \qquad \text{with} \qquad \alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}.$$

We will use the function g to define a term $\boldsymbol{J}_g$ (Eq. 2.4) such that $(\boldsymbol{J}_g\ \underline{0})$ is observationaly equal to the identity in Λ (Lemma 2.2.3.3) but can be denotationally distinguished in D (Lemma 2.2.3.6). This allows to refute the full abstraction:

Theorem 2.2.3.1 (Full abstraction imply Hyperimmunity). *If D is approximable but not hyperimmune, then it is not fully abstract for the λ-calculus.*

Basically, $(\boldsymbol{J}_g\ \underline{0})$ is a generalization of the term $\boldsymbol{J}$ used in [CDCZ87] to prove that the model D^*_∞ (Ex. 1.2.4.9) is not fully abstract. The idea is that $\boldsymbol{J}$ is the infinite η-expansion of the identity $\boldsymbol{I}$ where each level of the Böhm tree is η-expanded by one variable. Our term $(\boldsymbol{J}_g\ \underline{0})$ is also an infinite η-expansion of $\boldsymbol{I}$, but now, each level of the Böhm tree is η-expanded by $g(n)$ variables.[8]

Let $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$ be the sequence of closed λ-terms defined by:

$$\boldsymbol{G}_n := \lambda u e x_1 ... x_{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}) \tag{2.2}$$

The recursivity of g implies the one of the sequence $\boldsymbol{G}_n$. We can thus use Proposition 1.1.0.5: there exists a λ-term $\boldsymbol{G}$ such that:

$$\boldsymbol{G}\ \underline{n} \to^* \boldsymbol{G}_n. \tag{2.3}$$

Recall that $\boldsymbol{S}$ denotes the Church successor function and $\boldsymbol{\Theta}$ the Turing fixedpoint combinator. We define:

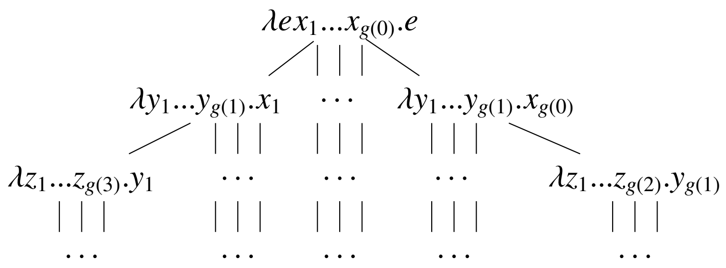
$$\boldsymbol{J}_g := \boldsymbol{\Theta}\ (\lambda uv.\boldsymbol{G}\ v\ (u\ (\boldsymbol{S}\ v))). \tag{2.4}$$

Then:

$$\boldsymbol{J}_g\ \underline{n} \to^* \boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n+1}), \tag{2.5}$$

which Böhm tree can be sketched as

[8] In the article [Bre13] of the same author, the reader may also find another counterexample based on the same kind of intuitions.



Lemma 2.2.3.3 proves that $\boldsymbol{J}_g\ \underline{0}$ is operationally equivalent to the identity $\boldsymbol{I}$. In fact it is an infinite η-expansion of $\boldsymbol{I}$. But first, we need the following auxiliary lemma.

Lemma 2.2.3.2. *For any terms $M, N \in \Lambda$ and any fresh z:*

$$(M\ z \geq_{\eta\infty} N\ z) \quad \Rightarrow \quad (M \geq_{\eta\infty} N).$$

Proof. If M diverges then so does $(M\ z)$, then $(N\ z)\Uparrow^{\mathrm{h}}$ and $N\Uparrow^{\mathrm{h}}$, so that $\boldsymbol{BT}(M) = \boldsymbol{BT}(N) = \Omega$. Otherwise we have $M \to_h^* \lambda x_1 \dots x_n.y\ M_1 \cdots M_k$:

- If $n = 0$ then $M\ z \to_h^* y\ M_1 \cdots M_k\ z$ and $N\ z \to_h^* y\ N_1 \cdots N_k\ z$ with $M_i \geq_{\eta\infty} N_i$, thus $M \geq_{\eta\infty} N$.
- Otherwise, $M\ z \to_h^* \lambda x_2 \dots x_n.y[z/x_1]\ M_1[z/x_1] \cdots M_k[z/x_1]$ and $N\ z \to_h^* N' \leq_\eta \lambda x_2 \dots x_n.y[z/x_1]\ N_1 \cdots N_k$ with $M_i[z/x_1] \geq_{\eta\infty} N_i$ for all i. Thus, since z is fresh, $N \to_h^* \lambda x_1.N'[x_1/z] \leq_\eta \lambda x_2 \dots x_n.y\ N_1[x_1/z] \cdots N_k[x_1/z]$ and $M_i \geq_{\eta\infty} N_i[x_1/z]$, so $M \geq_{\eta\infty} N$.

□

Lemma 2.2.3.3. *We have $\boldsymbol{J}_g\ \underline{0} \equiv_{\mathcal{H}^*} \boldsymbol{I}$.*

Proof. We prove that $(\boldsymbol{J}_g\ \underline{n}\ z) \geq_{\eta\infty} z$ (where z is fresh) for every n, by co-induction and unfolding of $\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z)$:

$$\begin{array}{lr}
\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z) & \\
= \boldsymbol{BT}(\boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n+1}\ z)) & \text{by (2.5)} \\
= \lambda \vec{x}^{g(n)}.z\ \boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n+1}\ x_1) \cdots \boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n+1}\ x_{g(n)}) & \text{by (2.2)} \\
\geq_{\eta\infty} \lambda \vec{x}^{g(n)}.z\ x_1 \cdots x_{g(n)} & \text{by co-Ind} \\
\geq_\eta z &
\end{array}$$

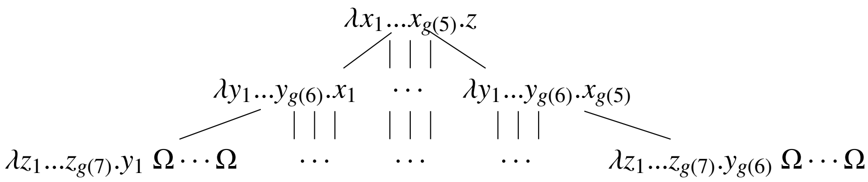
By applying Lemma 2.2.3.2, we know that $(\boldsymbol{J}_g\ \underline{n}) \geq_{\eta\infty} \boldsymbol{I}$ and by Corollary 1.2.2.14 that $\boldsymbol{J}_g\ \underline{0} \equiv_{\mathcal{H}^*} \boldsymbol{I}$. □

Denotational separation

In this section we show that $\boldsymbol{J}_g\ \underline{0}$ and $\boldsymbol{I}$ are denotationally separated (Lemma 2.2.3.6), despite being operationally equivalent.

Let $\boldsymbol{J}_g^{n,k}(z) \in \boldsymbol{BT}_f(\boldsymbol{J}_g\ \underline{n}\ z)$ be the truncation of $\boldsymbol{BT}(\boldsymbol{J}_g\ \underline{n}\ z)$ at depth k (in particular $\boldsymbol{J}_g^0 = \Omega$).

Example 2.2.3.4. *For example, the* $\boldsymbol{J}_g^{5,3}(z)$ *is the Böhm tree:*



We recall that the sequence $(\alpha_n)_{n\geq 0}$ is obtained from the refutation of the hyperimmunity and verify $\alpha_n = a_{n,1}\to\cdots\to a_{n,g(n)}\to\alpha'_n$ with $\alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}$.

Lemma 2.2.3.5. *For all n and k, and for all* $a \in \mathcal{A}_f(D)$ *such that* $\alpha_n \in a$*, we have*

$$(a, \alpha_n) \notin [\![\boldsymbol{J}_g^{n,k}(z)]\!]_D^z.$$

Proof. By induction on k:

- If $k = 0$, then $\boldsymbol{J}_n^k = \Omega$ and (approximation property) $(a, \alpha_n) \notin [\![\boldsymbol{J}_g^{n,k}(z)]\!]_D^z = \emptyset$.
- For $k + 1$:
  We first unfold $\alpha_n = a_1^n\to\cdots\to a_{g(n)}^n\to\alpha'_n$.
  Remark that $\boldsymbol{J}_n^{k+1}(z) = \lambda e x_1...x_{g(n)}.e\ (\boldsymbol{J}_{n+1}^k(x_1))\ \cdots\ (\boldsymbol{J}_{n+1}^k(x_{g(n)}))$ and that for all i, x_i is the only free variable of $\boldsymbol{J}_{n+1}^k(x_i)$.
  Thus (a, α_n) would be in $[\![\boldsymbol{J}_g^{n,k+1}(z)]\!]_D^z$ iff their where $\beta = b_1\to\cdots\to b_{g(n)}\to\alpha'_n \in a$ such that for all $i \leq g(n)$ and for all $\gamma \in b_i$, $(a_i^n, \gamma) \in [\![\boldsymbol{J}_g^{n+1,k}(x_i)]\!]_D^{x_i}$. The refutation has two cases:
  - For $\beta = \alpha_n$: there is $i \leq g(n)$ such that $\alpha_{n+1} \in b_i = a_i^n$, so that the induction hypothesis gives $(a_i^n, \alpha_{n+1}) \notin [\![\boldsymbol{J}_g^{n+1,k}(x_i)]\!]_D^{x_i}$.
  - For $\beta \neq \alpha_n$, since a is an anti-chain and $\alpha_n \in a, \beta \not\geq \alpha$. Thus there is $i \leq g(n)$ such that $b_i \not\leq a_i^n$ and, in particular, there is $\gamma \in b_i$ such that $\gamma \not\leq \delta$ for any $\delta \in a_i^n$, thus $(a_i^n, \gamma) \notin [\![I\ x_i]\!]^{x_i}$. Since $(I\ x_i) \leq_{\eta\infty} (A\ \underline{n+1}\ x_i)$, applying Lemma 2.2.1.13 gives that $[\![I\ x_i]\!]^{x_i} \supseteq [\![A\ \underline{n+1}\ x_i]\!]^{x_i} \supseteq [\![\boldsymbol{J}_{n+1}^k(x_i)]\!]$.

□

Lemma 2.2.3.6. *The term* $\boldsymbol{J}_g\ \underline{n}$ *(for any n) and the identity are denotationally separated in D:*

$$[\![\boldsymbol{J}_g\ \underline{n}]\!]_D \quad \neq \quad [\![\boldsymbol{I}]\!]_D$$

Proof. Using the approximation property and extensionality, it is sufficient to prove that

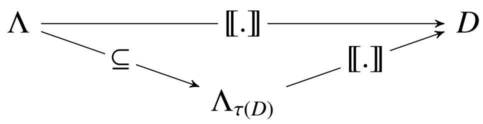
$$\{\alpha_0\}\to\alpha_0 \notin \bigcup_k [\![\lambda z.\boldsymbol{J}_g^{n,k}(z)]\!]_D = \bigcup_{U\in \boldsymbol{BT}_f(\boldsymbol{J}_g\ \underline{n})} [\![U]\!]_D = [\![\boldsymbol{J}_g\ \underline{n}]\!]_D,$$

which can be obtained by the application of Lemma 2.2.3.5. □

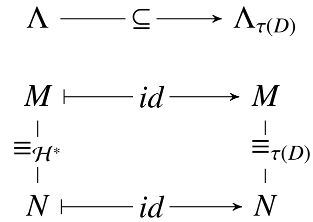
2.3. Syntactical proof using tests

In this section we will give a different proof (with respect to the proofs in Section 2.2) of the Theorem 2.1.0.6. The main idea is similar to Section 2.2, we will again use a middle step between our calculus and our models. However, this time the proxy will not be a kind of syntactical model (the Böhm trees), but a kind of semantical calculus, more exactly a set of calculi that we call λ-calculi with D-tests (Def. 2.3.1.1). Böhm trees were used since they were "syntactical models" directly inspired by the calculus (here the λ-calculus); thus, taking the opposite view, we will use "semantical calculi" that are directly inspired by the model (and that are dependent on the K-model D).

Given a K-model D, the λ-calculus with D-tests, denoted $\Lambda_{\tau(D)}$, is an extension of the untyped λ-calculus that can itself be interpreted in D (Def. 2.3.1.1):



The interest of $\Lambda_{\tau(D)}$ relies on the definition of sensibility for $\Lambda_{\tau(D)}$ (Def. 2.3.1.19), which easily implies the full abstraction of D for $\Lambda_{\tau(D)}$ (Th. 2.3.1.20), even if not for the λ-calculus. Therefore, it remains to understand when the observational equivalence is preserved from Λ to $\Lambda_{\tau(D)}$:



As for the semantical approach, the proof splits in the two directions: inequational full abstraction implies hyperimmunity (Sec. 2.3.2 and Th. 2.3.2.4) and the non-full abstraction for $\mathcal{H}^*$ given a counterexample to hyperimmunity (Sec. 2.3.3 and Th. 2.3.3.5). However, the proofs will rely on syntactical properties of $\Lambda_{\tau(D)}$ such as confluence (Th. 2.3.1.26) and standardization (Th. 2.3.1.29).

2.3.1. λ-calculi with D-tests

Syntax

The original idea of using *tests* to recover full abstraction (via a theorem of definability) is due to Bucciarelli *et al.* [BCEM11]. There we define variants of Bucciarelli *et al.*'s calculus adapted to our framework.

Directly dependent on a given K-model D, the λ-calculus with D-tests $\Lambda_{\tau(D)}$ is, to some extent, an internal calculus for D. In fact, we will see that, for D to be fully abstract for $\Lambda_{\tau(D)}$, it is sufficient to be sensible (Th. 2.3.1.20).

(term)	$\Lambda_{\tau(D)}$	M, N	$::= \quad x \quad \mid \quad \lambda x.M \quad \mid \quad M\,N \quad \mid \quad \sum_{i\leq n} \bar{\tau}_{\alpha_i}(Q_i)$	$,\forall (\alpha_i)_i \in D^n, n \geq 0$
(test)	$\boldsymbol{T}_{\tau(D)}$	P, Q	$::= \quad \sum_{i\leq n} P_i \quad \mid \quad \prod_{i\leq n} P_i \quad \mid \quad \tau_\alpha(M)$	$,\forall \alpha \in D, n \geq 0$

Figure 2.2.: Grammar of the calculus with D-tests

The idea is to introduce tests as a new kind in the syntax. Tests $Q \in \boldsymbol{T}_{\tau(D)}$ are sort of co-terms, in the sense that their interpretations are maps from the context to the dualizing object of the linear category ScottL ($\bot = \{*\}$):

$$[\![Q]\!]^{x_1\ldots x_n} \in D^n \Rightarrow \bot$$

The type $\bot$ is the unit type, having only one value representing the convergence of the evaluation, seen as a success. We will see in Remark 2.3.1.4 that in a polarized context, the behavior of test does not correspond to co-term (or stack), but to command, *i.e.*, to interactions between usual terms and fictive co-terms extracted from the semantics.

The interaction between terms and tests is carried out by two groups of operations indexed by the elements $\alpha \in D$:

$$\tau_\alpha : \Lambda_{\tau(D)} \to \boldsymbol{T}_{\tau(D)} \quad \text{and} \quad \bar{\tau}_\alpha : \boldsymbol{T}_{\tau(D)} \to \Lambda_{\tau(D)}.$$

The first operation, τ_α , will verify that its argument $M \in \Lambda_{\tau(D)}$ has the point α in its interpretation. Intuitively, this is performed by recursively unfolding the Böhm tree of M and succeeding (*i.e.*, converges) when α is in the interpretation of the finite unfolded Böhm tree. If $\alpha \notin [\![M]\!]$, the test $\tau_\alpha(M)$ will either diverge or refute (raising a $\mathbf{0}$ considered as an error). Concretely, it is an infinite application that feeds its argument with empty $\bar{\tau}$ operators.

The second operator, $\bar{\tau}_\alpha$, simply raises a term of interpretation $\downarrow\alpha$ if its argument succeeds and diverges otherwise. Concretely, it is an infinite abstraction that runs its test argument, but also tests each of its applicants using τ operators.

In addition to these operators, we use *sums* and *products* as ways to introduce may (for the addition) and must (for the multiplication) non-determinism; in the spirit of the λ+$\|$-calculus [DCdP98]. Indeed, these two forms of non-determinism are necessary to explore the branching of Böhm trees.

The idea of these two operators is to use the parametricity of our terms toward theirs intersection types. As a result, $\bar{\tau}_\alpha(\epsilon)$ (further on denoted $\bar{\boldsymbol{\epsilon}}_\alpha$), that transfers the always succeeding test ϵ into a term of interpretation $\downarrow\alpha$, constitutes the canonical term of type α; its behavior is exactly the common behavior of every term of type α. Symmetrically, the test $\tau_\alpha(M)$ will verify whether M behaves like a term of type α.

Hereinafter, D denotes a fixed extensional K-model.

Definition 2.3.1.1. *The* λ-calculus with D-tests*, for short* $\Lambda_{\tau(D)}$ *, is given by the grammar in Figure 2.2.*

We denote the empty sum by $\mathbf{0}$ *, and the empty product by* $\boldsymbol{\epsilon}$ *. Binary sums (resp. products) can be written with infix notation,* e.g. $M+N$ *(resp* $P{\cdot}Q$*).*

Moreover, we use the notation $\bar{\boldsymbol{\epsilon}}_\alpha := \bar{\tau}_\alpha(\boldsymbol{\epsilon})$ *and* $\bar{\boldsymbol{\epsilon}}_a := \sum_{\alpha\in a} \bar{\boldsymbol{\epsilon}}_\alpha$*; which are terms.*

Sums and products are considered as multisets, in particular we suppose associativity, *commutativity and neutrality with, respectively,* **0** *and* $\boldsymbol{\epsilon}$.

In the following, an abstraction *can refer either to a λ-abstraction or to a sum of $\bar{\tau}$ operators. This notation is justified by the behavior of $\Sigma_i\bar{\tau}_{\alpha_i}(Q_i)$ that mimics an infinite abstraction.*

The operational semantics is given by three sets of rules in Figure 2.3. The main rules *of Figure 2.3a are the effective rewriting rules. The* distributive rules *of Figure 2.3b implement the distribution of the sum over the test-operators and the product. The small step semantics $\to$ is the free contextual closure (*i.e., *by the rules of Figure 2.3d) of the rules of Figures 2.3a and 2.3b. The* contextual rules *of Figure 2.3c implement the* head reduction $\to_h$ *that is the specific contextual extension we are considering.*

Let us notice that this calculus enjoys the properties of confluence and standardization. These theorems are proved in the end of this section (Th. 2.3.1.26 and 2.3.1.29).

Example 2.3.1.2. *The operational behavior of D-tests depends on D. Recall the K-models of Example 1.2.4.9. In the case of Scott's D_∞ we have in $\Lambda_{\tau(D_\infty)}$:*

$$\begin{array}{llllll}
\tau_*(\underline{(\lambda xy.x\,y)\,\bar{\boldsymbol{\epsilon}}_*}) & \overset{\beta}{\to}_h & \underline{\tau_*(\lambda y}.\bar{\boldsymbol{\epsilon}}_*\,y) & \overset{\tau}{\to}_h & \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\,\bar{\boldsymbol{\epsilon}}_\emptyset}) & \\
 & \overset{\bar{\tau}}{\to}_h & \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*}) & = & \underline{\tau_*(\bar{\tau}_*}(\boldsymbol{\epsilon})) & \overset{\tau\bar{\tau}}{\to}_h \quad \boldsymbol{\epsilon},
\end{array}$$

$$\begin{array}{llllll}
\tau_*(\underline{(\lambda xy.y\,x)\,\bar{\boldsymbol{\epsilon}}_*}) & \overset{\beta}{\to}_h & \underline{\tau_*(\lambda y}.y\,\bar{\boldsymbol{\epsilon}}_*) & \overset{\tau}{\to}_h & \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_\emptyset}\,\bar{\boldsymbol{\epsilon}}_*) & \\
 & = & \tau_*(\underline{\mathbf{0}\,\bar{\boldsymbol{\epsilon}}_*}) & \overset{\bar{\tau}}{\to}_h & \underline{\tau_*(\mathbf{0})} & \overset{\tau\bar{\tau}}{\to}_h \quad \mathbf{0}.
\end{array}$$

In the case of Park P_∞:

$$\underline{\tau_*(\lambda x}.xx) \quad \overset{\tau}{\to}_h \quad \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\,\bar{\boldsymbol{\epsilon}}_*}) \quad \overset{\bar{\tau}}{\to}_h \quad \underline{\tau_*(\bar{\tau}_*}(\underline{\tau_*(\bar{\boldsymbol{\epsilon}}_*}))) \quad \overset{\tau\bar{\tau}}{\to}_h\overset{\tau\bar{\tau}}{\to}_h \quad \boldsymbol{\epsilon}.$$

In the case of Norm:

$$\underline{\tau_p(\lambda x}.x) \quad \overset{\tau}{\to}_h \quad \underline{\tau_p(\bar{\boldsymbol{\epsilon}}_q)} \quad \overset{\tau\bar{\tau}}{\to}_h \quad \boldsymbol{\epsilon}, \qquad\qquad \underline{\tau_q(\lambda x}.x) \quad \overset{\tau}{\to}_h \quad \underline{\tau_q(\bar{\boldsymbol{\epsilon}}_p)} \quad \overset{\tau\bar{\tau}}{\to}_h \quad \mathbf{0}.$$

Example 2.3.1.3. *In any K-model D, given $\alpha = a_1\to\cdots\to a_{n+1}\to\beta \in D$, and if we denote $\alpha' = a_2\to\cdots\to a_{n+1}\to\beta$ we have:*

$$\begin{array}{lll}
\bar{\boldsymbol{\epsilon}}_\alpha\, M_1\cdots M_{n+1} & \overset{\bar{\tau}}{\to}_h & \bar{\tau}_{\alpha'}(\Pi_{\gamma\in a_1}\tau_\gamma(M_1))\, M_2\cdots M_{n+1} \\
 & \overset{\bar{\tau}}{\to}{}^n_h & \bar{\tau}_\beta(\Pi_{i\leq n+1}\Pi_{\gamma\in a_i}\tau_\gamma(M_i))
\end{array}$$

Remark 2.3.1.4. *In a polarized (or classical) framework with explicit co-terms (or stacks) such that [MM09], tests would correspond to commands (or processes), or, more exactly, to conjunctions and disjunctions of commands. Indeed, a test $\tau_\alpha(M)$ is nothing else than the command $\langle M \mid \pi_\alpha\rangle$ where π_α would be the canonical co-term of interpretation $\uparrow\alpha$, the same way that $\bar{\boldsymbol{\epsilon}}_\alpha$ is the canonical term of interpretation $\downarrow\alpha$. Similarly, the term $\bar{\tau}(Q)$ can be seen as the canonical term $\bar{\boldsymbol{\epsilon}}_\alpha$ endowed with a parallel composition referring to the set of commands Q. To resume, we have:*

$$\tau_\alpha(M) \;\simeq\; \langle M \mid \uparrow\alpha\rangle \qquad\qquad \langle\bar{\boldsymbol{\epsilon}}_\alpha(Q)\mid\pi\rangle \;\simeq\; \langle\downarrow\alpha\mid\pi\rangle\cdot Q$$

$$
\begin{array}{llrcl}
(\beta) & & (\lambda x.M)\, N & \to & M[N/x] \\
(\tau) & \forall \beta = a{\to}\alpha, & \tau_\beta(\lambda x.M) & \to & \tau_\alpha(M[\bar{\epsilon}_a/x]) \\
(\bar\tau) & \forall \beta_i = a_i{\to}\alpha_i, & (\Sigma_i \bar\tau_{\beta_i}(Q_i))\, M & \to & \Sigma_i \bar\tau_{\alpha_i}(Q_i \cdot \Pi_{\gamma\in a_i}\tau_\gamma(M)) \\
(\tau\bar\tau) & \forall \alpha, \forall (\beta_i)_i, & \tau_\alpha(\Sigma_i \bar\tau_{\beta_i}(Q_i)) & \to & \Sigma_{\{i \mid \alpha \le \beta_i\}} Q_i
\end{array}
$$

(a) Main rules

$$
\begin{array}{lrcl}
(\cdot+) & \Pi_{i\le n}\Sigma_{j\le k_i} Q_{i,j} & \to & \Sigma_{j_1\le k_1,\ldots,j_n\le k_n}\Pi_{i\le n} Q_{i,j_i} \\
(\bar\tau+) & \bar\tau_\alpha(\Sigma_i Q_i) & \to & \Sigma_i \bar\tau_\alpha(Q_i)
\end{array}
$$

(b) Distribution of the sum

$$
\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'}\,(H\text{-}c\lambda)
\qquad
\frac{M \to_h M' \qquad M \text{ is not an abstraction}}{M\, N \to_h M'\, N}\,(H\text{-}c@)
$$

$$
\frac{M \to_h M' \qquad M \text{ is not an abstraction}}{\tau_\alpha(M) \to_h \tau_\alpha(M')}\,(H\text{-}c\tau)
\qquad
\frac{Q \to_h Q' \quad Q \text{ is not a sum}}{\bar\tau_\alpha(Q) \to_h \bar\tau_\alpha(Q')}\,(H\text{-}c\bar\tau)
$$

$$
\frac{M \to_h M'}{M + N \to_h M' + N}\,(H\text{-}cs)
\qquad
\frac{Q \to_h Q'}{Q + P \to_h Q' + P}\,(H\text{-}c{+})
\qquad
\frac{Q \to_h Q' \quad Q \text{ is not a sum}}{Q{\cdot}P \to_h Q'{\cdot}P}\,(H\text{-}c\cdot)
$$

(c) Contextual rules for the head reduction

$$
\frac{M \to M'}{\lambda x.M \to \lambda x.M'}\,(c\lambda)
\qquad
\frac{M \to M'}{M\, N \to M'\, N}\,(c@L)
\qquad
\frac{N \to N'}{M\, N \to M'\, N}\,(c@R)
$$

$$
\frac{M \to M'}{\tau_\alpha(M) \to \tau_\alpha(M')}\,(c\tau)
\qquad
\frac{Q \to Q'}{\bar\tau_\alpha(Q) \to \bar\tau_\alpha(Q')}\,(c\bar\tau)
$$

$$
\frac{M \to M'}{M + N \to M' + N}\,(cs)
\qquad
\frac{Q \to Q'}{Q + P \to Q' + P}\,(c{+})
\qquad
\frac{Q \to Q'}{Q{\cdot}P \to Q'{\cdot}P}\,(c\cdot)
$$

(d) Contextual rules for the full reduction

Figure 2.3.: Operational semantics of the calculus with *D*-tests

Remark 2.3.1.5. *In the conference version [Bre14], the rule ($\tau\bar{\tau}$) is decomposed in three rules (the distribution of the sum over τ, denoted ($\tau+$) and two versions of ($\tau\bar{\tau}$) depending whether $\alpha \le \beta$). This decomposition was easier to understand as more atomic, but ultimately it always reproduced our actual rule ($\tau\bar{\tau}$) and did not permit to use Theorem 2.3.1.32.*

Proposition 2.3.1.6. *A test is in* head-normal form *iff it has the following shape:*

$$\Sigma_{i\le k}\Pi_j\tau_{\alpha_{i,j}}(x_{i,j}\ M^1_{i,j}\cdots\ M^n_{i,j}),$$

with $k \ge 1$ and $M^k_{i,j}$ any term.
A term is in head-normal form *if it has one of the following shapes:*

$$\lambda x_1....x_n.y\ M_1\cdots\ M_m, \qquad\qquad or \qquad \lambda x_1...x_n.\Sigma_{i\le k}\bar{\tau}_{\alpha_i}(Q_i),$$

with $m,n \ge 0$, $k \ge 1$, $(\alpha_i)_i \in D^k$, M_i any term, and every Q_i any test in head-normal form without sums.

Proof. By structural induction on the grammar of $\Lambda_{\tau(D)}$. In particular, notice that any test of the shape $\tau_\alpha(\lambda x.M)$ is not a head-normal form because i_D is surjective and thus $\alpha = a{\to}\beta$ for some a,β and we can apply Rule (τ) □

Definition 2.3.1.7. *A term (resp. test) is* head-converging *if it head reduces to a* may-head-normal form *(denoted* mhnf *) that is either a head-normal form or a term (resp. test) of the form*

$$\lambda x_1...x_n.(\bar{\tau}_\alpha(Q)+N) \qquad\qquad resp.\ Q_1+Q_2$$

with $\bar{\tau}_\alpha(Q)$ (resp. Q_1) in head-normal form and N any term (resp. Q_2 any test). This corresponds to a may-convergence *for the sum. Coherently with the head convergence in λ-calculus, the convergence will be denoted by $\Downarrow^{\mathrm{h}}$ and the divergence by $\Uparrow^{\mathrm{h}}$.*

Example 2.3.1.8. *For any $n \in \mathbb{N}$, the term $\underline{n}\ (\lambda x.\bar{\tau}_\alpha(\tau_\alpha(x)+\tau_\beta(x)))\ \bar{\boldsymbol{\epsilon}}_\alpha$ may head converges.*

Remark 2.3.1.9. *Following Proposition A.2.1.34, we can coinductively prove the divergence of a term by constructing a tree of head reductions, with must and may nodes (corresponding to the reductions under sums and under products).*

Definition 2.3.1.10. *Grammars of* term-context*s* $\Lambda^{(\!|\cdot|\!)}_{\tau(D)}$ *and* test-context*s* $\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$ *are given in Figure 2.4.*

Definition 2.3.1.11. *The* observational preorder $\sqsubseteq_{\tau(D)}$ *of* $\Lambda_{\tau(D)}$ *is defined by:*

$$M \sqsubseteq_{\tau(D)} N\ \ iff\ \ (\forall K{\in}\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)},\ K(\!|M|\!)\Downarrow^{\mathrm{h}}\ \ implies\ \ K(\!|N|\!)\Downarrow^{\mathrm{h}}).$$

(term-context)	$\Lambda^{(\!|\cdot|\!)}_{\tau(D)}$	$C ::= x \mid (\!|.|\!) \mid C\,C' \mid \lambda x.C \mid \sum_{i\leq n}\bar{\tau}_{\alpha_i}(K_i)$	$,\forall(\alpha_i)_i\in D^n, n\geq 0$
(test-context)	$\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$	$K ::= \sum_{i\leq n} K_i \mid \prod_{i\leq n} K_i \mid \tau_\alpha(C)$	$,\forall\alpha\in D, n\geq 0$

Figure 2.4.: Grammar of the contexts in a calculus with D-tests

$$[\![x_i]\!]^{\vec{x}}_D = \{(\vec{a},\alpha)\mid \alpha\leq\beta\in a_i\}$$
$$[\![M\,N]\!]^{\vec{x}}_D = \{(\vec{a},\alpha)\mid \exists b, (\vec{a},(b{\to}\alpha))\in[\![M]\!]^{\vec{x}}_D \;\wedge\; \forall\beta{\in}b, (\vec{a},\beta)\in[\![N]\!]^{\vec{x}}_D\}$$
$$[\![\lambda y.M]\!]^{\vec{x}}_D = \{(\vec{a},(b{\to}\alpha))\mid(\vec{a}b,\alpha)\in[\![M]\!]^{\vec{x}y}_D\}$$

(a) Interpretation of Λ

$$[\![\Sigma_{i\leq k}\bar{\tau}_{\alpha_i}(Q_i)]\!]^{\vec{x}}_D = \bigcup_{i\leq k}\{(\vec{a},\beta)\mid \vec{a}\in[\![Q_i]\!]^{\vec{x}}_D \;\wedge\; \beta\leq_D\alpha_i\}$$
$$[\![\tau_\alpha(M)]\!]^{\vec{x}}_D = \{\vec{a}\mid(\vec{a},\alpha)\in[\![M]\!]^{\vec{x}}_D\}$$
$$[\![\Pi_{i\leq k}Q_i]\!]^{\vec{x}}_D = \bigcap_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D \qquad [\![\Sigma_{i\leq k}Q_i]\!]^{\vec{x}}_D = \bigcup_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D$$

(b) Interpretation of tests extensions

Figure 2.5.: Direct interpretation in D

We denote by $\equiv_{\tau(D)}$ *the* observational equivalence*,* i.e.*, the equivalence induced by* $\sqsubseteq_{\tau(D)}$.

Remark 2.3.1.12. *The observational preorder could have been defined using term-contexts rather than test-contexts, but this appears to be equivalent and test-contexts are easier to manipulate (because normal forms for tests are simpler).*
Proof. For any test Q and for any α, $Q\Downarrow^{\mathrm{h}}$ iff $\bar{\tau}_\alpha(Q)\Downarrow^{\mathrm{h}}$. Conversely, for all M, there is $n\in\mathbb{N}$ and $\alpha\in D$ such $M\Downarrow^{\mathrm{h}}$ iff $\tau_\alpha(M\underset{n}{x_0\cdots x_0})\Downarrow^{\mathrm{h}}$ (remark that if N diverges, then $\tau_\alpha(N\underbrace{x_0\cdots x_0}_{n\text{ times}})\Uparrow^{\mathrm{h}}$). □

Semantics

The standard interpretation of Λ into D (Fig. 1.2 and recalled here in Figure 2.5) can be extended to $\Lambda_{\tau(D)}$ (Fig. 2.5b).

Definition 2.3.1.13. *A term M with n free variables is* interpreted *as a morphism (Scott-continuous function) from D^n to D and a test Q with n free variables as a morphism from D^n to the dualizing object* $\{*\}$ *(singleton poset):*

$$[\![M]\!]^{x_1,\ldots,x_n}_D \;\subseteq\; (D\Rightarrow\cdots\Rightarrow D\Rightarrow D) \;\simeq\; (\mathcal{A}_f(D)^{op})^n\times D$$
$$[\![Q]\!]^{x_1,\ldots,x_n}_D \;\subseteq\; (D\Rightarrow\cdots\Rightarrow D\Rightarrow\{*\}) \;\simeq\; (\mathcal{A}_f(D)^{op})^n$$

This interpretation is given in Figure 2.5 by structural induction.

Proposition 2.3.1.14. *For any extensional K-model D, D is a model of the λ-calculus with D-tests,* i.e.*, the interpretation is invariant under the reduction.*

$$\frac{\alpha \in a}{x : a \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : a \vdash M : \alpha} \qquad \frac{\Gamma \vdash M : \beta \qquad \alpha \leq \beta}{\Gamma \vdash M : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a \to \alpha} \qquad \frac{\Gamma \vdash M : a \to \alpha \qquad \forall \beta \in a,\ \Gamma \vdash N : \beta}{\Gamma \vdash M\, N : \alpha}$$

$$\frac{\exists i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Sigma_{i \leq n} \bar{\tau}_{\alpha_i}(Q_i) : \alpha_i} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash \tau_\alpha(M)} \qquad \frac{\exists i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Sigma_{i \leq n} Q_i} \qquad \frac{\forall i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Pi_{i \leq n} Q_i}$$

Figure 2.6.: Intersection type system associated with tests extensions

Proof. The invariance under β-reduction is obtained, as usual, by the Cartesian closeness of ScottL$_!$. The other rules are easy to check directly. □

Proposition 2.3.1.15. *For any extensional K-model D, the interpretation is invariant by context,* i.e., $[\![M]\!]^{\vec{x}} = [\![N]\!]^{\vec{x}}$ *implies that for any test/term-context C,* $[\![C(\!|M|\!)]\!]^{\vec{y}} = [\![C(\!|N|\!)]\!]^{\vec{y}}$.

Proof. By easy induction on C. □

The idea of intersection types can be generalized to $\Lambda_{\tau(D)}$. We introduce in Figure 2.6 a type assignment system associating with any term $M \in \Lambda_{\tau(D)}$ an element of D under an environment $(x_i : a_i)_i$ with $a_i \in \mathcal{A}_f(D)$. The following theorem gives the equivalence between the interpretation of a term and the set of judgments derivable from the type system.

Theorem 2.3.1.16 (Intersection types). *Let M be a term of* $\Lambda_{\tau(D)}$*, (resp. Q be a test of* $\boldsymbol{T}_{\tau(D)}$*), the following statements are equivalent:*

- $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}_D$ *(resp.* $\vec{a} \in [\![Q]\!]^{\vec{x}}_D$*),*
- *the type judgment* $\vec{x} : \vec{a} \vdash M : \alpha$ *(resp.* $\vec{x} : \vec{a} \vdash Q$*) is derivable by the rules of Figure 2.6.*

Proof. By structural induction on the grammar of $\Lambda_{\tau(D)}$. □

Remark 2.3.1.17. *In particular, an easy induction gives that if* $\vdash M[N/x] : \alpha$ *then there is a such that* $N : a \vdash M : \alpha$*.*

Remark 2.3.1.18. *For more intuitions on tests, the reader is invited to look at Section 2.4.1, giving some analogies between tests and Böhm trees.*

Full abstraction and sensibility for tests

The main theorem (Th. 2.1.0.6) uses the assumption of the sensibility of D for $\Lambda_{\tau(D)}$. The sensibility is simply asking for the diverging terms $M \in \Lambda_{\tau(D)}$ to have empty interpretation as precised in Definition 2.3.1.19. Its interest is in implying directly the inequational full abstraction of D for $\Lambda_{\tau(D)}$ (*i.e.* for its observational preorder) as we will see in Theorem 2.3.1.20. The proof of Theorem 2.3.1.20 needs a technical counterpart that is basically the *definability* of $\Lambda_{\tau(D)}$ stated in Theorem 2.3.1.21. This definability theorem is not usual and appears to be stronger and more useful for future developments.

First we recall the definition of sensibility specified in the case of K-models (following Definition 1.2.1.7)

Definition 2.3.1.19. *An extensional K-model D is* sensible for $\Lambda_{\tau(D)}$ *whenever diverging terms (resp. tests) correspond exactly to the terms (resp. tests) having empty interpretation,* i.e., *for all $M \in \Lambda_{\tau(D)}$ and $Q \in \boldsymbol{T}_{\tau(D)}$:*

$$M{\Uparrow}^{\mathrm{h}} \quad \Leftrightarrow \quad [\![M]\!]^{\vec{x}}_D = \emptyset \qquad\qquad Q{\Uparrow}^{\mathrm{h}} \quad \Leftrightarrow \quad [\![Q]\!]^{\vec{x}}_D = \emptyset$$

Theorem 2.3.1.20 (full abstraction). *For any extensional K-model D, if D is sensible for $\Lambda_{\tau(D)}$, then D is inequationally fully abstract for the observational preorder of $\Lambda_{\tau(D)}$:*

$$[\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad \forall K \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}, K(\!|M|\!){\Downarrow}^{\mathrm{h}} \Rightarrow K(\!|N|\!){\Downarrow}^{\mathrm{h}}.$$

Proof. The left-to-right implication is the inequational adequation given by Lemma 1.2.1.10.
Conversely, if $\forall K \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}, K(\!|M|\!){\Downarrow}^{\mathrm{h}} \Rightarrow K(\!|N|\!){\Downarrow}^{\mathrm{h}}$ and if $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$:
Then by Theorem 2.3.1.21, $\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}]){\Downarrow}^{\mathrm{h}}$. Thus, stating $K = \tau_\alpha((\lambda x_. ..x_n.(\!|.|\!))\,\bar{\boldsymbol{\epsilon}}_{a_1}\cdots\bar{\boldsymbol{\epsilon}}_{a_n})$, we have $K(\!|M|\!) \to^n_h \tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}]){\Downarrow}^{\mathrm{h}}$ which implies that $K(\!|N|\!){\Downarrow}^{\mathrm{h}}$. However, the only n first head-reductions of $K(\!|N|\!)$ are forced into $K(\!|N|\!) \to^*_h \tau_\alpha(N[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}])$ so that this term is converging. Then by applying the reverse implication of Theorem 2.3.1.21 we conclude $(\vec{a},\alpha) \in [\![N]\!]^{\vec{x}}$. □

Theorem 2.3.1.21 (Definability). *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}} \quad \Leftrightarrow \quad \tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}]){\Downarrow}^{\mathrm{h}}.$$

Proof. If $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$ then $[\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}])]\!]$ is not empty by Lemma 2.3.1.22, thus it converges by sensibility. If $\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}]){\Downarrow}^{\mathrm{h}}$ then its interpretation is non empty, and necessarily $* \in [\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\le n}])]\!]$ (where $*$ denotes the only inhabitant of $\bot$) and thus, by Lemma 2.3.1.22, $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$. □

This proof make use of the following lemma that splits the problem in two.

Lemma 2.3.1.22. *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{a}b,\alpha) \in [\![M]\!]^{\vec{y}x} \quad \Leftrightarrow \quad (\vec{a},\alpha) \in [\![M[\bar{\boldsymbol{\epsilon}}_b/x]]\!]^{\vec{y}},$$

$$(\vec{a},\alpha) \in [\![M]\!]^{\vec{y}} \quad \Leftrightarrow \quad \vec{a} \in [\![\tau_\alpha(M)]\!]^{\vec{y}}.$$

Proof. For this proof we use the intersection type system of Figure 2.6. This replaces the statement by:

$$\Gamma, x : a \vdash M : \alpha \quad \Leftrightarrow \quad \Gamma \vdash M[\bar{\boldsymbol{\epsilon}}_a/x] : \alpha$$

$$\Gamma \vdash M : \alpha \quad \Leftrightarrow \quad \Gamma \vdash \tau_\alpha(M)$$

- $\Gamma, x : a \vdash M : \alpha \;\;\Rightarrow\;\; \Gamma \vdash M[\bar{\epsilon}_a/x] : \alpha$:
  and $\Gamma, x : a \vdash Q \;\;\Rightarrow\;\; \Gamma \vdash Q[\bar{\epsilon}_a/x]$:
  By structural induction on M and Q:
  - If $M = x$: then $\alpha \leq \beta \in a$ and by definition $\Gamma \vdash \bar{\epsilon}_a : \alpha$.
  - If $M = y \neq x$: trivial.
  - If $M = \lambda y.N$: then $\alpha = b{\to}\beta$ and $\Gamma, y : b, x : a \vdash N : \beta$ thus by IH, $\Gamma, y : b \vdash N[\bar{\epsilon}_a/x] : \beta$ and thus $\Gamma \vdash M[\bar{\epsilon}_a/x] : \alpha$.
  - If $M = N_1\ N_2$: then there exists b such that $\Gamma, x : a \vdash N_1 : b{\to}\alpha$ and for all $\beta \in b$, $\Gamma, x : a \vdash N_2 : \beta$. Thus by IH, $\Gamma \vdash N_1[\bar{\epsilon}_a/x] : b{\to}\alpha$ and for all $\beta \in b$, $\Gamma \vdash N_2[\bar{\epsilon}_a/x] : \beta$ and thus $\Gamma \vdash M[\bar{\epsilon}_a/x] : \alpha$.
  - If $M = \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$: then there exists i such that $\alpha = \alpha_i$ and $\Gamma, x : a \vdash Q_i$. Thus by IH, $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma \vdash M[\bar{\epsilon}_a/x] : \alpha$.
  - If $Q = \Sigma_i Q_i$: then there exists i such that $\Gamma, x : a \vdash Q_i$. Thus by IH, $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma \vdash Q[\bar{\epsilon}_a/x]$.
  - If $Q = \Pi_i Q_i$: then for all i, $\Gamma, x : a \vdash Q_i$. Thus by IH, for all i, $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma \vdash Q[\bar{\epsilon}_a/x]$.
  - If $Q = \tau_\beta(M)$: then $\Gamma, x : a \vdash M : \beta$. Thus by IH, $\Gamma \vdash M[\bar{\epsilon}_a/x] : \beta$ and thus $\Gamma \vdash Q[\bar{\epsilon}_a/x]$.
- $\Gamma, x : a \vdash M : \alpha \;\;\Leftarrow\;\; \Gamma \vdash M[\bar{\epsilon}_a/x] : \alpha$, and $\Gamma, x : a \vdash Q \;\;\Leftarrow\;\; \Gamma \vdash Q[\bar{\epsilon}_a/x]$:
  By structural induction on M and Q:
  - If $M = x$ then $\Gamma \vdash \bar{\epsilon}_a : \alpha$ and by definition $\Gamma, x : a \vdash x : \alpha$, *i.e*, $\Gamma, x : a \vdash M : \alpha$
  - If $M = y \neq x$: trivial.
  - If $M = \lambda y.N$: then $\alpha = i_D(b{\to}\beta)$ and $\Gamma, y : b \vdash N[\bar{\epsilon}_a/x] : \beta$ thus by IH, $\Gamma, y : b, x : a \vdash N : \beta$ and thus $\Gamma, x : a \vdash M : \alpha$.
  - If $M = N_1\ N_2$: then there exists b such that $\Gamma \vdash N_1[\bar{\epsilon}_a/x] : b{\to}\alpha$ and for all $\beta \in b$, $\Gamma \vdash N_2[\bar{\epsilon}_a/x] : \beta$. Thus by IH, $\Gamma, x : a \vdash N_1 : b{\to}\alpha$ and for all $\beta \in b$, $\Gamma, x : a \vdash N_2 : \beta$ and thus $\Gamma, x : a \vdash M : \alpha$.
  - If $M = \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$: then there exists i such that $\alpha = \alpha_i$ and $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, $\Gamma, x : a \vdash Q_i$ and thus $\Gamma, x : a \vdash M : \alpha$.
  - If $Q = \Sigma_i Q_i$: then there exists i such that $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, $\Gamma, x : a \vdash Q_i$ and thus $\Gamma, x : a \vdash Q$.
  - If $Q = \Pi_i Q_i$: then for all i, $\Gamma \vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, for all i, $\Gamma, x : a \vdash Q_i$ and thus $\Gamma, x : a \vdash Q$.
  - If $Q = \tau_\beta(M)$: then $\Gamma \vdash M[\bar{\epsilon}_a/x] : \beta$. Thus by IH, $\Gamma, x : a \vdash M : \beta$ and thus $\Gamma, x : a \vdash Q$.
- $\Gamma \vdash \tau_\alpha(M) \;\;\Leftrightarrow\;\; \Gamma \vdash M : \alpha$: by definition of the inference rule for τ_α

□

Confluence

This section is dedicated to the proof of Theorem 2.3.1.26 stating the confluence of the reduction $\to$ in $\Lambda_{\tau(D)}$. The proof is using the diamond property of the full parallel reduction, following the proof of [Tai67] for the λ-calculus.

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N'}{(\lambda x.M)\ N \Rightarrow M'[N'/x]}\ (P\text{-}\beta) \qquad \frac{M \Rightarrow M'}{\tau_{a\to\alpha}(\lambda x.M) \Rightarrow \tau_\alpha(M'[\bar{\epsilon}_a/x])}\ (P\text{-}\tau)$$

$$\frac{M \Rightarrow M' \qquad \forall i,\ Q_i \Rightarrow \Sigma_j Q'_{ij}}{\Sigma_i \bar{\tau}_{a_i\to\alpha_i}(Q_i)\ M \Rightarrow \Sigma_{ij}\bar{\tau}_{\alpha_i}(Q'_{ij} \cdot \Pi_{\gamma\in a_i}\tau_\gamma(M'))}\ (P\text{-}\bar{\tau}) \qquad \frac{\forall_i, Q_i \Rightarrow Q'_i}{\tau_\alpha(\Sigma_i \bar{\tau}_{\beta_i}(Q_i)) \Rightarrow \Sigma_{\{i|\alpha\le\beta_i\}} Q'_i}\ (P\text{-}\tau\bar{\tau})$$

(a) Main rules

$$\frac{\forall i,\ Q_i \Rightarrow Q'_i}{\bar{\tau}_\alpha(\Sigma_i Q_i) \Rightarrow \Sigma_i \bar{\tau}_\alpha(Q'_i)}\ (P\text{-}\bar{\tau}+) \qquad \frac{\forall ij,\ Q_{ij} \Rightarrow Q'_{ij}}{\Pi_{i\le n}\Sigma_{j\le k_i} Q_{ij} \Rightarrow \Sigma_{j_1\le k_1,\dots,k_n\le k_n}\Pi_{i\le n} Q'_{ij_i}}\ (P\text{-}\cdot\text{+})$$

(b) Distribution of the sum

$$\frac{}{x \Rightarrow x}\ (P\text{-}id) \qquad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}\ (P\text{-}c\lambda) \qquad \frac{M \Rightarrow M' \qquad N \Rightarrow N'}{M\ N \Rightarrow M'\ N'}\ (P\text{-}c@)$$

$$\frac{M \Rightarrow M'}{\tau_\alpha(M) \Rightarrow \tau_\alpha(M')}\ (P\text{-}c\tau) \qquad \frac{\forall i,\ M_i \Rightarrow M'_i}{\Sigma_i M_i \Rightarrow \Sigma_i M'_i}\ (P\text{-}cs)$$

(c) Contextual rules

Figure 2.7.: Operational Semantics of parallel reduction

We define first the *parallel reduction* $\Rightarrow$ in Figure 2.7, allowing the parallel reduction of independent redexes.

Lemma 2.3.1.23. *If $M \Rightarrow N$ then $M \to^* N$ and if $M \to^* N$ then $M \Rightarrow^* N$.*
In particular we have $\Rightarrow^* = \to^*$.

Proof. Firstly remark that $\Rightarrow$ is reflexive. Indeed, when we proceed by induction the only difficult case is $\epsilon \Rightarrow \epsilon$ that is obtained by rule P-$\cdot$+ for $n = 0$.
Rules with similar names are then simulating each other except for

- $(c@L)$ and $(c@R)$ that are simulated by $(P\text{-}c@)$.
- $(P\text{-}id)$ that is simulated by $\to^\epsilon$ (the reduction in 0 step).
- $(c+)$ that is a particular case of $(P\text{-}\cdot+)$ with $n = 1$ and $k_1 = 2$.
- $(c\cdot)$ that is a particular case of $(P\text{-}\cdot+)$ with $n = 2$ and $k_1 = k_2 = 1$.
- $(c\bar{\tau})$ that is a particular case of $(P\text{-}\bar{\tau}+)$ where the sum has one element.

□

For a term M (resp. a test Q) we define the *maximal parallel reduct* M^+ (resp. Q^+) by induction on M and Q in Figure 2.8. Recall that by abstractions, we not only mean λ-abstractions, but also terms of the form $\Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$.

$$\frac{}{((\lambda x.M)\ N)^+ := M^+[N^+/x]}\ (T\text{-}\beta) \qquad \frac{\forall i,\quad Q_i^+ = \Sigma_j Q'_{ij} \qquad \forall j,\ Q'_{i,j} \text{ are not sums}}{((\Sigma_i \bar{\tau}_{a_i\to\alpha_i}(Q_i))\ M)^+ := \Sigma_{ij}\bar{\tau}_{\alpha_i}(Q'_{ij}\cdot \Pi_{\gamma\in a_i}\tau_\gamma(M^+))}\ (T\text{-}\bar{\tau})$$

$$\frac{}{\tau_{a\to\alpha}(\lambda x.M)^+ := \tau_\alpha(M^+[\bar{\epsilon}_a/x])}\ (T\text{-}\tau) \qquad \frac{\forall i\in I,\ \alpha \leq_E \beta_i \qquad \forall i \in J,\ \alpha \not\leq_E \beta_i}{\tau_\alpha(\Sigma_{i\in I\cup J}\bar{\tau}_{\beta_i}(Q_i))^+ := \Sigma_{i\in I} Q_i^+}\ (T\text{-}\tau\bar{\tau})$$

(a) Main rules

$$\frac{\forall i,\ Q_i \text{ are not sums}}{\bar{\tau}_\alpha(\Sigma_i Q_i)^+ := \Sigma_i \bar{\tau}_\alpha(Q_i^+)}\ (T\text{-}\bar{\tau}+) \qquad \frac{n \neq 1 \text{ or } k_1 \neq 1 \qquad \text{the } Q_{ij} \text{ are not sums}}{(\Pi_{i\leq n}\Sigma_{j\leq k_i}Q_{ij})^+ := \Sigma_{j_1\leq k_1,\dots,k_n\leq k_n}\Pi_{i\leq n}Q_{ij_i}^+}\ (T\text{-}\cdot+)$$

(b) Distribution of the sum

$$\frac{}{x^+ := x}\ (T\text{-}id) \qquad \frac{}{(\lambda x.M)^+ \Rightarrow \lambda x.M^+}\ (T\text{-}c\lambda) \qquad \frac{M \text{ is not an abstraction}}{(M\ N)^+ := M^+\ N^+}\ (T\text{-}c@)$$

$$\frac{M \text{ is not an abstraction}}{\tau_\alpha(M) := \tau_\alpha(M^+)}\ (T\text{-}c\tau) \qquad \frac{k \neq 1}{(\Sigma_{i\leq k}M_i)^+ := \Sigma_{i\leq k}M_i^+}\ (T\text{-}cs)$$

(c) Contextual rules

Figure 2.8.: Full parallel reduction

Lemma 2.3.1.24. *For any M (resp. Q), M^+ (resp. Q^+) is well defined.*

Proof. By induction, since it is always the case that exactly one rule is applied. □

Lemma 2.3.1.25. *If $M \Rightarrow N$ (resp. $Q \Rightarrow P$) then $N \Rightarrow M^+$ (resp. $P \Rightarrow Q^+$).*

Proof. By induction on M:

- If $M = x$:
  Then $N = x \Rightarrow x = M^+$.
- If $M = \lambda x.M'$:
  Then $N = \lambda x.N'$ for some N' such that $M' \Rightarrow N'$.
  By IH, $N' \Rightarrow M'^+$ and thus $N \Rightarrow \lambda x.M'^+ = M^+$.
- If $M = M_1\ M_2$:
  - If M_1 is not an abstraction:
    Then $N = N_1\ N_2$ with $M_1 \Rightarrow N_1$ and $M_2 \Rightarrow N_2$.
    By IH, $N_1 \Rightarrow M_1^+$ and $N_2 \Rightarrow M_2^+$, thus $N \Rightarrow M_1^+\ M_2^+ = M^+$.
  - If $M_1 = \lambda x.M_0$:
    * Either $N = (\lambda x.N_0)\ N_2$ with $M_i \Rightarrow N_i$ (for $i \in \{0,2\}$).
      By IH, $N_i \Rightarrow M_i^+$ and $N \Rightarrow M_0^+[M_2^+/x] = M^+$.
    * Or $N = N_1[N_2/x]$ with $M_i \Rightarrow N_i$ (for $i \in \{0,2\}$).
      By IH, $N_i \Rightarrow M_i^+$ and $N \Rightarrow M_0^+[M_2^+/x] = M^+$.
  - If $M_1 = \Sigma_{i\in I}\bar{\tau}_{a_i\to\alpha_i}(Q_i)$:

* Either $N = (\Sigma_{i,j}\bar{\tau}_{a\to\alpha_i}(P_{i,j}))\ N_2$ with $M_2 \Rightarrow N_2$ and $Q_i = \Sigma_i P'_{i,j}$ and $P'_{i,j} \Rightarrow P_{i,j}$.
  By IH, $N_2 \Rightarrow M_2^+$ and, moreover,
  $P_{i,j} \Rightarrow Q_{i,j}^+ = \Sigma_k Q'_{i,j,k}$ where $Q'_{i,j,k}$ that are not sums.
  Thus $N \Rightarrow \Sigma_{i,j,k}\bar{\tau}_{\alpha_i}(Q'_{i,j,k}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(M_2^+)) = M^+$.
* Or $N = \Sigma_{i,j}\bar{\tau}_{\alpha_i}(P_{i,j}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(N_2))$ with $M_2 \Rightarrow N_2$ and $Q_i \Rightarrow \Sigma_j P_{i,j}$.
  By IH, $N_2 \Rightarrow M_2^+$ and, moreover,
  $\Sigma_j P_{i,j} \Rightarrow Q_i^+ = \Sigma_{j,k} Q'_{i,j,k}$ where $Q'_{i,j,k}$ that are not sums and $P_{i,j} \Rightarrow \Sigma_k Q'_{i,j,k}$.
  Thus $N \Rightarrow \Sigma_{i,j,k}\bar{\tau}_{\alpha_i}(Q'_{i,j,k}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(M_2^+)) = M^+$.

- If $Q = \tau_\alpha(M)$:
  - If M is not an abstraction:
    Then $P = \tau_\alpha(N)$ for some N such that $M \Rightarrow N$.
    By IH, $N \Rightarrow M^+$ and thus $P \Rightarrow \lambda x.M^+ = Q^+$.
  - If $\alpha = a\to\alpha$ and $M = \lambda x.M'$:
    * Either $P = \tau_{a\to\alpha}(\lambda x.N)$ with $M \Rightarrow N$.
      By IH, $N \Rightarrow M'^+$ and $P \Rightarrow \tau_\alpha(M'^+[\bar{\epsilon}_a/x]) = Q^+$.
    * Or $P = \tau_\alpha(N[\bar{\epsilon}_a/x])$ with $M' \Rightarrow N$.
      By IH, $N \Rightarrow M'^+$ and $P \Rightarrow \tau_\alpha(M'^+[\bar{\epsilon}_a/x]) = Q^+$.
  - If $M = \Sigma_i\bar{\tau}_{\beta_i}(Q_i)$:
    * Either $N = \tau_\alpha(\Sigma_{i,j}\bar{\tau}_{\beta_i}(P'_{i,j}))$ with $Q_i = \Sigma_j P_{i,j}$ and $P_{i,j} \Rightarrow P'_{i,j}$.
      By IH, $P'_{i,j} \Rightarrow P^+_{i,j}$. Thus, $N \Rightarrow \Sigma_{\{i|\alpha\leq\beta_i\}}\Sigma_j P^+_{i,j} = \Sigma_{\{i|\alpha\leq\beta_i\}} Q_i^+ = Q^+$.
    * Or $N = \Sigma_{\{i|\alpha\leq\beta_i\}} Q'_i$ with $Q_i \Rightarrow Q'_i$.
      By IH, $Q'_i \Rightarrow Q_i^+$. Thus, $N \Rightarrow \Sigma_{i|\alpha\leq\beta_i} Q_i^+ = Q^+$.

- If $M = \Sigma_i M_i$:
  Then $N = \Sigma_i N_i$ with $M_i \Rightarrow N_i$.
  By IH, $N_i \Rightarrow M_i^+$ and $N \Rightarrow \Sigma_i M_i^+ = M^+$.

- If $M = \bar{\tau}_\alpha(\Sigma_i Q_i)$ where none of the Q_i are sums:
  Then we can only apply rules (P-$\bar{\tau}$+) and (P-$\cdot$+). Thus there are J and a surjective function $\phi : I \to J$ such that $N = \Sigma_{j\in J}\bar{\tau}_\alpha(\Sigma_{i\in\phi^{-1}(j)} P_i)$ and $Q_i \Rightarrow P_i$.
  By IH, $P_i \Rightarrow Q_i^+$ and $N \Rightarrow \Sigma_{i\in I}\bar{\tau}_\alpha(Q_i^+) = M^+$.

- If $Q = \Pi_{i\leq n}\Sigma_{j\leq k_i} Q_{ij}$ where none of the Q_{ij} are sums and where either $n \neq 1$ or one of the $k_i \neq 1$:
  Then there are, for all $i \leq n$, J_i and $\phi_i : [\![1, k_i]\!] \to J_i$ such that $P = \Sigma_{(t_i)_i\in(J_i)_i}\Pi_{i\,\ln\,n}\Sigma_{j|\phi(j)=t_i} P_{ij}$ with $Q_{ij} \Rightarrow P_{ij}$.
  By IH, $P_{ij} \Rightarrow Q_{ij}^+$ and $P \Rightarrow \Sigma_{j_1\leq k_1\ldots j_n\leq k_n}\Pi_{i\leq n} Q^+_{ij_i} = Q^+$.

□

Theorem 2.3.1.26 (Confluence). *The calculus $\Lambda_{\tau(E)}$ with the reduction $\to$ is confluent*

$$\begin{array}{ccc} M & \to^* & M_2 \\ \downarrow_* & \rightsquigarrow & \downarrow_* \\ M_1 & \to^* & M^+ \end{array}$$

$$\frac{M \to_h^* x}{M \Rightarrow_{st} x}\,(S\text{-}x) \qquad \frac{M \to_h^* M_1\ M_2 \qquad M_1 \Rightarrow_{st} N_1 \qquad M_2 \Rightarrow_{st} N_2}{M \Rightarrow_{st} N_1\ N_2}\,(S\text{-}@)$$

$$\frac{M \to_h^* \lambda x.M_0 \qquad M_0 \Rightarrow_{st} N_0}{M \Rightarrow_{st} \lambda x.N_0}\,(S\text{-}\lambda) \qquad \frac{M \to_h^* \Sigma_i \bar{\tau}_{\alpha_i}(P_i) \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{M \Rightarrow_{st} \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)}\,(S\text{-}\bar{\tau})$$

$$\frac{P \to_h^* \Sigma_i P_i \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{P \Rightarrow_{st} \Sigma_i Q_i}\,(S\text{-}+) \qquad \frac{P \to_h^* \Pi_i P_i \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{P \Rightarrow_{st} \Pi_i Q_i}\,(S\text{-}\cdot)$$

$$\frac{P \to_h^* \tau_\alpha(M) \qquad M \Rightarrow_{st} N}{P \Rightarrow_{st} \tau_\alpha(N)}\,(S\text{-}\tau)$$

Figure 2.9.: Definition of the standard reduction

Proof. By Lemma 2.3.1.25, $\Rightarrow$ is strongly confluent, *i.e.*, for any $M_1 \Leftarrow M \Rightarrow M_2$, we have $M_1 \Rightarrow M^+ \Leftarrow M_2$. By chasing diagrams, we obtain the confluence of $\Rightarrow$ and we conclude by Lemma 2.3.1.23 that state that $\Rightarrow^* = \to^*$. □

Standardization theorem

This section is dedicated to the proof of Theorem 2.3.1.29 that state a version of the standardization theorem of $\Lambda_{\tau(D)}$. The proof is directly inspired from Kashima's proof [Kas01].

Definition 2.3.1.27. *The* standard reduction*, denoted by $\Rightarrow_{st}$ is defined in Figure 2.9.*

Proposition 2.3.1.28. *We have the following inclusions:*

- $\Rightarrow_{st} \subseteq \to^*$,
- $id \subseteq \Rightarrow_{st}$
- $\to_h^* \subseteq \Rightarrow_{st}$,
- $\Rightarrow_{st} \subseteq \to_h^* \to_{\not h}^*$ *where $\to_{\not h}^*$ is the reflexive transitive closure of $\to_{\not h} = \to - \to_h$.*

Proof.
- The inclusion $\Rightarrow_{st} \subseteq \to^*$ is obtain by easy induction (using each time the transitivity on $\to_h^* \subseteq \to^*$ and on the corresponding contextual rule of Figure 2.3d applied on the inductive hypothesis).
- The inclusion $id \subseteq \Rightarrow_{st}$ derives from an easy induction using $id \subseteq \to_h^*$.
- The inclusion $\to_h^* \subseteq \Rightarrow_{st}$ is obtained from a case analysis and the inclusion $id \subseteq \Rightarrow_{st}$.
- Let $M, N \in \Lambda_{\tau(D)}$ (rep. $P, Q \in \boldsymbol{T}_{\tau(D)}$) such that $M \Rightarrow_{st} N$ (resp. $P \Rightarrow_{st} Q$), we will show that $M \to_h^* \to_{\not h}^* N$ (resp. $P \to_h^* \to_{\not h}^* Q$) by induction on N (rep. Q):
  - If $N = x$ with $M \to_h^* x$: trivial.

- If $N = \lambda x.N_0$, then $M \rightarrow^*_h \lambda x.M_0$ and $M_0 \Rightarrow_{st} N_0$. By IH $M_0 \rightarrow^*_h \rightarrow^*_{\not h} N_0$ so that Rule (H-$c\lambda$) gives $M \rightarrow^*_h \lambda x.M_0 \rightarrow^*_h \rightarrow^*_{\not h} \lambda x.N_0$.
- If $N = N_1\ N_2$, then $M \rightarrow^*_h M_1\ M_2$, $M_1 \Rightarrow_{st} N_1$ and $M_2 \Rightarrow_{st} N_2$. By IH $M_1 \rightarrow^*_h M'_1 \rightarrow^*_{\not h} N_1$ for some $M'_1 \in \Lambda_{\tau(D)}$.
  - If M'_1 is not an abstraction, then there is no abstraction in the sequence $M_1 \rightarrow_h \cdots \rightarrow_h M'_1$ and by Rule (H-$c@$), $M \rightarrow^*_h M_1\ M_2 \rightarrow^*_h M'_1\ M_2 \rightarrow^*_{\not h} N_1\ M_2$.
  - Otherwise, there is a first abstraction M''_1 such that $M_1 \rightarrow^*_h M''_1 \rightarrow^* M'_1$ with no abstraction in the sequence $M_1 \rightarrow_h \cdots \rightarrow_h M''_1$.
    In this case, by Rule (H-$c@$), $M \rightarrow^*_h M_1\ M_2 \rightarrow^*_h M''_1\ M_2 \rightarrow^*_{\not h} M'_1\ M_2 \rightarrow^*_{\not h} N_1\ M_2 \rightarrow^*_{\not h} N_1\ N_2$.
- If $Q = \tau_\alpha(N)$, then the argument is similar:
  There is M such that $P \rightarrow^*_h \tau_\alpha(M)$ and $M \Rightarrow_{st} N$. By IH, there is M' such that $M \rightarrow^*_h M' \rightarrow^*_{\not h} N$. Either M' is not an abstraction and since there is no abstraction among the sequence $M \rightarrow_h \cdots \rightarrow_h M'$, we have, by Rule ($H$-$c\tau$), that $P \rightarrow_h \tau_\alpha(M) \rightarrow^*_h \tau_\alpha(M') \rightarrow^*_{\not h} \tau_\alpha(N)$. Otherwise there is a first abstraction M'' in the sequence $M \rightarrow_h \cdot \rightarrow_h M'' \rightarrow_h \cdots \rightarrow_h M'$, and we have, by Rule ($H$-$c\tau$), that $P \rightarrow_h \tau_\alpha(M) \rightarrow^*_h \tau_\alpha(M'') \rightarrow^*_{\not h} \tau_\alpha(N)$.
- If $N = \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$, there are $(P_i)_i$ such that $M \rightarrow^*_h \Sigma_i \bar{\tau}_{\alpha_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \rightarrow^*_h P'_i \rightarrow^*_{\not h} Q_i$ for some $P'_i \in \Lambda_{\tau(D)}$. For all i, if P'_i is not a sum (with $n \neq 1$ arguments) we set $P''_i = P'_i$, otherwise there is a first sum P''_i such that $P_i \rightarrow^*_h P''_i \rightarrow^*_h P'_i$.
  Then, using Rule (H-$c\bar{\tau}$) we have, for all i, $\bar{\tau}_{\alpha_i}(P_i) \rightarrow^*_h \bar{\tau}_{\alpha_i}(P''_i) \rightarrow^*_{\not h} \bar{\tau}_{\alpha_i}(Q_i)$.
  Thus, using Rule (H-cs), we have $M \rightarrow^*_h \Sigma_i \bar{\tau}_{\alpha_i}(P_i) \rightarrow^*_h \Sigma_i \bar{\tau}_{\alpha_i}(P''_i) \rightarrow^*_{\not h} \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$.
- If $Q = \Pi_i(Q_i)$ then the argument is similar:
  There are $(P_i)_i$ such that $P \rightarrow^*_h \Pi_i P_i$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \rightarrow^*_h P'_i \rightarrow^*_{\not h} Q_i$ for some $P'_i \in \Lambda_{\tau(D)}$. For all i, if P'_i is not a sum (with $n \neq 1$ arguments) we set $P''_i = P'_i$, otherwise there is a first sum P''_i such that $P_i \rightarrow^*_h P''_i \rightarrow^*_h P'_i$.
  Then, using Rule (H-$c\cdot$), we have $P \rightarrow^*_h \Pi_i P_i \rightarrow^*_h \Sigma_i P''_i \rightarrow^*_{\not h} \Sigma_i Q_i$.
- If $Q = \Sigma_i(Q_i)$, there are $(P_i)_i$ such that $P \rightarrow^*_h \Sigma_i P_i$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \rightarrow^*_h P'_i \rightarrow^*_{\not h} Q_i$ and, by Rule (H-$\cdot$), $\Sigma_i P_i \rightarrow^*_h \Sigma_i P'_i \rightarrow^*_{\not h} \Sigma_i Q_i$.

□

Theorem 2.3.1.29 (Standardization). *For any reduction $M \rightarrow^* N$ (resp. $P \rightarrow^* Q$), there is a standard reduction $M \Rightarrow_{st} N$ (resp. $P \Rightarrow_{st} Q$). In particular, any term M (resp. test Q) head converges iff it reduces to a may head-normal form:*

$$M\Downarrow^{\mathrm{h}} \quad\Leftrightarrow\quad \exists N \in mhnf, M \rightarrow^* N' \qquad\qquad P\Downarrow^{\mathrm{h}} \quad\Leftrightarrow\quad \exists Q \in mhnf, P \rightarrow^* Q'.$$

Proof. By applying successively Lemma 2.3.1.31. □

Lemma 2.3.1.30. *1. If $P \rightarrow^*_h \Sigma_{j\le k} Q_j$, then there is $(P_j)_{j\le k}$ such that $\bar{\tau}_\alpha(P) \rightarrow^*_h \Sigma_{j\le k} \bar{\tau}_\alpha(P_j)$ with $P_j \rightarrow^*_h Q_j$ for all $j \le k$.*

2. Similarly, if $P_i \rightarrow^*_h \Sigma_{j\le k_i} Q_{ij}$, then there is $(P_{ij})_{i,j}$ such that $\Pi_i P_i \rightarrow^*_h \Sigma_{(j_i)_i} \Pi_i P_{i,j}$ with $P_j \rightarrow^*_h Q_j$ for all $j \le k$.

3. *Similarly, if $M \to_h^* \Sigma_{j\leq k}\bar\tau_{\beta_j}(Q_j)$, then there is $(P_j)_{j\leq k}$ such that $\tau_\alpha(M) \to_h^* \Sigma_{\{j|\beta_j\geq\alpha\}}P_j$ with $P_j \to_h^* Q_j$ for all $j \leq k$.*

Proof. The proof follow the exact same pattern for each cases.

1. By induction on the lexicographicaly ordered (n, P) where n is the length $P \to_h^n \Sigma_{j\leq k}Q_j$:
   - If $n = 0$ then this is Rule $(\bar\tau+)$.
   - If $P = \Sigma_{i\leq k'}P'_i$ for $k' \neq 1$, there is a surjective $\phi : [1,k] \to [1,k']$ such that $P'_i \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)}Q_j$ with $n = \Sigma_i n_i$. By IH on each P'_i, there is $(P_j)_j$ such that, for all i, $\bar\tau_\alpha(P_i) \to_h^* \Sigma_{j\in\phi^{-1}(i)}\bar\tau_\alpha(P_j)$ with $P_j \to_h^* Q_j$. Thus $\bar\tau_\alpha(P) \xrightarrow{\bar\tau+}_h \Sigma_{i\leq k'}\bar\tau_\alpha(P_i) \to_h^* \Sigma_{i\leq k'}\Sigma_{j\in\phi^{-1}(i)}\bar\tau_\alpha(P_j)$.
   - Otherwise, we can decompose the reduction by $P \to_h P' \to_h^{n-1} \Sigma_{j\leq k}Q_j$. Since P is not a sum we can apply the rule $H\text{-}c\bar\tau$ so that $\bar\tau_\alpha(P) \to_h \bar\tau_\alpha(P')$ and we conclude since by IH, $\bar\tau_\alpha(P') \to_h^* \Sigma_{j\leq k}\bar\tau_\alpha(P_j)$.
2. We will prove the following simplest statement that trivially imply the main one:

   If $P \to_h^* \Sigma_{j\leq k}Q_j$, then there is (P_j) such that $Q{\cdot}P \to_h^* \Sigma_j(Q{\cdot}Q_j)$.

   We proceed by induction on the lexicographicaly ordered (n, P) where n is the length $P \to_h^n \Sigma_{j\leq k}Q_j$:
   - If $n = 0$ then this is Rule $(\cdot+)$.
   - If $P = \Sigma_{i\leq k'}P'_i$ for $k' \neq 1$, there is a surjective $\phi : [1,k] \to [1,k']$ such that $P'_i \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)}Q_j$ with $n = \Sigma_i n_i$. By IH on each P'_i, there is $(P_j)_j$ such that, for all i, $(Q{\cdot}P_i) \to_h^* \Sigma_{j\in\phi^{-1}(i)}(Q{\cdot}P_j)$ with $P_j \to_h^* Q_j$. Thus $Q{\cdot}P \xrightarrow{\cdot+}_h \Sigma_{i\leq k'}(Q{\cdot}P_i) \to_h^* \Sigma_{i\leq k'}\Sigma_{j\in\phi^{-1}(i)}(Q{\cdot}P_j)$.
   - Otherwise, we can decompose the reduction by $P \to_h P' \to_h^{n-1} \Sigma_{j\leq k}Q_j$. Since P is not a sum we can apply the rule $H\text{-}c{\cdot}$ so that $Q{\cdot}P \to_h Q{\cdot}P'$ and we conclude since by IH, $Q{\cdot}P' \to_h^* \Sigma_{j\leq k}{\cdot}P_j$.
3. By induction on the lexicographicaly ordered (n, M) where n is the length $M \to_h^n \Sigma_{j\leq k}\bar\tau_\alpha(Q_j)$:
   - If $n = 0$ then this is Rule $(\tau\bar\tau)$.
   - If $M = \Sigma_{i\leq k'}\bar\tau_{\gamma_i}(P'_i)$ for $k' \neq 1$, there is a surjective $\phi : [1,k] \to [1,k']$ such that $\bar\tau_{\gamma_i}(P'_i) \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)}\bar\tau_{\beta_j}Q_j$ with $n = \Sigma_i n_i$. By IH on each $\bar\tau_{\gamma_i}(P'_i)$, there is $(P_j)_j$ such that, for all i, $\tau_\alpha(\bar\tau_{\gamma_i}(P'_i)) \to_h^* \Sigma_{\{j\in\phi^{-1}(i)|\alpha\leq\beta_j\}}P_j$ with $P_j \to_h^* Q_j$. Since the only head reduction that can be applyed on each $\tau_\alpha(\bar\tau_{\gamma_i}(P'_i))$ is $(H\text{-}\tau\bar\tau)$, we have that $\tau_\alpha(M) \to_h \Sigma_{\{1|\alpha\leq\gamma_i\}}P_i \to_h^* \Sigma_j Q_j$.
   - If $M = \lambda x.M'$: impossible since $M \to^* \Sigma_j\bar\tau_{\beta_j}(Q_j)$ and no rule can erase a λ in first position.
   - Otherwise, we can decompose the reduction by $M \to_h M' \to_h^{n-1} \Sigma_{j\leq k}\bar\tau_{\beta_j}(Q_j)$. Since P is not an abstraction we can apply the rule $(H\text{-}\tau)$ so that $\tau_\alpha(M) \to_h \tau_\alpha(M')$ and we conclude since by IH, $\tau_\alpha(M') \to_h^* \Sigma_{\{j|\beta_j\geq\alpha\}}P_j$.

□

Lemma 2.3.1.31. *For all $M, N, N' \in \Lambda_{\tau(D)}$ (resp. $P, Q, Q' \in \boldsymbol{T}_{\tau(D)}$) such that $M \Rightarrow_{st} N \to N'$ (resp. $P \Rightarrow_{st} Q \to Q'$), there is M' (resp. P') such that $M \Rightarrow_{st} N'$ (resp. $P \Rightarrow_{st} Q'$).*

Proof. We proceed by structural induction on N:

- If $N = x$: this is impossible since x is a normal form.

- If $N = \lambda x.N_0$ then $N_0 \to N'_0$ with $N' = \lambda x.N'_0$. By definition of $\Rightarrow_{st}$, $M \to_h^* \lambda x.M_0$ and $M_0 \Rightarrow_{st} N_0$. By IH, $M_0 \Rightarrow_{st} N'_0$, thus $M \Rightarrow_{st} \lambda x.N'_0$.

- If $N = \mathbf{0}$: this is impossible since $\mathbf{0}$ is a normal form.

- If $N = \bar{\tau}_\alpha(Q)$ then the only rule that change the form of the expression is $(\bar{\tau}+)$ applied in head position:

  - Either $N = \bar{\tau}_\alpha(\Sigma_j Q_j) \xrightarrow{\bar{\tau}+}_h N' = \Sigma_j \bar{\tau}_\alpha(Q_j)$. By definition of $\Rightarrow_{st}$, $M \to_h^* \bar{\tau}_\alpha(P)$ and $P \to_h^* \Sigma_j P_j$ with $P_j \Rightarrow_{st} Q_j$. Thus, by Lemma 2.3.1.30, there is $(P'_j)_j$ such that $M \to_h^* \Sigma_j \bar{\tau}_\alpha(P'_j)$ with $P'_j \to^* P_j \Rightarrow_{st} Q_j$, so that $M \Rightarrow_{st} N'$.
  - Otherwise, $Q \to Q'$ and $N' = \bar{\tau}_\alpha(Q')$. In this case, since $M \to_h^* \bar{\tau}_\alpha(P)$ and $P \Rightarrow_{st} Q \to Q'$, we can apply the IH so that $P \Rightarrow_{st} Q'$ and $M \Rightarrow_{st} \bar{\tau}_\alpha(Q')$.

- If $N = \Sigma_{i\leq n} N_i$ for $n > 0$. Then, modulo commutativity of the sum, we can assume that $N_n \to N'_n$, so that $N' = \Sigma_{i<n} N_i + N'_n$. By definition of $\Rightarrow_{st}$, $M \to_h^* \Sigma_{i\leq n} M_i$ with $M_i \Rightarrow_{st} N_i$. By induction hypothesis, $M_n \Rightarrow_{st} N'_n$ and we can set $M \Rightarrow_{st} N'$.

- If $N = N_1\ N_2$, then $M \to_h^* M_1\ M_2$ with $M_1 \Rightarrow_{st} N_1$ and $M_2 \Rightarrow_{st} N_2$. There are different cases:

  - Either $N_1 \to_h N'_1$ and $N' = N'_1\ N_2$. In this case, the IH on $M_1 \Rightarrow_{st} N_1 \to_h N'_1$ gives $M_1 \Rightarrow_{st} N'_1$, so that $M \Rightarrow_{st} N'$.
  - Or $N_2 \to_h N'_2$ and $N' = N_1\ N'_2$. In this case, the IH on $M_2 \Rightarrow_{st} N_2 \to_h N'_2$ gives $M_2 \Rightarrow_{st} N'_2$, so that $M \Rightarrow_{st} N'$.
  - Or $N_1 = \lambda x.N_0$ and $N' = N_0[N_2/x]$. By definition of $\Rightarrow_{st}$, $M_1 \to_h^* \lambda x.M_0$ with $M_0 \Rightarrow_{st} N_0$. By easy induction on $\Rightarrow_{st}$, one can see that $M_0[M_2/x] \Rightarrow_{st} N_0[N_2/x]$, we can conclude since $\to^* \Rightarrow_{st} \subseteq \Rightarrow_{st}$.
  - Or $N_1 = \Sigma_{i\leq n} \bar{\tau}_{a_i\to\alpha_i}(Q_i)$ and $N' = \Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(Q_i \cdot \Pi_{\gamma\in a_i} \tau_\gamma(N_2))$. By definition of $\Rightarrow_{st}$, $M_1 \to_h^* \Sigma_{i\leq n} \bar{\tau}_{a_i\to\alpha_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By definition of $\Rightarrow_{st}$, one can see that $\Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(P_i \cdot \Pi_{\gamma\in a_i} \tau_\gamma(M_2)) \Rightarrow_{st} \Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(\Pi_{\gamma\in a_i} \tau_\gamma(N_2))$ so that $M \Rightarrow_{st} N$.

- If $Q = \tau_{a\to\alpha}(N)$, then $P \to_h^* \tau_{a\to\alpha}(M)$ with $M \Rightarrow_{st} N$ and there are different cases:

  - Either $N \to N'$ and $Q' = \tau_{a\to\alpha}(N')$. In this case, the IH on $M \Rightarrow_{st} N \to N'$ gives $M \Rightarrow_{st} N'$, so that $P \Rightarrow_{st} Q'$.
  - Or $N = \lambda x.N_0$ and $Q' = \tau_\alpha(N_0[\bar{\boldsymbol{\epsilon}}_a/x])$. By definition of $\Rightarrow_{st}$, $M \to_h^* \lambda x.M_0$ with $M_0 \Rightarrow_{st} N_0$. By easy induction on $\Rightarrow_{st}$, one can see that $M_0[\bar{\boldsymbol{\epsilon}}_a/x] \Rightarrow_{st} N_0[\bar{\boldsymbol{\epsilon}}_a/x]$, we can conclude since $\to^* \Rightarrow_{st} \subseteq \Rightarrow_{st}$.
  - Or $N = \Sigma_{i\leq n} \bar{\tau}_{\beta_i}(Q_i)$ and $N' = \Sigma_{i\leq n | \beta_i\geq\alpha} Q_i$. By definition of $\Rightarrow_{st}$, $M \to_h^* \Sigma_{i\leq n} \bar{\tau}_{\beta_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By Lemma 2.3.1.30, there is $(P'_i)_i$ such that $\tau_\alpha(M) \to_h^* \Sigma_{i\leq n|\beta_i\geq\alpha} P'_i$ and $P'_i \Rightarrow_{st} Q_i$ so that $P \Rightarrow_{st} Q'$.

- If $Q = \Sigma_{i\leq n} Q_i$ then (up to commutativity of the sum) $Q_n \to Q'_n$ and $Q' = \Sigma_{i<n} Q_i + Q'_n$. By definition of $\Rightarrow_{st}$, $P \to_h^* \Sigma_{i\leq n} P_i$ with $P_i \Rightarrow_{st} Q_i$ for all i. By IH on $P_n \Rightarrow_{st} Q_n \to Q'_n$, $P_n \Rightarrow_{st} Q'_n$ so that $P \Rightarrow_{st} Q'$.

- If $Q = \Pi_{i\leq n} Q_i$ then the only rule that change the form of the expression is $(\cdot+)$ applied in head position, there are two cases:

- Either $Q = \Pi_i\Sigma_{j\leq k_i}Q_{ij} \xrightarrow{\cdot +}_h Q' = \Sigma_{(j_i)_i}\Pi_i Q_{ij_i}$. By definition of $\Rightarrow_{st}$ (used 2 times), $P \to_h^* \Pi_i P_i$ and $P_i \to_h^* \Sigma_{j\leq k_i}P_{ij}$ with $P_{ij} \Rightarrow_{st} Q_{ij}$ for all i, j. Thus, by Lemma 2.3.1.30, there is $(P'_{ij})_{ij}$ such that $P \to_h^* \Sigma_{(j_i)_i}\Pi_i P'_{ij}$ with $P'_{ij} \to^* P_{ij} \Rightarrow_{st} Q_{ij}$, so that $M \Rightarrow_{st} N'$.
- Otherwise (and up to commutativity of the sum), $Q_n \to Q'_n$ and $Q' = \Pi_{i<n}Q_i{\cdot}Q'_i$). By definition of $\Rightarrow_{st}$, $P \to_h^* \Pi_i P_i$ and $P_i \Rightarrow_{st} Q_i$. We can apply the IH on $P_n \Rightarrow_{st} Q_n \to_h Q'_n$ so that $P \Rightarrow_{st} Q'$.

□

Invariance of the convergence

We will see in this section that the head convergence in at most n steps is invariant wrt the reduction. This means that performing a non-head reduction can only reduce the length of convergence.

Theorem 2.3.1.32 (Invariance of the convergence). *For any terms $M \to N$ (resp. test $P \to Q$) and any $n \in \mathbb{N}$:*

$$M{\Downarrow}^{\mathrm{h}}_n \quad \Rightarrow \quad N{\Downarrow}^{\mathrm{h}}_n \qquad\qquad P{\Downarrow}^{\mathrm{h}}_n \quad \Rightarrow \quad Q{\Downarrow}^{\mathrm{h}}_n$$

Proof. By recursive invocations of Lemma 2.3.1.34, for any (n, k) we can close the diagram:

$$\begin{array}{ccc} M & \to_h^n & M_1 \\ \downarrow_k & \rightsquigarrow & \downarrow_* \\ M_2 & \to_h^{\leq n} & M' \end{array} \qquad\qquad \begin{array}{ccc} Q & \to_h & Q_1 \\ \downarrow_k & \rightsquigarrow & \downarrow_* \\ Q_2 & \to_h^{\leq n} & Q' \end{array}$$

where $\to_h^{\leq n} = \bigcup_{i\leq n} \to_h^i$.
In particular, if $M \to_h^* M'$ with $M' \in mhnf$ (*i.e.* M converges), since $M \to N$, there is N_0 such that $N \to_h^{\leq n} N_0$ and $N \to^* N_0$, from the last we deduce that $N_0 \in mhnf$ and conclude. The same goes for tests. □

Remark 2.3.1.33. *We have seen in Proposition A.2.1.37 that this property allows to coinductively treat head-divergence modulo non-head reductions.*

In order to prove this theorem we need a stronger notion of confluence for the cases where one of the reduction is a head reduction.

Lemma 2.3.1.34. *Any pick, $M \to_h M_1$ and $M \to^* M_2$ (resp. $Q \to_h Q_1$ and $Q \to^* Q_2$), between a head reduction and any reduction verify the diamond:*

$$\begin{array}{ccc} M & \to_h & M_1 \\ \downarrow & \rightsquigarrow & \downarrow_* \\ M_2 & \to_h^? & M' \end{array} \qquad\qquad \begin{array}{ccc} Q & \to_h & Q_1 \\ \downarrow & \rightsquigarrow & \downarrow_* \\ Q_2 & \to_h^? & Q' \end{array}$$

where $\to_h^? := (\to_h \cup id)$ is either a head reduction or an equality.

Proof. By induction on M and Q:

- If $M = x$ or $M = 0$: it is impossible that $M \to_h M_1$.
- If $M = \lambda x.N$: then $M_1 = \lambda x.N_1$ and $M_2 = \lambda x.N_2$ so that $N_1 \, {}_h\!\leftarrow N \to N_2$, thus, by induction, there is N' such that $N_1 \to^* N' \, {}_h^?\!\leftarrow N_2$, finally we can fix $M' = \lambda x.N'$.
- If $M = \Sigma_{i \le n+2} N^i$: then, modulo commutativity of the sum, $M_1 = N_1^{n+2} + \Sigma_{i \le n+1} N^i$ with $N^{n+2} \to_h N_1^{n+2}$.
  - Either (modulo commutativity of the sum), $M_2 = N_2^{n+2} + \Sigma_{i \le n+1} N^i$ with $N^{n+2} \to N_2^{n+2}$ and by induction there is $N_1^{n+2} \to^* N_0^{n+2} \, {}_h^?\!\leftarrow N_2^{n+2}$ so that $M' = N_0^{n+2} + \Sigma_{i \le n+1} N^i$.
  - Or (modulo commutativity of the sum), $M_2 = N^{n+2} + N_2^{n+1} + \Sigma_{i \le n+1} N^i$ with $N^{n+1} \to N_1^{n+1}$, so that $M' = N_1^{n+2} + N_2^{n+1} + \Sigma_{i \le n+1} N^i$.
- If $M = \bar{\tau}_\alpha(Q)$ with Q that is not a sum: then $M_1 = \bar{\tau}_{\alpha_i}(Q_1)$ and $M_2 = \bar{\tau}_\alpha(Q_2)$ with $Q_1 \, {}_h\!\leftarrow Q \to Q_2$, thus, by induction, there is Q' such that $Q_1 \to^* Q' \, {}_h^?\!\leftarrow Q_2$, finally we can fix $M' = \bar{\tau}_\alpha(Q')$.
- If $M = \bar{\tau}_\alpha(\Sigma_{i \le n+1} Q^i)$ and $M_1 = \Sigma_{i \le n+1} \bar{\tau}_\alpha(Q^i)$:
  - Either $M_2 = \bar{\tau}_\alpha(Q_2^{n+1} \Sigma_{i \le n} Q^i)$ and $M' = \bar{\tau}_\alpha(Q^{n+1}) \Sigma_{i \le n} \bar{\tau}_\alpha(Q^i)$.
  - Or $Q^i = \Sigma_j P^{i,j}$ and $M_2 = \Sigma_j \bar{\tau}_\alpha(P^{i,j})$, then $M' = \Sigma_{i,j} \bar{\tau}_\alpha(P^{i,j})$.
- If $M = N\, L$:
  - If N is not an abstraction: then $M_1 = N_1\, L$ with $N \to_h N_1$. Moreover
    - Either $M_2 = N_2\, L$ with $N \to N_2$ and N_2 that is not an abstraction. By induction there is N' such that $N_1 \to^* N'^?_h \leftarrow N_2$, and $M' = N'\, L$.
    - Or $M_2 = (\lambda x.N_2)\, L$ with $N \to N_2$ and N_2 that is an abstraction: since N is not an abstraction, this can only be the result of a (β) or a $\bar{\tau}$ reduction in outermost position in N. In both cases, necessary $M_1 = M_2$.
    - Or $M_2 = N\, L_2$ with $L \to L_2$: then $M' = N_1\, L_2$.
  - If $N = \lambda x.N'$: then $M_1 = N'[L/x]$ and
    - Either $M' = M_2 = M_1$.
    - Or $M_2 = \lambda x.N_2'\, L$ with $N' \to N_2$, thus $M' = N_2'[L/x]$.
  - If $N = \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$: idem.
- If $Q = \tau_\alpha(M)$:
  - If M is not an abstraction: then $Q_1 = \tau_\alpha(M_1)$ and $Q_2 = \tau_\alpha(M_2)$ with $M_1 \, {}_h\!\leftarrow M \to M_2$ and by induction hypothesis, there is M' so that $M_1 \to^* M'^?_h \leftarrow M_2$.
    - Either M_2 is not an abstraction and $Q' = \tau_\alpha(M')$.

* Or $M \to M_2$ is an abstraction created by a (β) or a $(\bar{\tau})$ outermost reduction. In both cases, necessary $M_1 = M_2$.

– If $M = \lambda x.N$: then $Q_1 = \tau_{\alpha'}(N[\bar{\epsilon}_a/x])$ and

  * Either $Q_2 = Q_1 = Q'$.
  * Or $Q_2 = \tau_\alpha \lambda x.N_2$ with $N \to N_2$, thus $Q' = \tau_{\alpha'}(N_2[\bar{\epsilon}_a/x])$.

– If $M = \Sigma_{i\leq n+1}\bar{\tau}_{\beta_i}(P^i)$: then $Q_1 = \Sigma_{\{i\leq n+1|\alpha\leq\beta_i\}}P^i$ and

  * Either $Q_2 = Q_1 = Q'$.
  * Or $Q_2 = \tau_\alpha\Big(\Sigma_{i\leq n}\bar{\tau}_{\alpha_i}(P^i) + \Sigma_j\bar{\tau}_{\beta_n}(R^j)\Big)$ with $\bar{\tau}_{\alpha_{n+1}}(P^{n+1}) \to \Sigma_j\bar{\tau}_{\beta_j}(R^j)$, thus $Q' = \Sigma_{\{j|\alpha\leq\beta_n\}}R^j + \Sigma_{\{i\leq n|\alpha\leq\beta_i\}}P^i$.

- If $Q = P{+}R$: then, modulo commutativity of the sum, $Q_1 = P_1{+}R$ with $P \to_h P_1$.
  – Either $Q_2 = P_2{+}R$ with $P \to P_2$ and the induction hypothesis gives P' so that $M' = P'{+}R$.
  – Or $Q_2 = P{+}R_2$ and $M' = P_1{+}R_2$.
- If $Q = P{\cdot}R$: same as for $Q = P{+}R$ except if a rule $(\cdot{+})$ is used in outermost position. In this case, either only one of the reduction is a $(\cdot{+})$ and the two reductions are independents, or both of them are $(\cdot{+})$, which is similar to $M = \bar{\tau}_\alpha(\Sigma_{i\leq n+1}Q^i)$.

□

2.3.2. Hyperimmunity implies full abstraction

In this subsection we show that if D is sensible for $\Lambda_{\tau(D)}$ and is hyperimmune, D is inequationally fully abstract for Λ, that is Theorem 2.3.2.4. We use the full abstraction of D for $\Lambda_{\tau(D)}$ of Theorem 2.3.1.20 (or rather its technical counterpart: Theorem 2.3.1.21) in order to express the problem in a purely syntactical form. We prove Theorem 2.3.2.3 stating the (inequational) full abstraction of D for Λ. The purpose of the second subsection is to prove Lemma 2.3.2.2, which is a restricted version of Theorem 2.3.2.3. This lemma states that $N \sqsupseteq_{\mathcal{H}^*} x$ implies that $[\![M]\!]^{x_0} \supseteq [\![x]\!]^x$; this case being the key-point where the hypothesis of hyperimmunity is used. But, before that, we need the technical Lemma 2.3.2.1 in order to refute the operational equivalence between two λ-terms in easy cases.

Technical lemma

Lemma 2.3.2.1. *Let $M = \lambda x_1...x_n.y\ M_1 \cdots M_k \in \Lambda$ and let $N = \lambda x_1...x_{n'}.y'\ N_1 \cdots N_{k'} \in \Lambda$ be such that $M \sqsubseteq_{\mathcal{H}^*} N$. Then:*

1. *$y = y'$,*
2. *$n - k = n' - k'$,*
3. *if $i \leq k$ and $i \leq k'$ then $M_i \sqsubseteq_{\mathcal{H}^*} N_i$,*
4. *if $i > k$ and $i \leq k'$ then $x_{i-k} \sqsubseteq_{\mathcal{H}^*} N_i$,*

5. *if* $i \leq k$ *and* $i > k'$ *then* $M_{i-k} \sqsubseteq_{\mathcal{H}^*} x_i$.

Proof. From each $i \leq 5$, assuming statements (1)...(i-1) and refuting statement (i), we can exhibit a context $C \in \Lambda^{(\!|\cdot|\!)}$ such that $C[\![M]\!]\Downarrow^{\mathrm{h}}$ and $C[\![N]\!]\Uparrow^{\mathrm{h}}$.

In the following, $M = \lambda x_1...x_n.y\ M_1 \cdots M_k$ and $N = \lambda x_1...x_{n'}.y'\ N_1 \cdots N_{k'}$.

If $y \neq y'$, then $M \not\sqsubseteq_{\mathcal{H}^*} N$:

- If y' is free in N then by setting $C(\!|.|\!) = (\lambda y'.(\!|.|\!))\ \Omega$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y' = x_j$ for $j \leq n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1\ \cdots\ x_{j-1}\ \Omega$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

Now we suppose that $M = \lambda x_1...x_n.y\ M_1 \cdots M_k$ and $N = \lambda x_1...x_{n'}.y\ N_1 \cdots N_{k'}$.

If $n - k \neq n' - k'$, then $M \not\sqsubseteq_{\mathcal{H}^*} N$:

- If y is free in N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots x_{n'+k})\ (\lambda z_1...z_{k'+k}u.u)\ \mathbf{\Omega}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$:
- If $y = x_j$ for $j \leq n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k'+k}u.u)\ x_{i+1} \cdots x_{n'+k}\ \mathbf{\Omega}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

Now we suppose that $n - k = n' - k'$.

If there is i such that $i \leq k$, $i \leq k'$ and $M_i \not\sqsubseteq_{\mathcal{H}^*} N_i$ then $M \sqsubseteq_{\mathcal{H}^*} i$, by hypothesis, there is $C'(\!|.|\!)$ such that $C'(\!|M_i|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|N_i|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!))\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \leq n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

If there is i such that $i > k$, $i \leq k'$ and $x_{i-k} \not\sqsubseteq_{\mathcal{H}^*} N_i$ then $M \sqsubseteq_{\mathcal{H}^*} N$, by hypothesis, there is $C'(\!|.|\!)$ such that $C'(\!|x_{i-k}|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|N_i|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots\ x_{n+k})\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \leq n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))\ x_{j+1} \cdots\ x_{n+k}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

If there is i such that $i \leq k$, $i > k'$ and $M_i \not\sqsubseteq_{\mathcal{H}^*} x_{i-k'}$ then $M \sqsubseteq_{\mathcal{H}^*} N$, by hypothesis, there is $C'(\!|.|\!)$ such that $C'(\!|M_i|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|x_{i-k'}|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots\ x_{n+k})\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \leq n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))\ x_{j+1} \cdots\ x_{n+k}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

□

The key-lemma

From now on, we consider an extensional K-model D that is both hyperimmune and sensible for $\Lambda_{\tau(D)}$.

The following lemma is a key lemma that introduces the hyperimmunity in the picture:

Lemma 2.3.2.2. *Let $\alpha \in D$ and $a_0, \dots, a_k \in \mathcal{A}_f(D)$ be such that $\alpha \in a_0$.*
Let $N \in \Lambda$ and $x_0, \dots, x_k$ be such that $\tau_\alpha(N[s])\Uparrow^{\mathrm{h}}$ with $s = [\bar{\epsilon}_{a_0}/x_0, \dots, \bar{\epsilon}_{a_k}/x_k]$. Then $N \not\sqsupseteq_{\mathcal{H}^*} x_0$.

Proof. We define the recursive function $g_{N'}$ for any $N' \in \Lambda$ such that $N' \sqsupseteq_{\mathcal{H}^*} x_0$, it is done by recursivelly defining $g_{N'}(k)$ for $k \in \mathbb{N}$:
Since $N' \sqsupseteq_{\mathcal{H}^*} x_0$, N' is converging, and by Lemma 2.3.2.1 $N' \to_h^* \lambda y_1...y_n.x_0\ N_1 \cdots N_n$ with $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ for all $m \leq n$. We then define $g_{N'}(0) = n$ and $g_{N'}(k+1) = max_{i\leq n} g_{N_i}(k)$.
We will show that assuming $N \sqsupseteq_{\mathcal{H}^*} x_0$ contradicts the hyperimmunity of D by showing that:

There exists $(\alpha_n)_{n\geq 0}$ with $\alpha_0 = \alpha$ and for all n, $\alpha_n = a_1^n \to \cdots a_{g_N(n)}^n \to \alpha'_n$ and $\alpha_{n+1} \in \bigcup_{i\leq g_N(n)} a_i^n$.

We are constructing $(\alpha_n)_n$ by co-induction.
Since $N \sqsupseteq_{\mathcal{H}^*} x_0$, it is converging, and by Lemma 2.3.2.1, $N \to^* \lambda y_1...y_n.x_0\ N_1 \cdots N_n$ with $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ for all $m \leq n$.
We will assume that $\alpha = b_1 \to \cdots \to b_n \to \alpha'$ and $a_0 = \{\alpha, \beta_1, \dots, \beta_t\}$ with $\beta_i = c_1^i \to \cdots \to c_n^i \to \beta'_i$ (always possible since "$\to$" is a bijection).
Then

$$
\begin{aligned}
\tau_\alpha(N[s]) &\to^* \tau_\alpha(\lambda y_1...y_n.\bar{\epsilon}_{a_0}\ N_1[s] \cdots N_n[s]) \\
&\overset{\tau}{\to}{}_h^* \tau_{\alpha'}(\bar{\epsilon}_{a_0}\ N_1[s,s'] \cdots N_n[s,s']) \\
&\overset{Ex2.3.1.3}{\longrightarrow}{}^* \tau_{\alpha'}(\Sigma_{d_1\to\cdots d_n\to\delta\in a_0} \bar{\tau}_\delta(\Pi_{m\leq n}\Pi_{\gamma\in d_m}\tau_\gamma(N_m[s,s']))) \\
&\overset{\tau\bar{\tau}}{\to}{}_h \Pi_{m\leq n}\Pi_{\gamma\in b_m}\tau_\gamma(N_m[s,s']) \quad + \quad \Sigma_{\{i\leq t|\alpha'\leq\beta'_i\}}\Pi_{m\leq n}\Pi_{\gamma\in c_m^i}\tau_\gamma(N_m[s,s'])
\end{aligned}
$$

with $s' = [\bar{\epsilon}_{b_1}/y_1, \dots, \bar{\epsilon}_{b_n}/y_n]$.
Since $\tau_\alpha(N[s])$ diverges, by standardization theorem (Th. 2.3.1.29), the test $\Pi_{m\leq n}\Pi_{\gamma\in b_m}\tau_\gamma(N_m[s,s'])$ diverges. In particular there is $m \leq n$ and $\gamma \in b_m$ such that $\tau_\gamma(N_m[s,s'])$ diverges.
Since $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ and $\tau_\gamma(N_m[s,s'])\Uparrow^{\mathrm{h}}$, the co-induction gives $(\gamma_k)_k$ such that $\gamma_0 = \gamma$ and for all k, $\gamma_k = c_1^k \to \cdots c_{g_{M_m}(k)}^k \to \gamma'_k$ and $\gamma_{k+1} \in \bigcup_{i\leq g_{M_m}(k)} a_i^k$. In this case we can define $(\alpha_k)_k$ as follows:

$$\alpha_0 = \alpha \qquad\qquad \forall k,\ \alpha_{k+1} = \gamma_k$$

This is sufficient since:

$$m \leq n = g_N(0) \qquad\qquad g_{M_m}(k) \leq sup_{j\leq n} g_{M_j}(k) = g_N(k+1)$$

□

Inequational completeness

Theorem 2.3.2.3 (Inequational completeness). *For all $M, N \in \Lambda$,*
if there exists $(a_0...a_k, \alpha) \in [\![M]\!]^{x_0...x_k}$ such that $(a_0...a_k, \alpha) \notin [\![N]\!]^{x_0...x_k}$, then $M \not\sqsubseteq_{\mathcal{H}^*} N$.

Proof. We will prove the equivalent (by Theorem 2.3.1.21) statement:

Let $\alpha \in D$ and $a_0, \dots, a_k \in \mathcal{A}_f(D)$.
Let a set of variables $\{x_0, \dots, x_k\} \supseteq \mathrm{FV}(M)$, and let $[s] = [\bar{\epsilon}_{a_0}/x_0 \cdots \bar{\epsilon}_{a_k}/x_k]$.
If[9] $\tau_\alpha(M[s])\Downarrow_l^{\mathrm{h}}$ and $\tau_\alpha(N[s])\Uparrow^{\mathrm{h}}$ then $M \not\sqsubseteq_{\mathcal{H}^*} N$.

[9] Recall that $M\Downarrow_l^{\mathrm{h}}$ means that M head converges in at most l steps

The statement is proven by induction on the length l of the reduction $\tau_\alpha(M[s])\Downarrow^{\mathrm{h}}_l$:

- The case $l = 0$:
  Then $\tau_\alpha(M[s])$ is in normal form without free variables, which is impossible.

- The case $l \geq 1$:
  Since $\tau_\alpha(M[s])\Downarrow^{\mathrm{h}}_l$, by applying the sensibility for $\Lambda_{\tau(D)}$, the interpretation of $\tau_\alpha(M[s])\Downarrow^{\mathrm{h}}_l$ is non empty. By Remark 2.3.1.17, the interpretation of M is also non empty. Thus, reapplying the sensibility, M is converging to a head-normal form $M \to_h^* \lambda y_1...y_n.z\ M_1 \cdots M_m$. We can then make some assumptions:

  - We can assume that $N \to_h^* \lambda y_1...y_{n'}.z'\ N_1 \cdots N_{m'}$:
    In fact, if N does not converge then trivially $M \not\sqsubseteq_{\mathcal{H}^*} N$.
  - We can assume that $n' \geq n$:
    In fact, if $n' < n$ then we can always define $N' = \lambda y_1...y_{n'} y_{n'+1}...y_n.z'\ N_1 \cdots N_{m'}\ y_{n'+1} \cdots y_n$ (with $y_{n'+1}...y_n \notin \mathrm{FV}(z'\ N_1 \cdots N_{m'})$), and we would have $N' \equiv_{\mathcal{H}^*} N$ and $\tau_\alpha(N'[s])\Uparrow^{\mathrm{h}}$.
  - We can assume that $n = 0$:
    In fact, let $a_0 \to \cdots a_n \to \alpha' = \alpha$, $[s'] = [\bar{\epsilon}_{a_0}/y_1, \ldots, \bar{\epsilon}_{a_n}/y_n]$, $N' = \lambda y_{n+1}...y_{n'}.z'\ N_1 \cdots N_{m'}$ and $M' = z\, M_1 \cdots M_m$. Since $\tau_\alpha(M[s]) \to^* \tau_{\alpha'}(M'[s,s'])$ (resp. $\tau_\alpha(N[s]) \to^* \tau_{\alpha'}(N'[s,s'])$), by confluence and standardization theorems (Th. 2.3.1.26 and Th.2.3.1.29), the convergences of $\tau_\alpha(M[s])$ (resp. $\tau_\alpha(N[s])$) and $\tau_{\alpha'}(M'[s,s'])$ (resp. $\tau_{\alpha'}(N'[s,s'])$) are equivalent. Applying Theorem 2.3.1.32, we thus have $\tau_{\alpha'}(M'[s,s'])\Downarrow^{\mathrm{h}}_l$ and $\tau_{\alpha'}(N'[s,s'])\Uparrow^{\mathrm{h}}$.
    Moreover $M' \sqsubseteq_{\mathcal{H}^*} N' \Leftrightarrow M \sqsubseteq_{\mathcal{H}^*} N$ so that the property on M' and N' is equivalent to the same property on M and N.
  - We can assume that $z' = z = x_0$:
    Since $\{x_0 \ldots x_k\} \supseteq \mathrm{FV}(M)$, there is $j \leq k$ such that $z = x_j$, for simplicity we assume that $j = 0$. Then we can remark that by Item (1) of Lemma 2.3.2.1, either $M \not\sqsubseteq_{\mathcal{H}^*} N$ or $z' = z = x_0$, we will thus continue with the second case.

  Altogether we have:

  $$M \to_h^* x_0\ M_1 \cdots M_m \qquad\qquad N \to_h^* \lambda y_1...y_{n'}.x_0\ N_1 \cdots N_{m'}$$

  The case $M = x_0$ corresponds exactly to the hypothesis of Lemma 2.3.2.2 that concludes by $M = x_0 \not\sqsubseteq_{\mathcal{H}^*} N$. We are now assuming that $m \geq 1$.

  By Lemma 2.3.2.1, either $M \not\sqsubseteq_{\mathcal{H}^*} N$ or the following holds:

  - $m = m' - n'$, and in particular $m \leq m'$
  - for $i \leq m$, $M_i \sqsubseteq_{\mathcal{H}^*} N_i$
  - for $m < i \leq m'$, $y_{i-m} \sqsubseteq_{\mathcal{H}^*} N_i$.

  We will assume that $m = m' - n'$ and then refute $M_i \sqsubseteq_{\mathcal{H}^*} N_i$ or $y_i \sqsubseteq_{\mathcal{H}^*} N_{m+i}$ for some $i \leq n'$; we then conclude that $M \not\sqsubseteq_{\mathcal{H}^*} N$.

  In the following we unfold

  - $\alpha = b_1 \to \cdots \to b_{n'} \to \alpha'$,
  - $a_0 = \{\beta_0 \ldots \beta_r\}$,
  - for all $t \leq r, \beta_t = c^t_1 \to \cdots c^t_m \to \beta'_t$,
  - and for all $t \leq r, \beta'_t = c^t_{m+1} \to \cdots c^t_{m'} \to \beta''_t$.

Moreover we set $[s'] = [\bar{\epsilon}_{b_1}/y_1 \ldots \bar{\epsilon}_{b_{n'}}/y_{n'}]$.

Then we have:

$$\begin{aligned} \tau_\alpha(M[s]) &\to^* \tau_\alpha(\bar{\epsilon}_{a_0}\ M_1[s]\cdots M_m[s]) &(2.6)\\ &\overset{\bar\tau}{\to}{}^m_h \overset{\tau\bar\tau}{\to}_h \Sigma_{\{t\leq r|\alpha\leq\beta'_t\}}\Pi_{i\leq m}\Pi_{\gamma\in c_i^t}\tau_\gamma(M_i[s]). &(2.7)\end{aligned}$$

By Theorem 2.3.1.32, $\tau_\alpha(\bar{\epsilon}_{a_0}\ M_1[s]\cdots M_m[s])\Downarrow^{\mathrm{h}}_l$. Moreover, since the head reduction (2.7) is prefix of any head reduction sequence starting from $\tau_\alpha(\bar{\epsilon}_{a_0}\ M_1[s]\cdots M_m[s])$, the test $\Sigma_{\{t\leq r|\alpha\leq\beta'_t\}}\Pi_{i\leq m}\Pi_{\gamma\in c_i^t}\tau_\gamma(M_i[s])$ head converges in $(l-m-1)$ steps so that there exists $t_0\leq r$ such that $\alpha\leq\beta'_{t_0}$ and for all $i\leq m$ and all $\gamma\in c_i^{t_0}$, we have $M_i[s]\Downarrow^{\mathrm{h}}{}_{l-1}$.

Similarly we have:

$$\begin{aligned} \tau_\alpha(N[s]) &\to^* \tau_\alpha(\lambda y_1...y_{n'}.\bar{\epsilon}_{a_0}\ N_1[s]\cdots N_{m'}[s])\\ &\overset{\tau}{\to}{}^{n'} \tau_{\alpha'}(\bar{\epsilon}_{a_0}\ N_1[s,s']\cdots N_{m'}[s,s']))\\ &\overset{\bar\tau}{\to}{}^{m'} \tau_{\alpha'}(\Sigma_{t\leq r}\bar\tau_{\beta''_t}(\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s'])))\\ &\overset{\tau\bar\tau}{\to} \Sigma_{t\leq r|\alpha'\leq\beta''_t}\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s']).\end{aligned}$$

Thus, by standardization (Th. 2.3.1.29), $\Sigma_{t\leq r|\alpha'\leq\beta''_t}\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s'])$ diverges. Thus there are two cases:

- Either $\alpha'\not\leq\beta''_{t_0}$: which is impossible since $\alpha\leq\beta'_{t_0}$.
- Or there is $i\leq m'$ and $\gamma\in c_i^{t_0}$ such that $\tau_\gamma(N_i[s,s'])$ diverges.
  * Either $i\leq m$:
    Then since $\tau_\gamma(M_i[s,s']) = \tau_\gamma(M_i[s])\Downarrow^{\mathrm{h}}{}_{l-1}$, the induction hypothesis yields that $M_i \not\sqsubseteq_{\mathcal{H}^*} N_i$.
  * Or $m<i$:
    Since $\alpha\leq\beta'_{t_0}$ we have $b_{i-m}\geq c_i^{t_0}$ and $\gamma\leq\gamma'\in b_{i-m}$. Moreover, using Theorem 2.3.1.21 and $\gamma\leq\gamma'$, we have that $\tau_{\gamma'}(N_i[s,s'])$ diverges. Thus we can apply Lemma 2.3.2.2 that results in $y_{i-m}\not\sqsubseteq_{\mathcal{H}^*} N_i$.

□

Theorem 2.3.2.4 (Hyperimmunity implies full abstraction).
Any extensional K-model D that is hyperimmune and sensible for $\Lambda_{\tau(D)}$ is inequationally fully abstract for the pure λ-calculus.

Proof. Inequational adequacy: inherited from the inequational sensibility of D for $\Lambda_{\tau(D)}$. Indeed, for any $M,N\in\Lambda$ and $C\in\Lambda^{(\!|\cdot|\!)}$, if $[\![M]\!]^{\vec{x}}_D\subseteq[\![N]\!]^{\vec{x}}_D$ and if $C(\!|M|\!)\Downarrow^{\mathrm{h}}$, then by sensibility

$$[\![C(\!|N|\!)]\!]^{\vec{x}'}_D\supseteq[\![C(\!|M|\!)]\!]^{\vec{x}'}_D\neq\emptyset$$

and (still by sensibility) $[\![C(\!|N|\!)]\!]^{\vec{x}'}_D$ converges.

Inequational completeness: for all $M,N\in\Lambda$ such that $[\![M]\!]^{\vec{x}}\not\subseteq[\![N]\!]^{\vec{x}}$, there is $(\vec{a},\alpha)\in[\![M]\!]^{\vec{x}}-[\![N]\!]^{\vec{x}}$, thus by Theorem 2.3.2.3, $M\not\sqsubseteq_{\mathcal{H}^*} N$. □

2.3.3. Full abstraction implies hyperimmunity

The counterexample

In this section, we are assuming that D is sensible for $\Lambda_{\tau(D)}$ but is not hyperimmune. Then we will construct a counterexample $(\boldsymbol{J}_g\ \underline{0})$ for the full abstraction such that $(\boldsymbol{J}_g\ \underline{0}) \equiv_{\mathcal{H}^*} \boldsymbol{I}$ and $[\![\boldsymbol{J}_g\ \underline{0}]\!] \neq [\![\boldsymbol{I}]\!]$ resulting in Theorem 2.3.3.5. Remark that this counterexample is exactly the same as in Section 2.2.3, only the sub-proofs justifying the non-full abstraction are different as they are syntactic proofs.

By Definition 2.1.0.1, if D is hyperimmune, then there exist a recursive $g : (\mathbb{N} \to \mathbb{N})$ and a family $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$ such that $\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n$ with $\alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}$.

We will use the function g for defining a term $\boldsymbol{J}_g$ (Eq. 2.10) such that $(\boldsymbol{J}_g\ \underline{0})$ is observationally equal to the identity in Λ (Lm. 2.3.3.2) but can be distinguished in $\Lambda_{\tau(D)}$. From this latter statement and the full abstraction for $\Lambda_{\tau(D)}$ (Th. 2.3.1.20), we will obtain that $[\![\boldsymbol{J}_g\ \underline{0}]\!]_D \neq [\![\boldsymbol{I}]\!]_D$, and thus we conclude with Theorem 2.3.3.5.

Let $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$ be the sequence of closed λ-terms defined by:

$$\boldsymbol{G}_n := \lambda u e x_1 ... x_{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}) \tag{2.8}$$

The recursivity of g implies that of the sequence $\boldsymbol{G}_n$. We can thus use the Proposition 1.1.0.5 that set a $\boldsymbol{G} \in \Lambda$ such that:

$$\boldsymbol{G}\ \underline{n} \to^* \boldsymbol{G}_n. \tag{2.9}$$

Recall that $\boldsymbol{S}$ denotes the Church successor function and $\boldsymbol{\Theta}$ the Turing fixedpoint combinator. We define:

$$\boldsymbol{J}_g := \boldsymbol{\Theta}\ (\lambda uv.\boldsymbol{G}\ v\ (u\ (\boldsymbol{S}\ v))). \tag{2.10}$$

Then:

$$\boldsymbol{J}_g\ \underline{n} \to^* \boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1}). \tag{2.11}$$

Lemma 2.3.3.1. *For all $n \in \mathbb{N}$, all $\alpha \in D$ and all $b = \{\beta_1, ..., \beta_k\} \subseteq D$, let:*

- $\alpha = a_1 \to \cdots \to a_{g(n)} \to \alpha'$,
- *for all $j \leq k$, $\beta_j = b_{j1} \to \cdots \to b_{jg(n)} \to \beta'_j$,*

we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\ \to^* \to_h\ \Sigma_{\{j\leq k \mid \alpha' \leq \beta'_j\}} \Pi_{i\leq g(n)} \Pi_{\gamma \in b_{ji}} \tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

Proof. We can reduce:

$$\begin{aligned}
\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b) &\overset{Eq(2.11)}{\longrightarrow}_* \tau_\alpha(\boldsymbol{G}\ \underline{n}\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \bar{\boldsymbol{\epsilon}}_b)\\
&\overset{Eq(2.8)}{\longrightarrow}_* \tau_\alpha(\boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \bar{\boldsymbol{\epsilon}}_b)\\
&\overset{Eq(2.9)}{\longrightarrow}_* \tau_\alpha\Big((\lambda ue\vec{x}^{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}))\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \bar{\boldsymbol{\epsilon}}_b\Big)\\
&\overset{\beta}{\to}{}^2_h \tau_\alpha\Big(\lambda\vec{x}^{g(n)}.\bar{\boldsymbol{\epsilon}}_b\ (\boldsymbol{J}_g\ \underline{n{+}1}\ x_1)\ \cdots\ (\boldsymbol{J}_g\ \underline{n{+}1}\ x_{g(n)})\Big)\\
&\overset{\tau}{\to}{}^{g(n)}_h \tau_{\alpha'}\Big(\bar{\boldsymbol{\epsilon}}_b\ (\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_1})\ \cdots\ (\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_{g(n)}})\Big)\\
&\overset{\bar{\tau}}{\to}{}^{g(n)}_h \tau_{\alpha'}(\Sigma_{j\leq k}\bar{\tau}_{\beta'_j}(\Pi_{i\leq g(n)}\Pi_{\gamma\in b_{ji}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_i)))\\
&\overset{\tau\bar{\tau}}{\to}_h \Sigma_{\{j\leq k|\alpha'\leq\beta'_j\}}\Pi_{i\leq g(n)}\Pi_{\gamma\in b_{ji}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})
\end{aligned}$$

□

Lemma 2.3.3.2. *For all n, we have* $\boldsymbol{J}_g\ \underline{n} \equiv_{\mathcal{H}^*} \boldsymbol{I}$.

Proof. Let D_∞ be defined as in Example 1.2.4.9. We have seen in Example 2.1.0.4 that D_∞ is hyperimmune and we will see in Example 2.4.2.13 Section 2.4.2 that it is sensible for $\Lambda_{\tau(D_\infty)}$. Thus by Theorem 2.3.2.4, D_∞ is fully abstract for Λ. It results that it is sufficient to verify that $[\![\boldsymbol{J}_g\ \underline{n}]\!]_{D_\infty} = [\![\boldsymbol{I}]\!]_{D_\infty}$, or equivalently (Th. 2.3.2.4) to verify that :

$$\forall\alpha\in D_\infty, \tau_\alpha(\boldsymbol{J}_g\ \underline{n})\Downarrow^{\mathrm{h}} \Leftrightarrow \tau_\alpha(\boldsymbol{I})\Downarrow^{\mathrm{h}}.$$

Trivially $\tau_{a_0\to\alpha}(\boldsymbol{I})$ converges iff there is β such that $\alpha\leq\beta\in a_0$. Conversely we can prove by induction on a_0 that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0})$ converges iff there is β such that $\alpha\leq\beta\in a_0$ and conclude by extensionality.

If we denotes $\alpha = a_1{\to}\cdots{\to}a_{g(n)}{\to}\alpha'$, Lemma 2.3.3.1 gives that:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0}) \quad \to^*\to_h \quad \Sigma_{\{b_1\to\cdots b_{g(n)}\to\beta'\in a_0|\alpha'\leq\beta'\}}\Pi_{i\leq g(n)}\Pi_{\gamma\in b_i}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

By induction hypothesis and standardisation, this test converges iff there is $\beta = b_1{\to}\cdots b_{g(n)}{\to}\beta'\in a_0$ such that $\alpha'\leq\beta'$ and for all $i\leq g(n)$ and all $\gamma\in b_i$, $\gamma\leq\delta\in a_i$, *i.e.*, for all i, $b_i\leq a_i$. Equivalently, this test converges iff $\alpha\leq\beta\in a_0$. Thus, using the standardization (Th. 2.3.1.29), $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0})$ converges iff $\alpha\leq\beta\in a_0$. □

Lemma 2.3.3.3. *For all $n\in\mathbb{N}$, all $\alpha\in D$ and all $b\in\mathcal{A}_f(D)$, if $\beta\not\geq\alpha$ for all $\beta\in b$, then:*

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}$$

Proof. Let $\{\beta_1,....,\beta_k\} = b$ and, for all $j\leq k$, let $b_{j,1}{\to}\cdots{\to}b_{j,k}{\to}\beta'_j = \beta_j$.

We are proceeding by coinductively constructing the proof of $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}$ following Proposition A.2.1.37 (that we can apply due to Theorem 2.3.1.32).

From Lemma 2.3.3.1, we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\ \to^*\to_h\ \Sigma_{\{j\leq l|\beta'_j\leq\alpha'\}}\Pi_{i\leq g(n)}\Pi_{\gamma\in b_{j,i}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$$

with the last reduction that is the only possible head reduction.

By applying Proposition A.2.1.37, we get that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)$ head-diverges if the above reduct head-diverges. In particular, it is sufficent to prove that for any $j \leq l$ such that $\beta'_j \leq \alpha'$ there exists $i \leq g(n)$ and $\gamma \in b_{j,i}$ such that $\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ head diverges.

Let $j \leq l$ be such that $\beta'_j \geq \alpha'$.

Since $\beta_j \ngeq \alpha$, there is i such that $b_{j,i} \nleq a_i$, *i.e.*, there is $\gamma \in b_{j,i}$ such that for all $\delta \in a_i$, $\gamma \nleq \delta$ and by coinduction we get that $\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ head diverges. □

We recall that $(\alpha_n)_n$ is given by the counterexample of the hyperimmunity, and that for all n, $\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n$ and $\alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}$.

Lemma 2.3.3.4. *For any $n \in \mathbb{N}$ and any anti-chain $b = \{\alpha_n, \beta_1, \ldots., \beta_k\}$, then:*

$$\tau_{\alpha_n}((\boldsymbol{J}_g\ \underline{n})\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}.$$

In particular, $\tau_{\alpha_0}(\boldsymbol{J}_g\ \underline{0}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})\Uparrow^{\mathrm{h}}$.

Proof. Let $\beta_j = b_{j,1} \to \cdots \to b_{j,g(n)} \to \beta'_j$.

We are proceeding by coinductivelly constructing the proof of $\tau_{\alpha_n}((\boldsymbol{J}_g\ \underline{n})\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}$ following Proposition A.2.1.37 (that we can apply due to Theorem 2.3.1.32).

From Lemma 2.3.3.1, we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b) \quad \to^*\to_h \quad \Pi_{i \leq g(n)}\Pi_{\gamma \in a_{ni}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}) \quad + \quad \Sigma_{\{j \leq l | \alpha'_n \leq \beta'_j\}}\Pi_{i \leq g(n)}\Pi_{\gamma \in b_{ji}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

with the last reduction that is the only possible head reduction.

By applying Proposition A.2.1.37, we get that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)$ diverges if the above reduct head-diverges. In particular, it is sufficent to prove that both addens diverge.

- The fist member $\Pi_{i \leq g(n)}\Pi_{\gamma \in a_{ni}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ head-diverges since there is $i \leq g(n)$ such that $\alpha_{n+1} \in a_{n_i}$ and by coinduction, $\tau_{\alpha_{n+1}}(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ diverges.
- The second member of the sum diverges by Lemma 2.3.3.3.
  For any $j \leq l$ such that $\beta'_j \geq \alpha'_n$ we know that $\beta_j \ngeq \alpha_n$ since $\{\alpha_n, \beta_1, \ldots, \beta_l\}$ is an anti-chain. Thus there is always $i \leq g(n)$ such that $b_{j,i} \nleq a_{n,i}$, *i.e.*, there is $\gamma \in b_{j,i}$ such that for all $\delta \in a_{n,i}$, $\gamma \nleq \delta$.
  We can conclud by Lemma 2.3.3.3 that $\tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ diverges.

□

Theorem 2.3.3.5 (Full abstraction implies Hyperimmunity). *If D is not hyperimmune, but sensible for $\Lambda_{\tau(D)}$, then it is not fully abstract for the λ-calculus.*

Proof. Since $\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0}) \xrightarrow{\beta}_h \xrightarrow{\tau\bar{\tau}}_h \epsilon$, we have that $[\![\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!] \neq \emptyset$, while by Lemma 2.3.3.4 we have that $[\![\tau_{\alpha_0}(\boldsymbol{J}_g\ \underline{0}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!] = \emptyset$, and thus $[\![\boldsymbol{J}_g\ \underline{0}]\!] \neq [\![\boldsymbol{I}]\!]$. Hence, by Lemma 2.3.3.2, D is not fully abstract. □

2.4. More on D-tests

In this section we investigate further on the interest of tests and their links with other tools, in particular with Böhm trees. This is done by achieving two quite interesting theorems.

The first is Theorem 2.4.1.9 that is the purpose of Section 2.4.1. This theorem states, for any model D, the equivalence between the approximation property and the sensibility for $\Lambda_{\tau(D)}$. As such, it also states the equivalence between the syntactical and semantical versions of the main theorem (Th. 2.1.0.5 and Th. 2.1.0.6). In this section we will see how the reduction of a test $\tau_\alpha(M)$ can be seen as a kind of kind of type inference over the Böhm tree $\boldsymbol{BT}(M)$.

The second is Theorem 2.4.2.12 that is the purpose of Section 2.4.2. This theorem proposes an easy-to-verify condition for the sensibility of tests that is the *stratified positivity* (Def. 2.4.2.2). This is a rather fine property, maximizing the potential of the methods using logical relations[10] that are used for the proof. Indeed, we will see that usual proofs, using logical relations to prove the sensibility for the untyped λ-calculus, generalize naturally to tests. Moreover, the new formulation seems to be even more precise.

2.4.1. D-tests and Böhm trees

Recall that the approximation property relies on the inductive interpretation of Böhm trees (Def. 2.2.1.10). Recall also that this inductive interpretation corresponds to the type assertion that are finitely derivable by the rules of Figure 2.1 (Prop. 2.2.1.11).

We will see that for any λ-term $M \in \Lambda$, the reduction of $\tau_\alpha(M)$ can be seen as a (non-necessarily terminating) type inference procedure that tests whether $\boldsymbol{BT}(M)$ is of type α in the inductive intersection-type system of Figure 2.1.

Lemma 2.4.1.1. *For any term $M \in \Lambda$ with free variables $\mathrm{FV}(M) \subseteq \{x_1...x_k\}$ and any $(\vec{a}, \alpha)$ so that $[s] = [\bar{\boldsymbol{\epsilon}}_{a_1}/x_1...\bar{\boldsymbol{\epsilon}}_{a_k}/x_k]$, the test $\tau_\alpha(M[s])$ converges iff the statement $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(M) : \alpha$ has a finite derivation in the system of Figure 2.1.*

Proof. Assuming that $\tau_\alpha(M[s]) \Downarrow^{\mathrm{h}}_n$, we can do an induction on the lexicographicaly ordered (n, M):

- If M is a λ-abstraction $\lambda y.N$ and if $\alpha = b \to \alpha'$, then $\tau_\alpha(M[s]) \to_h \tau_{\alpha'}(N[s, \bar{\boldsymbol{\epsilon}}_b/y])$ and it is the only possible head reduction so that $(N[s, \bar{\boldsymbol{\epsilon}}_a/y]) \Downarrow^{\mathrm{h}}_{n-1}$. By induction hypothesis we know that the statement $\vec{x} : \vec{a}, y : b \vdash \boldsymbol{BT}(N) : \alpha'$ is derivable, so is $\vec{x} : \vec{a} \vdash \lambda y.\boldsymbol{BT}(N) : b \to \alpha'$ by the rule $(BT\text{-}\lambda)$.
- If $M = x_i\ M_1 \cdots M_m$ then $\tau_\alpha(M[s]) \to_h^{m+1} \Sigma_{\{b_1\to\cdots\to b_m\to\beta\in a_i \mid \alpha\leq\beta\}} \Pi_{j\leq m} \Pi_{\gamma\in b_j} \tau_\gamma(M_j[s]))$ and there in $b_1 \to \cdots \to b_m \to \beta \in a_i$ such that $\alpha \leq \beta$ and for all $j \leq m$ and $\gamma \in b_j$, $\tau_\gamma(M_j[s])) \Downarrow^{\mathrm{h}}_{n-m-1}$. Thus, after applying the induction hypothesis, we can conclude with the rule $BT\text{-}@$.
- Otherwise M is not a head-normal form nor an abstraction so that $M \to_h N \in \Lambda$ and $\tau_\alpha(M[s]) \to_h \tau_\alpha(N[s])$. Moreover, this head reduction is prefix to any head-reduction starting from $\tau_\alpha(M[s])$ so that $\tau_\alpha(N[s]) \Downarrow^{\mathrm{h}}_{n-1}$. By induction hypothesis, $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(N) : \alpha$ which conclude since $\boldsymbol{BT}(M) = \boldsymbol{BT}(N)$.

Conversely, assuming that the statement $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(M) : \alpha$ has a finite derivation in the system of Figure 2.1, we can proceed by induction on the derivation tree:

[10] or equivalently realisability candidates

- If The last rule is $(BT\text{-}\lambda)$, then $\boldsymbol{BT}(M) = \lambda y.U$, *i.e.*, there is N such that $M \to_h^* \lambda y.N$ with $\boldsymbol{BT}(N) = U$, in particular there is a first one along the reduction. Thus, assuming that $\alpha = b \to \beta$,

$$\tau_\alpha(M[s]) \quad \to_h^* \quad \tau_\alpha(\lambda y.N[s]) \quad \to_h \quad \tau_\beta(N[s, \bar{\boldsymbol{\epsilon}}_b/y]).$$

  However, the derivation also gives that $\vec{x} : \vec{a}, y : b \vdash U : \beta$ where we can apply the induction hypothesis to get the convergence of $\tau_\beta(N[s, \bar{\boldsymbol{\epsilon}}_b/y])$.

- If The last rule is $(BT\text{-}@)$, then $\boldsymbol{BT}(M) = x_i\ U_1 \cdots U_m$, *i.e.*, there is $N_1, ..., N_m$ such that $M \to_h^* y\, N_1 \cdots N_m$ with $\boldsymbol{BT}(N_j) = U_j$ for all j. No abstraction appearing along the reduction, we have that

$$\begin{aligned} \tau_\alpha(M[s]) \quad &\to_h^* \quad \tau_\alpha(\bar{\boldsymbol{\epsilon}}_{a_i}\ N_1[s] \cdots N_m[s]) \\ &\to_h^* \sum_{\{b_1 \to \cdots \to b_m \to \beta \in a_i \mid \alpha \leq \beta\}} \Pi_{j \leq m} \Pi_{\gamma \in b_i} \tau_\gamma(N_j[s]). \end{aligned}$$

  However, the derivation gives exactly the existence of $b_1 \to \cdots \to b_m \to \beta \in a_i$ such that $\alpha \leq \beta$ and for all $j \leq m$ and $\gamma \in b_i$, $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(N_i) : \gamma$, *i.e.*, using the IH, that each $\tau_\gamma(N_j[s])$ converges.

□

Remark 2.4.1.2. *For $M \in \Lambda$, the derivation of $\tau_\alpha(M[s])$, seen as type inference procedure, has three elementary kinds of divergence in $\Lambda_{\tau(D)}$:*

- *either M diverges: then $\boldsymbol{BT}(M) = \Omega$ which is non-typable by definition,*
- *or $\tau_\alpha(M[s]) \to_h^* 0$: the $\boldsymbol{BT}(M)$ may be typable, but not by α (under $\vec{x} : \vec{a}$),*
- *finally, it is possible that the type inference run into an infinite tree (*i.e.*, a co-inductive derivation). This is the most interesting case for us as it corresponds to an infinite nesting of alternating (τ) and $(\bar{\tau})$ reductions in the idea of what we did in Section 2.3.3.*

Example 2.4.1.3. *The following reduction and derivation are in correspondence:*

$$\begin{aligned} &\tau_{\{\alpha,\alpha\to\alpha\}\to\alpha}(\lambda x.x\ x) \\ \to_h &\tau_\alpha(\bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) \\ \to_h &\tau_\alpha(\bar{\epsilon}_{\alpha\to\alpha}\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) + \tau_\alpha(\bar{\epsilon}_\alpha\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) \\ \to_h &\tau_\alpha(\bar{\tau}_\alpha(\tau_\alpha(\bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}))) + \tau_\alpha(\bar{\epsilon}_\alpha\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) \\ \to_h &\tau_\alpha(\bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) + \tau_\alpha(\bar{\epsilon}_\alpha\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) \\ \to_h &\epsilon + \tau_\alpha(\bar{\epsilon}_\alpha\ \bar{\epsilon}_{\{\alpha,\alpha\to\alpha\}}) \end{aligned}$$

$$\dfrac{\dfrac{\overline{x : \{\alpha, \alpha\to\alpha\} \vdash x : \alpha} \qquad \overline{\alpha \leq \alpha}}{x : \{\alpha, \alpha\to\alpha\} \vdash x\ x : \alpha}}{\vdash \lambda x.x\ x : \{\alpha, \alpha\to\alpha\}\to\alpha}$$

In the following example, we fix $D = D_\infty^*$ so that $p = q \to p$ and $q = p \to q$. This gives rise to an infinite derivation:

$$\begin{aligned} \tau_p(J\ \bar{\epsilon}_p) \to^* &\ \tau_p(\lambda y.\bar{\epsilon}_p\ (J\ y)) \\ \to_h &\ \tau_p(\bar{\epsilon}_p\ (J\ \bar{\epsilon}_q)) \\ \to_h &\ \tau_p(\bar{\tau}_p(\tau_q(J\ \bar{\epsilon}_q))) \\ \to_h &\ \tau_q(J\ \bar{\epsilon}_q) \\ \to &\ \ldots \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{\ldots}{y : q \vdash J\ y : q} \qquad \overline{p \leq p}}{x : p, y : q \vdash x\ (J\ y) : p}\ J\ x \to_h^* \lambda y.x\ (J\ y)}{x : p \vdash J\ x : p}$$

Lemma 2.4.1.4. *If D is sensible for $\Lambda_{\tau(D)}$ (Def. 2.3.1.19), then it is approximable (Def: 2.2.1.12).*

Proof. Let D an extensional K-model sensible for $\Lambda_{\tau(D)}$.
Let $M \in \Lambda$ be a λ-term and $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$. We want to show that $(\vec{a}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^{\vec{x}}_{ind}$.
By Proposition 2.2.1.11, it suffices to show that $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(M) : \alpha$ is finitely derivable in the system of Figure 1.3. By Lemma 2.4.1.1, it suffices to show that $\tau_\alpha(M[\overrightarrow{\bar{\epsilon}_a}/\vec{x}])\Downarrow^{\mathrm{h}}$. However, this is given by the sensibility for $\Lambda_{\tau(D)}$ and Theorem 2.3.1.21. □

Lemma 2.4.1.5. *If D is approximable then for any $M \in \Lambda$ we have:*

$$(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}} \;\Rightarrow\; \tau_\alpha(M[\overrightarrow{\bar{\epsilon}_a}/\vec{x}])\Downarrow^{\mathrm{h}}$$

Proof. Let D an extensional and approximable K-model. Let $M \in \Lambda$ be a λ-term and $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$.
By the approximation property, $(\vec{a}, \alpha) \in [\![\boldsymbol{BT}(M)]\!]^{\vec{x}}_{ind}$.
By Proposition 2.2.1.11, the statement $\vec{x} : \vec{a} \vdash \boldsymbol{BT}(M) : \alpha$ is finitely derivable in the system of Figure 1.3.
By Lemma 2.4.1.1, we have $\tau_\alpha(M[\overrightarrow{\bar{\epsilon}_a}/\vec{x}])\Downarrow^{\mathrm{h}}$. □

To show that being sensible for $\Lambda_{\tau(D)}$ and being approximable are equivalent requirements, the only remaining part is to extends Lemma 2.4.1.5 to terms $M \in \Lambda_{\tau(D)} - \Lambda$. This will be done by exhibiting, for any $M \in \Lambda_{\tau(D)}$, a term $M^\Lambda \in \Lambda$ and a substitution $[s_M] = [\bar{\epsilon}_{a_1}/x_1 ... \bar{\epsilon}_{a_n}/x_n]$ such that $M^\Lambda[s_M] \to^* N^* \leftarrow M$ for some N.

Definition 2.4.1.6. *For an $M \in \Lambda_{\tau(D)}$, we define a term without test $M^\Lambda \in \Lambda$ and the corresponding substitution (with tests) $[s_M]$ over some free variables $x_1 ... x_n \in \mathrm{FV}(M^\Lambda) - \mathrm{FV}(M)$:*

- *$x^\Lambda = x$ and $[s_x] = []$.*
- *$(\lambda x.M)^\Lambda = \lambda x.M^\Lambda$ and $[s_{\lambda x.M}] = [s_M]$.*
- *$(M\ N)^\Lambda = M^\Lambda\ N^\Lambda$ and $[s_{(M\ N)}] = [s_M][s_N]$*
- *Otherwise $M \to^* \Sigma_{i \leq n} \bar{\tau}_{\alpha_i}(\Pi_{j \leq k_i} \tau_{\beta_{i,j}}(M_{i,j}))$ (by only using the distributions of the sums). Then $M^\Lambda = u\ M^\Lambda_{1,1}\ M^\Lambda_{1,2} \cdots M^\Lambda_{1,k_1}\ M^\Lambda_{2,1} \cdots M^\Lambda_{n,k_n}$ where u is fresh; and $[s_M] = [\bar{\epsilon}_a/u] \bigcup_{i,j} [s_{M_{i,j}}]$ where $a = \{\gamma_i \mid i \leq n\}$ and $\gamma_i = b^i_{1,1} \to \cdots \to b^i_{n,k_n} \to \alpha_i$ and $b^i_{i',j} = \emptyset$ if $i' \neq i$ and $b^i_{i,j} = \{\beta_{i,j}\}$ otherwise.*

Remark that the substitution is linear (exactly one occurrence of each x_i will appears)

Proposition 2.4.1.7. *For all $M \in \Lambda_{\tau(D)}$, $M^\Lambda[s_M] \to^* N^* \leftarrow M$ for some N.*

Proof. By induction on M, the only non trivial case is when $M = \Sigma_i \bar{\tau}_{\alpha'_i}(Q)$ (using the linearity of the substitution for the application case).
Since Q is an alternance of sums and products with atoms of the form $\tau(\alpha_\gamma)(N)$, we can perform every

possible distributions of the sum over the products and the τ's so that $M \to^* \Sigma_{i\leq n}\bar{\tau}_{\alpha_i}(\cdot\Pi_{j\leq k_i}\tau_{\beta_{i,j}}(M_{i,j}))$. By applying the induction hypothesis on each $M_{i,j}$ we get that M has a common reduct with

$$\Sigma_{i\leq n}\bar{\tau}_{\alpha_i}(\cdot\Pi_{j\leq k_i}\tau_{\beta_{i,j}}(M^{\Lambda}_{i,j}))[s_{i,j} \mid i, j]$$

which is a reduct of $M^{\Lambda}[s_M]$. □

Lemma 2.4.1.8. *If D is approximable then D is sensible for $\Lambda_{\tau(D)}$.*

Proof. Let M be a term with tests such that $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}} = [\![M^{\Lambda}[s_M]]\!]^{\vec{x}}$:
Then, by Lemma 2.4.1.5, $\tau_\alpha(M^{\Lambda}[s_M][\overrightarrow{\epsilon}_a/\vec{x}])\Downarrow^{\mathrm{h}}$.
In particular $M^{\Lambda}[s_M]\Downarrow^{\mathrm{h}}$, thus by confluence and standardization (Th 2.3.1.26 and Th. 2.3.1.29), we obtain that $M\Downarrow^{\mathrm{h}}$. □

Theorem 2.4.1.9 (approximability and sensibility).
An extensional K-model D is approximable iff D is sensible for $\Lambda_{\tau(D)}$.

Proof. Lemmas 2.4.1.8 and 2.4.1.4. □

2.4.2. A sufficient condition for the sensibility of tests

So far we could not find a generic proof of the approximation property in the literature for standard K-models.[11] Hence, we give a sufficient condition (Def 2.4.2.2) for a K-model D to be approximable (Cor. 2.4.2.14). Indeed, we use this condition in Example 2.4.2.13 for stating the approximation property of the models of Example 1.2.4.9 (save for P_∞).

The proof of Corollary 2.4.2.14 uses the equivalence between approximability and sensibility with tests (Th. 2.4.1.9) proven in the previous chapter. In fact, we prove that stratified positivity of D implies sensibility for $\Lambda_{\tau(D)}$. Most proofs of sensibility for untyped λ-calculus extend realizability methods of Tait [Tai67]. We do the same for sensibility with tests with a condition (Def 2.4.2.2) extending the positivity of [Ber00].

Definition 2.4.2.1. *A preorder $(S, \leq)$ is said* well founded *if the order induced in the quotient space $(S, \leq)/(\leq \cap \geq)$ is well founded.*

Definition 2.4.2.2. *A (partial) K-model D is* stratified positive *if there exist*

- *a function $\mathcal{V}$ from D in $(\{-, +\}, - \leq_{\mathcal{V}} +)$,*
- *a well founded preorder $\leq$ in D,*

[11] Save Chapter 17.3 of the book of Barendregt, Dekkers ans Statman [BDS13] where this proof is done in parallel for several models of different classes.

such that for all $a \in \mathcal{A}_f(D)$, $\alpha \in D$ and all $\beta \in a$:

$$a{\to}\alpha \geq \alpha, \qquad a{\to}\alpha \simeq \alpha \;\Rightarrow\; \mathcal{V}(a{\to}\alpha) = \mathcal{V}(\alpha),$$
$$a{\to}\alpha \geq \beta, \qquad a{\to}\alpha \simeq \beta \;\Rightarrow\; \mathcal{V}(a{\to}\alpha) \neq \mathcal{V}(\beta),$$
$$\alpha \leq_D \beta \Rightarrow (\alpha \geq \beta \;\vee\; \alpha \leq \beta), \qquad \alpha \simeq \beta \;\Rightarrow\; \alpha \leq_D \beta \;\Rightarrow\; \mathcal{V}(\alpha) \leq_{\mathcal{V}} \mathcal{V}(\beta),$$

where $\simeq := (\leq \cap \geq)$

This condition can be seen as a stratification given by $\leq$, where the quotient $D/\simeq$ represents the different levels of the stratification, and a positive valuation $\mathcal{V}$ of every level. This stratification improves the condition of [Ber00] that only considers completions of positive partial K-models. This condition is the invariant by completion, which simplify the proof of stratified positivity of K-models of Example 1.2.4.9 (save for P_∞).

Proposition 2.4.2.3. *A partial K-model E is stratified positive iff its completion $\bar{E}$ is stratified positive.*

Proof. The "if" part is trivial.
If E is stratified positive as a partial K-model with an evaluation $\mathcal{V}$ and a preorder $\leq$, then it is stratified positive as a K-model with the an evaluation $\mathcal{V}'$ defined by

$$\forall \alpha \in E,\; \mathcal{V}'(\alpha) = \mathcal{V}(\alpha), \qquad \forall \alpha \notin E,\; \mathcal{V}(\alpha) = +,$$

and the preorder $\leq'$ given by the reflexive transitive closure of:

$$\leq'_{\restriction E \times E} = \leq, \qquad \forall (a,\alpha), \forall \beta \in a, \quad a{\to}\alpha \geq' \alpha, \text{ and } a{\to}\alpha \geq' \beta.$$

□

Example 2.4.2.4. *All the K-models of Example 1.2.4.9 are stratified positive except P_∞:*

- *For D_∞: We have to show the stratified positivity of the partial K-model $(\{*\}, id, \{(\emptyset, *) \mapsto *\})$ which is trivial.*
- *For D_∞^*: Idem, we set $\mathcal{V}(p) = +$, $\mathcal{V}(q) = -$ and $p \simeq q$.*
- *For $\bar{\omega}$: We set $\leq$ to be the usual order on $\mathbb{N}$ and the valuation can be anything as there is one element per stratification level.*
- *For $\overline{\mathbb{Z}}$: There are two choices, either to take the natural order on $\mathbb{N}$ and any valuation, or any order and the even/odd valuation...*
- *For H^f: a constant valuation will do with the order $\alpha_k^n \leq \alpha_{k'}^{n'}$ iff $n \leq n'$ and always $\alpha_k^n \leq *$.*

In the following we suppose given an extensional K-model D that is stratified positive.

We will now show (Th. 2.4.2.12) that the stratified positivity is a sufficient condition for the sensibility for $\Lambda_{\tau(D)}$. Therefore we will build a maximal S^D-realizer. A S^D-realizer $\mathfrak{R}$ is, basically, the supply, for each $\alpha \in D$, of a set $\mathfrak{R}(\alpha)$ of terms M such that $\tau_\alpha(M)$ head converges.

Definition 2.4.2.5. *A* saturated set *is a set of term $G \subseteq \Lambda_{\tau(D)}$ that is close by backward reduction.*
Given two saturated sets G, H, we denotes by $G \mapsto H$ the saturated set of terms M such that $\forall N \in G,\ M\ N \in H$.
Given an antichain $a \in \mathcal{A}_f(D)$, we will write $\Re(a)$ for $\bigcap_{\alpha\in a} \Re(\alpha)$.

Definition 2.4.2.6. *We will denote, for all $\alpha \in D$:*

- $\mathcal{N}^+_\alpha = \{M \in \Lambda_{\tau(D)} \mid \forall\beta \leq_D \alpha, \tau_\beta(M){\Downarrow}^{\mathrm{h}}\}$,
- $\mathcal{N}^-_\alpha = \{M \in \Lambda_{\tau(D)} \mid \exists\beta \geq_D \alpha, \exists L \in \Lambda_{\tau(D)}, M \to^* \bar{\boldsymbol{\epsilon}}_\beta + L\}$,
- *S^D_α is the set of saturated subsets of $\mathcal{N}^+_\alpha$ that contains $\mathcal{N}^-_\alpha$,*
- $S^D = (S^D_\alpha)_{\alpha\in D}$.

For any partial K-model $J \subseteq D$, we write S^J for the restriction of S^D to J.

Definition 2.4.2.7. *A* realizer *in D is a function $\Re$ from D to saturated subsets of $\Lambda_{\tau(D)}$ such that:*

1. *for all $b{\to}\alpha \in D$, we have*
$$\Re(b{\to}\alpha) = \Re(b) \mapsto \Re(\alpha),$$
2. *for all $\alpha,\beta \in D$, if $\alpha \leq_D \beta$ then $\Re(\alpha) \supseteq \Re(\beta)$.*

A realizer $\Re$ is a S^D-realizer if for all α, $\Re(\alpha) \in S^D_\alpha$.
Given a partial K-model $J \subseteq D$, a function $\Re : J \to \mathcal{P}(\Lambda_{\tau(J)})$ is a S^J-realizer if it respects the same property when restricted to J.

Definition 2.4.2.8. *For any functions $X, Y \in S^D$ and any evaluation $\mathcal{V} : D \to \{+,-\}$, we will say that $X \subseteq_{\mathcal{V}} Y$ if for all α:*

- *$X_\alpha \subseteq Y_\alpha$ whenever $\mathcal{V}(\alpha) = +$,*
- *and $X_\alpha \supseteq Y_\alpha$ whenever $\mathcal{V}(\alpha) = -$.*

Lemma 2.4.2.9. *There exists a S^D-realizer $\Re$ in D*

Proof. By induction on the well founded order formed by the set of initial segments of D by the preorder $\preceq$ of the stratified positivity, we can construct, for every initial segment $J \in \mathcal{I}(D)$, a S^J-realizer $\Re_J : J \to \mathcal{P}(\Lambda_{\tau(D)})$ so that $\Re_J = \Re_{\restriction_J}$ (where $\Re_{\restriction_J}$ is the restriction of $\Re$ to J). In particular it gives $\Re = \Re_D$.

- If J is empty: trivial
- If $J = \bigcup_{k\in K} J_k$ with $J_k \subset J$ for all k:
  Then for all $k, k' \in K$, $\Re_{J_k \cap J_{k'}} = \Re_{J_k}{\restriction_{J_{k'}}} = \Re_{J_{k'}}{\restriction_{J_k}}$. Thus, $\Re_J(\alpha) = J_{J_k}(\alpha)$ for $\alpha \in J_k$ is well defines (it does not depend on the choice of J_k). The conditions of Definition 2.4.2.7 are universal conditions and thus still verified:
  1. For all $b{\to}\alpha \in J$, there is k such that $a{\to}\alpha \in J_k$ and
  $$\forall N \in \Re_J(b) = \Re_{J_k}(b), \quad M\, N \in \Re_{J_k}(\alpha) = \Re_J(\alpha).$$
  2. For all $\alpha \leq \beta$ with $\alpha,\beta \in J$, since $\alpha \geq \beta$ or $\alpha \leq \beta$, there is k such that $\alpha,\beta \in J_k$ and thus
  $$\Re_J(\alpha) = \Re_{J_k}(\alpha) \supseteq \Re_{J_k}(\beta) = \Re_J(\beta).$$
- If $J =\downarrow_{\preceq} \alpha$, *i.e.*, it is the set elements $\beta \preceq \alpha$ (with the stratification order):
  Let $[\alpha]$ be the equivalence class of α for the preorder $\preceq$.
  Let J' be the initial segment $J - [\alpha]$, and $\Re_{J'}$ the realizer of J' given by the IH.
  Let $\Re'_J : J \to \mathcal{P}(\Lambda_{\tau(D)})$ be a function (not yet a realizer) defined by
  $$\begin{aligned} \Re'_J(\beta) &= \Re_{J'}(\beta) && \text{if } \beta \in J', \\ \Re'_J(\beta) &= \mathcal{N}^-_\beta \cup \bigcup_{\gamma\in J',\gamma\geq_D\beta} \Re_{J'}(\gamma) && \text{if } \beta{\simeq}\alpha \text{ and } \mathcal{V}(\beta){=}{+}, \\ \Re'_J(\beta) &= \mathcal{N}^+_\beta \cap \bigcap_{\gamma\in J',\gamma\leq_D\beta} \Re_{J'}(\gamma) && \text{if } \beta{\simeq}\alpha \text{ and } \mathcal{V}(\beta){=}{-}. \end{aligned}$$
  Let $H_J : \Lambda^{[\alpha]}_{\tau(D)} \to \Lambda^{[\alpha]}_{\tau(D)}$ be defined by
  $$H_J((X_\gamma)_{\gamma\simeq a})_{c\to\beta} = \Big(\bigcap_{\gamma\in c} X_\gamma \;\mapsto\; X_\beta\Big) \qquad \text{where } X_\gamma = \Re_{J'}(\gamma) \text{ for } \gamma \in J'$$
  When its component in J' is fixed to $\Re_{J'}$, $H_J \cup id_{J'}$ is monotone for $\subseteq_E$. Moreover, $H_J(\Re'_J) \supseteq_E \Re'_J$. Thus H_J has a least fixpoint $\Re_J$ above $\Re'_J$. The collection $\Re_J$ is a realizer since the conditions of Definition 2.4.2.7 are verified:
  1. For all $c{\to}\beta \in J'$ the IH gives $\Re_J(c{\to}\beta) = \Re_{J'}(c{\to}\beta) = \Re_J(c) \mapsto \Re_J(\beta)$ and for all $c{\to}\beta \in [\alpha]$, we have $\Re_J(c{\to}\beta) = \Re_J(c) \mapsto \Re_J(\beta)$ by definition of H_J
  2. $\Re'_J$ respects Item (2) and H_J conserves the property.

  It is moreover a S^J-realizer:
  - For all $\beta \in J$, $\Re'_J(\beta) \in S^J_\beta = S^D_\beta$:
    * If $\beta \in J'$: since $\Re_{J'}$ is a $S^{J'}$-realizer
    * If $\beta \simeq \alpha$ and $\mathcal{V}(\beta) = +$. Then trivially $\mathcal{N}^-_\beta \subseteq \Re'_J(\beta)$. Moreover, if $M \in \Re'_J(\beta)$ then either $M \in \mathcal{N}^-_\beta \subseteq \mathcal{N}^+_\beta$, or there is $\gamma \geq_D \beta$ such that $M \in \Re_{J'}(\gamma)$. In the last case, for any $\delta \leq_D \beta$, since $\beta \leq_D \gamma$ and $\Re_{J'}$ is a $S^{J'}$-realizer, $\tau_\delta(M){\Downarrow}^{\mathrm{h}}$, so that $M \in \mathcal{N}^+_\beta$.
    * If $\beta \simeq \alpha$ and $\mathcal{V}(\beta) = -$. Then trivially $\Re'_J(\beta) \subseteq \mathcal{N}^-_\beta$. Moreover, for all $M \in \mathcal{N}^-_\beta$, there is $\delta \geq_D \beta$ such that $M \to^* \bar{\epsilon}_\delta + L$. Then for all $\gamma \in J'$ such that $\gamma \leq_D \beta$, we have $\delta \geq_D \beta$ and since $\Re_{J'}$ is a $S^{J'}$-realizer $M \in \Re_{J'}(\gamma)$, so that $M \in \Re'_J(\beta)$.

– S^J is an atractor of $H_J \cup id_{J'}$:
Suppose that $\Re''_J \in S^J$ and $\Re''_{J\upharpoonright_{J'}} = \Re_{J'}$.
Then for all $\beta \in J'$, trivially $(H_J \cup id_{J'})(\Re''_J)(\beta) = \Re''_J(\beta) \in S^{J'}(\beta) = S^J(\beta)$ (by induction hypothesis).
For all $b{\to}\beta \simeq \alpha$:

  * Let $M \in (H_J \cup id_{J'})(\Re''_J)(b{\to}\beta)$, we have to show that $M \in \mathcal{N}^+_{b\to\beta}$.
  Let $b' \geq_D b$ and $\beta' \leq_D \beta$, we have to show that $\tau_{b'\to\beta'}(M)\Downarrow^{\mathrm{h}}$. By extensionality, it is sufficient to show that $\tau_{\beta'}(M\ \bar{\boldsymbol{\epsilon}}_{b'})$ converges. But since $\bar{\boldsymbol{\epsilon}}_{b'} \in \bigcap_{\gamma\in b}\mathcal{N}^-_\gamma \subseteq \Re''_J(b)$, the hypothesis gives that $(M\ \bar{\boldsymbol{\epsilon}}_{b'}) \in \Re''_J(\beta) \subseteq \mathcal{N}^+_\gamma$, and thus that $\tau_{\beta'}(M\ \bar{\boldsymbol{\epsilon}}_{b'})$ converges.
  * Let $M \in \mathcal{N}^-_{b\to\beta}$, *i.e.*, $M \to^* \bar{\boldsymbol{\epsilon}}_{b'\to\beta'} + L$ with $b' \leq_D b$ and $\beta' \geq_D \beta$.
  We have to prove that $M \in (H_J \cup id_{J'})(\Re''_J)(b{\to}\beta)$, *i.e.*, that for any $\gamma \in b$ and $N \in \Re''_J(\gamma)$, the application $M\ N$ is in $\Re''_J(\beta)$. Since $N \in \Re''_J(\gamma) \subseteq \mathcal{N}^+_\gamma$, we know that for any $\gamma' \in b' \leq_D b$, the close test $\tau_{\gamma'}(M)$ converges to ϵ.
  Thus $M\ N \to^* \bar{\tau}_{\beta'}(\Pi_{\gamma'\in b'}\tau_{\gamma'}(M)) + L' \to^* \bar{\boldsymbol{\epsilon}}_{\beta'} + L'$ so that $M\ N \in \mathcal{N}^-_\beta \subseteq \Re''_J(\beta)$.

□

Remark 2.4.2.10. *This realizer is in fact the inverse of the interpretation* $[\![.]\!]_D$ *in the sens that* $\alpha \in [\![M]\!]_D$ *iff* $M \in \Re_D(\alpha)$.

Lemma 2.4.2.11. *Let* $\Re$ *be a* S^D*-realizer in D.*

$$\begin{array}{llllll} \text{if} & (\vec{a},\alpha) \in [\![M]\!]^{\vec{x}} & \text{and} & (\forall i, L_i \in \Re(a_i)) & \text{then} & M[\vec{L}/\vec{x}] \in \Re(\alpha) \\ \text{if} & \vec{a} \in [\![Q]\!]^{\vec{x}} & \text{and} & (\forall i, L_i \in \Re(a_i)) & \text{then} & Q[\vec{L}/\vec{x}] \to^* \epsilon \end{array}$$

Proof. By induction on M and Q:

- $M = x_i$:
If $(\vec{a};\alpha) \in [\![M]\!]^{\vec{x}}$, there exists $\beta \in a_i$ such that $\alpha \leq \beta$. Thus if $L_i \in \Re(a_i) \subseteq \Re(\beta) \subseteq \Re(\alpha)$, we have $M[\vec{L}/\vec{x}] = L_i \in \Re(\alpha)$.

- $M = N_1\ N_2$:
If $(\vec{a};\alpha) \in [\![M]\!]^{\vec{x}}$ there exists $b = \{\beta_1, ..., \beta_n\}$ such that $(\vec{a};b{\to}\alpha) \in [\![N_1]\!]^{\vec{x}}$ and $(\vec{a};\beta_j) \in [\![N_2]\!]^{\vec{x}}$ for all j. Thus, by IH, if for all i, $L_i \in \Re(a_i)$, $N_1[\vec{L}/\vec{x}] \in \Re(b{\to}\alpha) = (\bigcap_j \Re(\beta_j) \Rightarrow \Re(\alpha))$ and $N_2[\vec{L}/\vec{x}] \in \bigcap_j \Re(\beta_j)$.

- $M = \lambda y.N$:
If $(\vec{a};\alpha) \in [\![M]\!]^{\vec{x}_1}$ then $\alpha = c{\to}\beta$ and $((\vec{a},c);\beta) \in [\![N]\!]^{\vec{x}y}$.
We want to show that whenever $\forall i \leq |\vec{x}|$, $L_i \in \Re(a_i)$, we have

$$\lambda y.N[\vec{L}/\vec{x}] \in \Re(c{\to}\beta) = \bigcup_{\gamma\in c} \Re(\gamma) \Rightarrow \Re(\beta),$$

but if $L \in \Re(\gamma)$ for all $\gamma \in c$, the IH give us that $P[\vec{L}/\vec{x}][L/y] \in \Re(\beta)$

- $M = \Sigma_j \bar{\tau}_{\gamma_j}(Q_j)$:
  If $(\vec{a};\alpha) \in [\![\Sigma_j \bar{\tau}_{\gamma_j}(Q_j)]\!]^{\vec{x}}$, there is j such that $\alpha \leq \gamma_j$ and $\vec{a} \in [\![Q_j]\!]^{\vec{x}}$.
  By IH, if $\forall i \leq |\vec{x}|,\ L_i \in \Re(a_i)$ then $Q_i[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$ and $M[\vec{L}/\vec{x}] \to^* M' + \bar{\boldsymbol{\epsilon}}_{\gamma_j} \in \Re(\gamma_j) \subseteq \Re(\alpha)$, we can conclude by saturation of $\Re(\alpha)$

- $Q = \tau_\alpha(M)$:
  If $\vec{a} \in [\![\tau_\alpha(M)]\!]^{\vec{x}}$, we have $(\vec{a};\alpha) \in [\![M]\!]^{\vec{x}}$ and by IH, if $\forall i \leq |\vec{x}|,\ L_i \in \Re(a_i)$ then $M[\vec{L}/\vec{x}] \in \Re(\alpha)$, thus by definition, $\tau_\alpha(M[\vec{L}/\vec{x}]) \to^* \boldsymbol{\epsilon}$

- $Q = Q_1 \cdot Q_2$:
  If $\vec{a} \in [\![Q_1 \cdot Q_2]\!]^{\vec{x}}$, Then $\vec{a} \in [\![Q_1]\!]^{\vec{x}} \cap [\![Q_2]\!]^{\vec{x}}$ and by IH whenever $\forall i \leq |\vec{x}|,\ L_i \in \Re(a_i)$, $Q_1[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$ and $Q_2[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$, thus trivially $Q_1 \cdot Q_2 \to^* \boldsymbol{\epsilon}$

□

Theorem 2.4.2.12 (positivity implies sensibility).
Any stratified positive and extensional K-model D is sensible for $\Lambda_{\tau(D)}$.

Proof. Let D be a stratified positive extensional K-model.
Let $M \in \Lambda_{\tau(D)}$ be such that $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$ and $Q \in \boldsymbol{T}_{\tau(D)}$ such that $(\vec{a}) \in [\![Q]\!]^{\vec{x}}$.
By Lemma 2.4.2.9, there is an S^D-realizer $\Re$ in D.
Since for all $i \leq n$, $\bar{\boldsymbol{\epsilon}}_{a_i} \in \bigcap_{\gamma \in a_i} \mathcal{N}^-_\gamma \subseteq \Re(a_i)$, by Lemma 2.4.2.11 there is $M[\bar{\boldsymbol{\epsilon}}_{a_1}/x_1 ... \bar{\boldsymbol{\epsilon}}_{a_n}/x_n] \in \Re(\alpha)$. We know that $\Re(\alpha) \subseteq \mathcal{N}^+_\alpha$ contains only converging terms, thus $M[\bar{\boldsymbol{\epsilon}}_{a_1}/x_1 ... \bar{\boldsymbol{\epsilon}}_{a_n}/x_n]$ converges, in particular M converges (Rk. 2.3.1.17).
Similarly we can prove that Q converges. □

Example 2.4.2.13. *By Theorem 2.4.2.12 and Example 2.4.2.4, all the K-models D of Example 1.2.4.9 are sensible for* $\Lambda_{\tau(D)}$ *except* P_∞.

Here is a bypassing corollary:

Corollary 2.4.2.14. *Any stratified positive extensional K-model D is approximable.*

Proof. By Theorems 2.4.2.12 and 2.4.1.9. □

3. Quantitative subexponentials and their models

We have seen in Section 1.3 that the logics $B_{\mathcal{S}}LL$ offer a logical ground to the design of type systems allowing to express various co-effects.

In this section, we present various denotational models for $B_{\mathcal{S}}LL$. In the literature, there is a categorical axiomatisation of what is a model of $B_{\mathcal{S}}LL$ known by the name of *bounded exponential situation* (recalled here in Definition 3.1.1.2). This notion has been presented at first in [BGMZ14], but it originates from Melliès' works on parametrized monads [Mel13].[1] However, there is no known concrete category satisfying the axioms of a bounded exponential situation: the article [BGMZ14] gives only a realisability model for a few specific examples of semirings $\mathcal{S}$.

Our approach is based on a very simple intuition. We believe that behind any model of linear logic (*i.e.*, in any linear category Sec. A.3.2) lies a model of $B_{\mathcal{S}}LL$ for some *internal lax-semiring* $\mathcal{S}$. Moreover, we expect this *internal lax-semiring* to be non trivial for most models of interest. The goal of this chapter will then be to track the *internal lax-semiring* of a given linear category and to study its structure.

The first step is Section 3.1. We proceed by analyzing which kind of models of $B_{\mathcal{S}}LL$ can be found in a degenerated but rich structure such as the relational models of linear logic (Th. 3.1.3.8).

In Section 3.1.1, we recall and explain the notion of $\mathcal{S}$*-bounded exponential situation*, for $\mathcal{S}$ a bounded (lax-)semiring. Then, using our assumption on the existence of an internal lax-semiring we define the notion of *stratification* (Def 3.1.1.13) over a linear category. This stratification consists of decomposing the exponential comonad of the ambient linear category into a $\mathcal{S}$-bounded exponential situation.

Then, we study the relational model (Def. A.3.4.4) endowed with the free exponential (Sec. 3.1.2) and other possible non-free exponential comonads (Sec. 3.1.3). We will see that Rel can form a $\mathcal{S}$-bounded exponential situation for many different semirings $\mathcal{S}$. By choosing an adequate generalized exponential, we will see in Proposition 3.1.3.13 that we can form a $\mathcal{S}$-bounded exponential situation for any semiring $\mathcal{S}$.

Unfortunately, it is clear from the categorical definition of stratification (Def 3.1.1.13) that we loss of generality. Indeed, stratification relies on the existence of a degenerated morphism (the differential ∂). Nonetheless, this semantics is a first step and despite giving quite unsatisfactory results, it still gives insight and intuitions over the whole theory we are trying to build up.

In Section 3.2, we use these results to explicitly track the internal lax-semiring. Indeed, we will see that any linear category (or more exactly its 2-categorical counterpart Def. 3.2.1.2)

[1] See also Melliès' presentation "Sharing and Duplication in Tensorial Logic" at the workshop *Developments in Implicit Computational complExity* 2013.

intrinsically contains a natural *internal lax-semiring* hidden inside its hom-category $\mathcal{L}[!1, 1]$ (Th. 3.2.2.4).

Moreover, the linear category itself can be transformed to give a $\mathcal{S}$-bounded exponential situation relatively to any refining its internal semiring (Th.3.2.3.3). This transformation consists of using the slice category around the tensorial unit $\mathbb{1}$.

Going further, we will see that the notion of internal semiring naturally expands to a dependent version (Def. 3.2.4.5) that can be used to generalize $\mathrm{B}_{\mathcal{S}}$LL (Def. 3.2.5.1). This step is important to understand more elaborate bounded logics such as the original BLL [GSS92b] or DFuzz [GHH+13a] that carry some notions of dependency. Indeed, the major weakness of $\mathrm{B}_{\mathcal{S}}$LL is the lack of dependent types, in particular $\mathrm{B}_{\mathcal{S}}$LL is not an extension of the original bounded linear logic. The interest in this latter has been recently renovated by a series of works, like Dal Lago and Gaboardi's D_{ℓ}PCF [dLG11] or Gaboardi et al.'s DFuzz [GHH+13a]. These systems use parameters depending on variables which can be bounded and instantiated in the type derivation.

Our long-term perspective is to investigate whether this notion of internal semiring can be used to classify linear categories. For example, it seems that the linear categories whose Kleisli category is well-pointed is associated with the Boolean semiring. A general study of internal semirings for concrete examples of linear categories may be necessary, both for the understanding of quantitative semantics, and for the selection of suitable semirings regarding interesting applications of $\mathrm{B}_{\mathcal{S}}$LL.

3.1. Models of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

We give a general recipe for getting a bounded exponential situation out of a model of linear logic (Theorem 3.1.1.16). The main point of our construction is to specify the intuition that $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ corresponds to a stratification of the linear logic exponential comonad along the semiring $\mathcal{S}$: any model of linear logic admitting such a stratification (and one model can admit more than one) defines a model also of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$. From our point of view, this result, although simple, is meant to be a first step in relating the semantical notion of "approximants" (or "stratus") of the linear logic exponential, with a notion of co-effect annotation in a type system.

In Section 3.1.2, we apply our recipe to the category Rel of sets and relations, showing various examples of bounded exponential situations. The category Rel provides one of the simplest models of linear logic, where the exponential comonad is given by the finite multiset functor. We also extend the result to other linear categories such that ScottL, $\mathrm{Coh}^{\mathbb{B}}$ or $\mathrm{Coh}^{\mathbb{N}}$.

Further, in Section 3.1.3, in order to show the diversity of the models of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$, we consider a generalization of the comonad in Rel given by the notion of *multiplicity semiring* in Carraro *et al.*'s [CES10]. A multiplicity semiring $\mathcal{R}$ is a semiring satisfying some conditions (Definition 3.1.3.1) which generalize the natural number semiring $\mathbb{N}$. This generalization has been introduced in [CES10] for proving the existence of non-sensible models of the untyped λ-calculus in the category Rel. There, one can define a "universal" interpretation for any semiring $\mathcal{S}$ (Prop. 3.1.3.13).

3.1.1. Stratifying Linear Logic Exponentials

The notion of *linear category* (Sec. A.3.2) has been introduced [BBdPH93] as a categorical axiomatisation of a model of the multiplicative, exponential fragment of intuitionistic linear logic. This definition has been recently revisited [BGMZ14] with the notion of *$\mathcal{S}$-bounded exponential situation*, which roughly corresponds to a variant of linear category where the exponential comonad is parameterized by the elements of a partially ordered semiring $\mathcal{S}$, and which gives a categorical model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$.

In this section, we are discussing this characterization: explaining its intuitions, strengths and weaknesses. Our main contribution is the definition of the *stratification* of an exponential comonad along a semiring $\mathcal{S}$ (Definition 3.1.1.13). We use the fact that any linear category is a special case of a $\mathbb{1}$-bounded exponential situation (for the trivial semiring $\mathbb{1}$). We give a general recipe for extracting a bounded exponential situation from a linear category (Theorem 3.1.1.16). Section 3.1.2 will apply this recipe to the concrete case of the relational category.

Semirings as categories

In categories, formulas/types are the objects while derivations/terms are morphisms. What is the place of the semiring $\mathcal{S}$ bounding $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ along this scheme?

The semiring elements I, J... definitively live at the level of formulas since they are part of them (for example, in A^I), thus they should be objects of a certain category.

What about morphisms? In the rule:

$$\frac{\Gamma, A^I \vdash B \qquad J \geq I}{\Gamma, A^J \vdash B,}\ \mathcal{S}\mathtt{weak}$$

the right assertion $J \geq I$ appears like an axiom, which means that it has the status of a morphism. In fact, morphisms of $\mathcal{S}$ are defined by $\mathcal{S}(J, I)$ being empty if $J \not\geq I$ and a singleton if $J \geq I$.

In this framework, the sum and product are bifunctors $+ : (\mathcal{S} \times \mathcal{S}) \to \mathcal{S}$,$\cdot : (\mathcal{S} \times \mathcal{S}) \to \mathcal{S}$ so that their monotonicity corresponds to the bifunctoriality.

Similarly we can see that 0 and 1 are 0-ary functors $0 : \mathtt{unt} \to \mathcal{S}$ and $1 : \mathtt{unt} \to \mathcal{S}$.

The other axioms (associativity, distributivity...) are natural isomorphisms called $\mathtt{unt}^{+}$, $\mathtt{untL}^{\cdot}$, $\mathtt{untR}^{\cdot}$, $\mathtt{as}^{+}$, $\mathtt{as}^{\cdot}$, $\mathtt{com}^{+}$, $\mathtt{absR}$ and $\mathtt{dstL}$; the left absorption and the right distributivity are just natural transformations[2] $\mathtt{absL}_I : 0 \longrightarrow 0{\cdot}I$ and $\mathtt{dstR}_{I,J,K} : (I{\cdot}J) + (I{\cdot}K) \longrightarrow I{\cdot}(J + K)$.

Any coherence diagram will automatically be verified since the category is an order (there is at most one morphism between two objects). Thus we can say that the resulting category is a *bimonoidal category* (more exactly its lax version).

Remark 3.1.1.1. *The definition of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ *(Def. 1.3.1.1) could be naturally generalized to any lax-bimonoidal category, with richer morphism structure. Forcing the category to be an order means that we consider every proofs of $J \geq I$ to be the same. Nonetheless, we do not see (for the moment) any example of interest for such a refinement so that we will stay with semirings. In fact, the categorical view of the semiring as lax-bimonoidal categories will only serve for defining the categorical semantics, allowing to reason with functors and natural transformations (along Section 3.1 at least).*

The bounded exponential situation

A linear category $\mathcal{L}$ (Sec. A.3.2) gives a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ with $\mathcal{S}$ the trivial semiring (i.e. the usual intuitionistic MELL). In this case we have just one exponential modality, which is interpreted by the functor ! of $\mathcal{L}$ and its associated structures. When $\mathcal{S}$ is non-trivial, one has to parameterize the exponential modality ! by the elements of $\mathcal{S}$ and to propagate this parameterization along the whole structure; the interaction between these various modalities will follow the laws of the semiring $\mathcal{S}$. Such a structure has been suggested by Melliès and formally introduced [BGMZ14] along with the notion of *bounded exponential situation* which is a parameterized version of the linear category:

Definition 3.1.1.2 ([BGMZ14]). *Let $\mathcal{S}$ be a lax-semiring.*

*A $\mathcal{S}$-*bounded exponential situation *consists of:*

- *a symmetric monoidal closed category* $(\mathcal{L}, \otimes, \mathbb{1}, \multimap)$*, used to interpret the multiplicative fragment of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$*;*
- *a bifunctor* $(\_)^{-} : \mathcal{S} \times \mathcal{L} \to \mathcal{L}$ *together with six natural transformations:*

$$\mathtt{p}'_{I,J,a} : a^{I\cdot J} \longrightarrow a^{J^{I}}, \qquad \mathtt{d}'_{a} : a^{1} \longrightarrow a,$$
$$\mathtt{c}'_{I,J,a} : a^{I+J} \to a^{I} \otimes a^{J}, \qquad \mathtt{w}'_{a} : a^{0} \longrightarrow \mathbb{1},$$
$$\mathtt{m}'_{I,a,b} : a^{I} \otimes b^{I} \longrightarrow (a \otimes b)^{I}, \qquad \mathtt{m}'_{I,\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1}^{I},$$

[2] Recall that left absorption and right distributivity are directed in lax-semirings

> *which should satisfy parameterized versions of the diagrams of linear category. These*
>
> *diagrams are parameterized version of diagrams of linear category (Sec. A.3.2), except for the diagrams of Figure 3.1*

Remark 3.1.1.3. *A surprising point, regarding this semantics, is that each axiom of semiring corresponds exactly to a coherence diagram. We consider this fact as a hint for a strong relation between semirings and linear categories, which we will explicit in Section 3.2.*

Remark 3.1.1.4. *The question of the right distributivity* `DistR'` *and left absorption* `AbsR'` *in Figure 3.1 is crucial here. Since we relax the previous definition of* B$_\mathcal{S}$LL *[BGMZ14] with lax-semiring, we had to significantly modify the notion of bounded exponential situation.*

Indeed, if the directions of the other arrows of the form ${id_a}^{natTrans}$ *are not significant (they are isomorphisms); the directions for* ${id_a}^{\mathtt{absL}_I}$ *and* ${id_a}^{\mathtt{dstR}_{I,J,K}}$ *are forced by the orientation of the left absorption and right distribution. One could ask whether it is a safe choice.*

Along the remaining of the chapter, we will try to explain our choice of using a lax-semiring but here is a first explanation.[3] *When using* B$_\mathcal{S}$LL *as a type system in the style of Brunel* et al *[BGMZ14], the promotion of a term t is denoted* $!t$ *and the cut between two banged terms is denoted by a let rule:*

$$\frac{\Delta \vdash s : A^K \qquad \Gamma, x : A^K \vdash t : B}{\Gamma, \Delta \vdash \textbf{\textit{let}}\ !x := s\ \textbf{\textit{in}}\ t : B}$$

The reduction of the let rule is the following:

$$\textbf{\textit{let}}\ !x :=!s\ \textbf{\textit{in}}\ t \quad \to \quad t[s/x].$$

This rule have to be verified at logical level. In particular, the subject reduction forces the following reduction:

$$\frac{\dfrac{y : C^I \vdash s : A}{y : C^{I\cdot K} \vdash !s : A^K} \qquad \dfrac{y : C^J, x : A \vdash t : B}{y : C^{J\cdot K}, x : A^K \vdash !t : B^K}}{y : C^{I\cdot K + J\cdot K} \vdash \textbf{\textit{let}}\ !x :=!s\ \textbf{\textit{in}}\ !t : B^K} \quad \to \quad \frac{\dfrac{y : C^I \vdash s : A \qquad y : C^J, x : A \vdash t : B}{y : C^{I+J} \vdash t[s/x] : B}}{y : C^{(I+J)\cdot K} \vdash !(t[s/x]) : B^K}$$

Of course, this rule is not correct (the conclusions are different). This shows that we need to perform an approximation in the right sequent (corresponding exactly to the direction of the lax right distribution):

$$\frac{\dfrac{y : C^I \vdash s : A}{y : C^{I\cdot K} \vdash !s : A^K} \qquad \dfrac{y : C^J, x : A \vdash t : B}{y : C^{J\cdot K}, x : A^K \vdash !t : B^K}}{y : C^{I\cdot K + J\cdot K} \vdash \textbf{\textit{let}}\ !x :=!s\ \textbf{\textit{in}}\ !t : B^K} \quad \to \quad \frac{\dfrac{\dfrac{y : C^I \vdash s : A \qquad y : C^J, x : A \vdash t : B}{y : C^{I+J} \vdash t[s/x] : B}}{y : C^{(I+J)\cdot K} \vdash !(t[s/x]) : B^K}}{y : C^{I\cdot K + J\cdot K} \vdash !(t[s/x]) : B^K}$$

In conclusion, the right distribution means that commuting a contraction with a promotion is possible but only one way: the earlier the contraction is, the more accurate we are.

Remark 3.1.1.5. *This axiomatisation has been obtained by parameterization of the linear category. However, it is not clear whether it is sufficient or not.*

Indeed the soundness proof of the linear category [Bie94] strongly refers to the Eilenberg-Moore category over $(!, \mathtt{d}, \mathtt{p})$.

[3] I have to thanks Dominic Orchard for this remark.

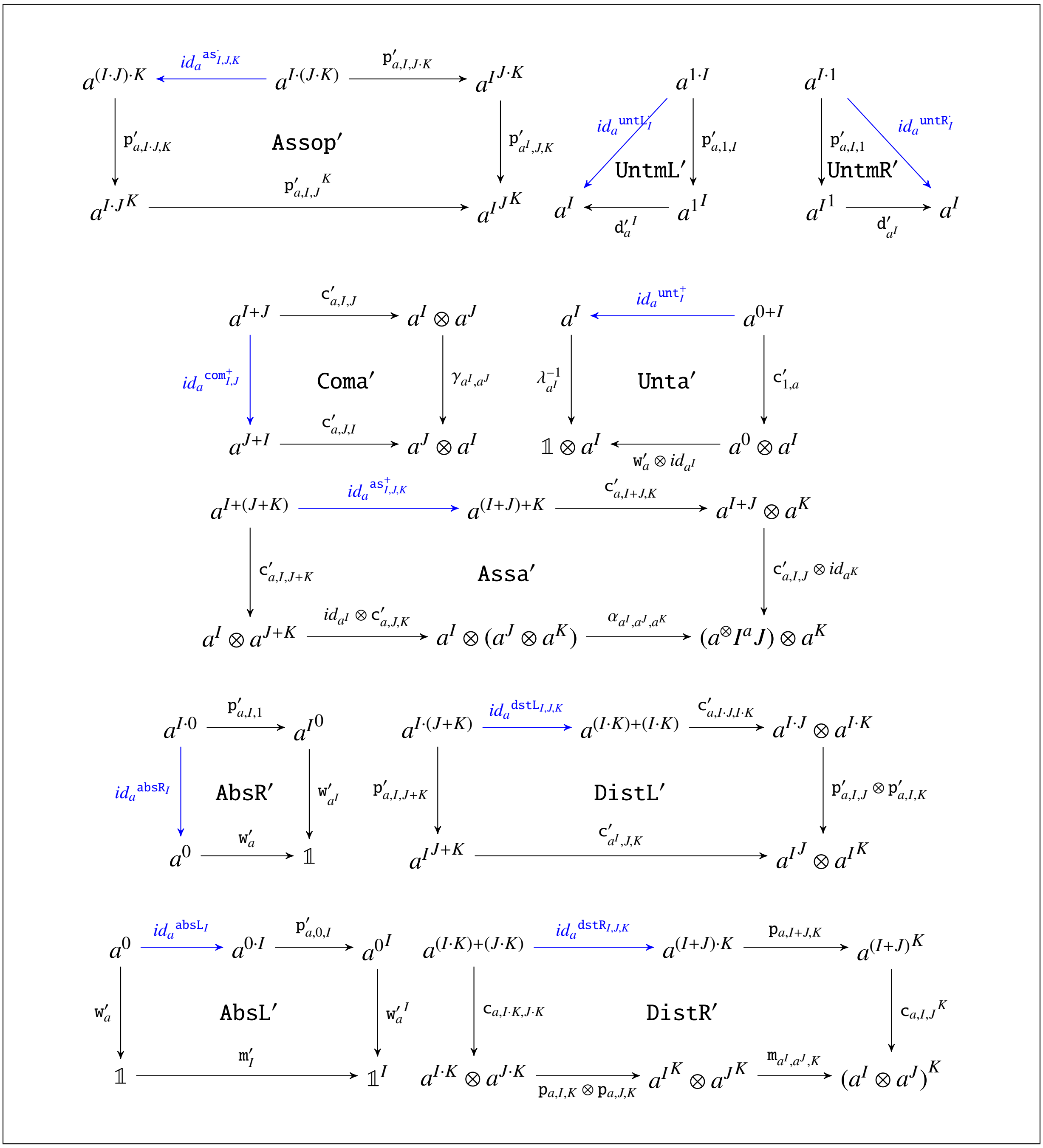


Figure 3.1.: Principal diagrams of the $\mathcal{S}$-bounded exponential situation. Here we denote a^{I^J} for $(a^I)^J$.

For example, the diagrams AbsR *and* DistR *(Sec.A.3.2) are supposed to be a consequence of a stronger requirement that every free coalgebra should be a co-monoid morphism. This requirement is not fully translated here and is only replaced by diagrams* AbsR′ *and* DistR′*.*

In the parametric version, a notion of parameterized Eilenberg-Moore category would be the category of $(a, (h_I)_{I\in\mathcal{S}})$ *with* h_I *natural in* I *and* $h_J ; {h_I}^J = h_{I\cdot J} ; \mathtt{p}_{a,I,J}$*. However with such a parameterized notion,* $(a, I_{a,I,J})$ *is not a parametric coalgebra, and* AbsR′ *and* DistR′ *do not follow from requiring all free parametric coalgebras to be co-monoid morphisms.*

A last remark is that AbsL *and* DistL *are supposed to offer a structure of coalgebra morphism to* w *and* c*, which does not make sense here (domains* w′ *and* c′ *are not parameterized coalgebras). This is only a reminiscence of Remark 3.1.1.4.*

Remark 3.1.1.6. *Another reasonable choice of categorical semantic would be to parameterize the new-Seely semantics [Bie94] which is equivalent in the non parameterized case. The new-Seely semantics is based on the isomorphism* $!a\otimes !b \simeq !(a \,\&\, b)$ *that has to be parameterized by* $s_{a,I,b,J} : a^I \otimes b^J \simeq (a \,\&\, b)^{I+J}$ *(in order to encode contraction).*

If the two semantics are equivalent in the non-parametric setting, it seems that they are different when parameterized. One can recover the bounded exponential situation from a parameterized new-Seely semantics via the following interpretation of $\mathtt{m}_{a,b,I}$*:*

$$a^I \otimes b^I \xrightarrow{s_{a,I,b,I}} (a \,\&\, b)^{I+I} \xrightarrow{id_{a\otimes b}^{I+I\geq 2\cdot I}} (a \,\&\, b)^{2\cdot I} \xrightarrow{\mathtt{p}'_{a\&b,2,I}} (a \,\&\, b)^{2^I} \xrightarrow{(\mathtt{c}';(\mathtt{d}'\otimes\mathtt{d}'))^I} ((a \,\&\, b)\otimes(a \,\&\, b))^I \xrightarrow{\pi_1\otimes\pi_2} (a\otimes b)^I,$$

where $(I{+}I \geq 2{\cdot}I) := (\mathtt{untL}^{\cdot -1}, \mathtt{untL}^{\cdot -1}); \mathtt{dstR}$ *uses the right distributivity on along the allowed direction.*

However, it is not possible to go the other way around. In fact, the parameterized new-Seely semantic can be described as too strong. Indeed, it implies the existance of the following morphism:

$$a^I \otimes b^J \xrightarrow{\quad s_{a,I,b,J}\quad} (a \,\&\, b)^{I+J} \xrightarrow{\quad s^{-1}_{a,J,b,I}\quad} a^J \otimes b^I$$

which is not so natural. In particular, for $\mathcal{S} = \mathbb{N}$*, we get for every* $m, n \in \mathbb{N}$ *the morphism:*

$$a^m \xrightarrow{\lambda} a^m\otimes\mathbb{1} \xrightarrow{id\otimes\mathtt{m}_n\otimes\cdots\otimes\mathtt{m}_n} a^m\otimes\mathbb{1}^n \xrightarrow{s_{a,m,\mathbb{1},n}} (a \,\&\, \mathbb{1})^{m+n} \xrightarrow{s^{-1}_{a,n,\mathbb{1},m}} a^n\otimes\mathbb{1}^m \xrightarrow{id_{a^n}\otimes(\mathtt{c};\ldots;\mathtt{c})} a^n\otimes\underbrace{\mathbb{1}^1\otimes\cdots\mathbb{1}^1}_{m\text{ times}} \xrightarrow{id\otimes(\mathtt{d}_{\mathbb{1}}\otimes\cdots\mathtt{d}_{\mathbb{1}})} a^n\otimes\underbrace{\mathbb{1}\otimes\cdots\mathbb{1}}_{m\text{ times}} \xrightarrow{\lambda;\ldots;\lambda} a^n$$

which is clearly not desired in general.

Notice, however, that this semantics is promising for the semantics of syntaxical extensions using "shapes" in the idea of Petricek et al *[POM13].*

Finally, we can express the conjecture form Brunel *et. al.* [BGMZ14] that foresee the correction of this axiomatization. Unfortunately, despite the semantic being reasonable, we did not check all the details.

Conjecture 3.1.1.7. *A* $\mathcal{S}$*-bounded exponential situation gives a model of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$*.*

Interpretation of semiring

We do not just study one logic $B_{\mathcal{S}}LL$, but a class of logics indexed by semirings. From this generalization emerges a structure resulting from simulations/functors between the logics. Indeed, among all these logics there is some that are finer than others. For example, $B_{\mathbb{B}}LL$ (Ex. of Sec. 1.3) is finer that $B_{\mathbb{1}}LL$, in the sense that any derivation in $B_{\mathbb{1}}LL$ can be injectively simulated by the same derivation in $B_{\mathbb{B}}LL$: the only possible amount of resource is tt and any structural weakening (Weak) is simulated by structural weakening followed by a resource weakening ($\mathcal{S}$weak) (that collapses ff into tt). It is only natural for the semantic to respect this structure.

Definition 3.1.1.8. *An* interpretation *of the lax-semiring $\mathcal{S}$ into the lax-semiring $\mathcal{R}$ is a bimonoidal functor.*[4] *This means that there is a function $[\![-]\!] : \mathcal{S} \mapsto \mathcal{R}$ such that (for all $I, J \in \mathcal{S}$):*

$$I \leq_{\mathcal{S}} J \;\Rightarrow\; [\![I]\!] \leq_{\mathcal{R}} [\![J]\!], \qquad [\![I]\!] +_{\mathcal{R}} [\![J]\!] \leq_{\mathcal{R}} [\![I +_{\mathcal{S}} J]\!], \qquad [\![I]\!] \cdot_{\mathcal{R}} [\![J]\!] \leq_{\mathcal{R}} [\![I \cdot_{\mathcal{S}} J]\!],$$
$$0_{\mathcal{R}} \leq_{\mathcal{R}} [\![0_{\mathcal{S}}]\!], \qquad 1_{\mathcal{R}} \leq_{\mathcal{R}} [\![1_{\mathcal{S}}]\!].$$

The embedding $[\![-]\!]$ is said to be a refinement *of $\mathcal{S}$.*

Remark 3.1.1.9. *If bimonoidal categories were considered in place of semirings, $[\![-]\!]$ should be a functor and several diagrams should commute.*

As a result, any interpretation of $\mathcal{S}$ into $\mathcal{R}$ defines a simulation of $B_{\mathcal{S}}LL$ into $B_{\mathcal{R}}LL$, *i.e.*, a coherent translation of formulas, derivation trees and cut-eliminations.

The simulation of formulas is the identity except for the exponential:

$$[\![A^I]\!] := [\![A]\!]^{[\![I]\!]}$$

The interpretation of derivation rules are also immediate except for those that use exponentials (and are subject to equations of Definition 3.1.1.8) :

$$\left[\!\!\left[\dfrac{\Gamma \vdash B \qquad I \geq 0_{\mathcal{S}}}{\Gamma, A^I \vdash B}\text{Weak} \right]\!\!\right] \quad := \quad \dfrac{[\![\Gamma]\!] \vdash [\![B]\!] \qquad \dfrac{\dfrac{I \geq 0_{\mathcal{S}}}{[\![I]\!] \geq [\![0_{\mathcal{S}}]\!]} \qquad [\![0_{\mathcal{S}}]\!] \geq 0_{\mathcal{R}}}{[\![I]\!] \geq 0_{\mathcal{R}}}}{[\![\Gamma]\!], [\![A]\!]^{[\![I]\!]} \vdash [\![B]\!]}\text{Weak}$$

where the right side of the equation is strictly equivalent to:

$$\dfrac{[\![\Gamma]\!] \vdash [\![B]\!] \qquad [\![I]\!] \geq 0_{\mathcal{R}}}{[\![\Gamma]\!], [\![A]\!]^{[\![I]\!]} \vdash [\![B]\!]}\text{Weak}$$

Similarly, each rule is simulated by the same rule where every formula and every semiring annotation is translated.

It is trivial to check that each step of cut elimination is simulated by the same step in the co-domain system.[5]

[4] In fact we should say colax-lax-bimonoidal functor: "lax" because those are lax-semiring and "colax" because we only ask for natural transformation mapping $[\![0_{\mathcal{S}}]\!]$ to $0_{\mathcal{R}}$...etc...

[5] Here we have a trivial translation step by step so that we do not need to expand further.

Proposition 3.1.1.10. *Given any interpretation* $[\![-]\!]$ *of* $\mathcal{S}$ *into* $\mathcal{R}$*, any* $\mathcal{R}$*-bounded exponential situation gives a* $\mathcal{S}$*-bounded exponential situation by pre-composition.*

Proof. Given a $\mathcal{R}$-bounded exponential situation $(\mathcal{L}, (\_)^{-}, \mathtt{p}', \mathtt{d}', \mathtt{c}', \mathtt{w}', \mathtt{m}')$, we can get a $\mathcal{S}$-bounded exponential situation by setting $a^I := a^{[\![I]\!]}$ and translating the natural transformations, for example:

$$\mathtt{d}''_a := \quad a^1 = a^{[\![1]\!]} \xrightarrow{id_a^{[\![1]\!] \geq [\![1]\!]}} a^1 \xrightarrow{\mathtt{d}'_a} a$$

$$\mathtt{p}''_{a,I,J} := \quad a^{I \cdot J} = a^{[\![I \cdot J]\!]} \xrightarrow{id_a^{[\![I \cdot J]\!] \geq [\![I]\!] \cdot [\![J]\!]}} a^{[\![I]\!] \cdot [\![J]\!]} \xrightarrow{\mathtt{p}'_{a,[\![I]\!],[\![J]\!]}} a^{[\![I]\!][\![J]\!]} = a^{IJ}$$

The coherence diagrams are then trivially obtained using the fact that any diagram commutes in $\mathcal{R}$ (since it is an order). One has to be more careful with the extension to any bimonoidal category. □

Example 3.1.1.11.

- *The trivial semiring is final,* i.e., *any* $\mathrm{B}_{\mathcal{S}}$LL *is simulated by* LL.
- *Setting* $[\![*]\!] = tt$ *gives an interpretation of the trivial semiring into the Boolean semiring* $\mathbb{B}$*. However, the trivial semiring cannot be interpreted into* $\mathbb{N}$ *since* $\mathrm{B}_{\mathbb{1}}$LL *is equivalent to* LL *which is stronger (in term of logical power) that* $\mathrm{B}_{\mathbb{N}}$LL.
- *Recalling Example A.4.1.3, the semiring* $\mathcal{P}(\mathbb{N})$ *interprets* $\mathbb{N}$*,* $\mathbb{N}_f$*,* $\mathbb{B}$ *and* $\mathbb{Z}_2$*:*

  $\mathbb{N}_f$: $[\![n]\!] = \{n\}$,

  $\mathbb{N}$: $[\![n]\!] = \{m \leq n\}$,

  $\mathbb{B}$: $[\![tt]\!] = \{n \mid n \neq 0\}$ *and* $[\![ff]\!] = \{0\}$,

  $\mathbb{Z}_2$: $[\![0]\!] = \{2n \mid n \in \mathbb{N}\}$ *and* $[\![1]\!] = \{2n+1 \mid n \in \mathbb{N}\}$.

  Remark, for example, that for $\mathbb{B}$ *we have*

  $$[\![tt \vee tt]\!] = \{n > 0\} \supsetneq \{n_1 + n_2 \mid n_1, n_2 > 0\} = [\![tt]\!] + [\![tt]\!].$$

From the first example we have the following:

Proposition 3.1.1.12. *Any linear category* $\mathcal{L}$ *defines a* $\mathcal{S}$*-bounded situation for any semiring* $\mathcal{S}$ *by setting* $a^I := !a$.

However, models mentioned in Proposition 3.1.1.12 are critically degenerated as they collapse all bounding parameters $I \in \mathcal{S}$ to the linear logic exponential.

Stratification

Following the idea that in any linear category $\mathcal{L}$ is hidden an internal semiring, one should expect the interpretation of the usual exponential of LL to result from an interpretation of the trivial semiring into the internal semiring. There are some assumptions we will use to describe intuitively our positions:

- First, we presume the existence of some internal semiring $\mathcal{S}_{\mathcal{L}}$,

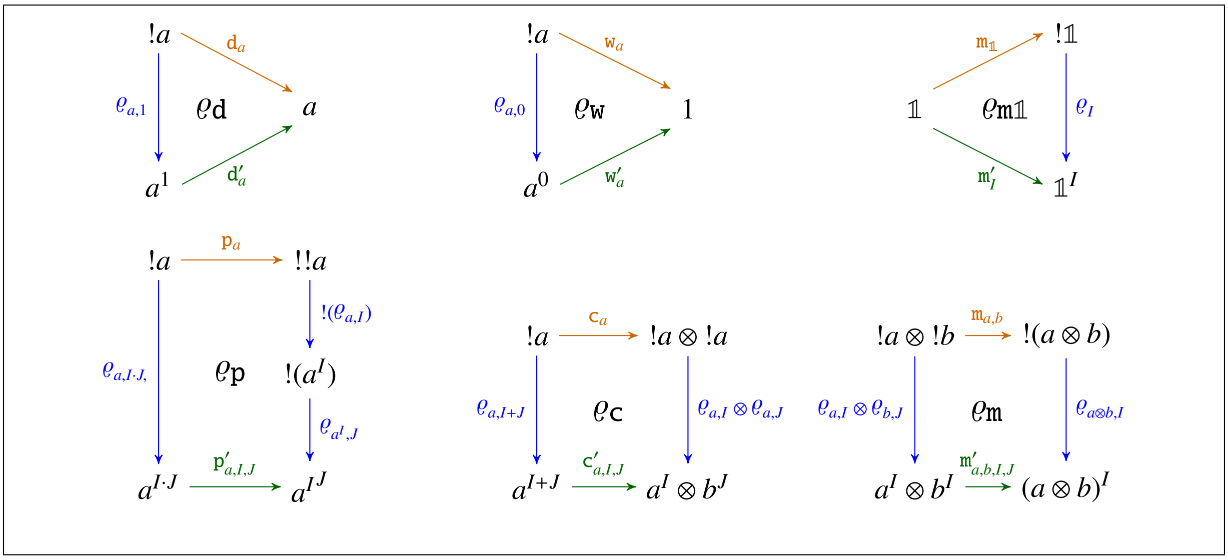


Figure 3.2.: Coherence diagrams between the natural transformation ϱ and the exponential structure $(!, \mathtt{d}, \mathtt{p}, \mathtt{w}, \mathtt{c}, \mathtt{m})$ of a linear category.

- then we assume that $\mathcal{S}_{\mathcal{L}}$ interprets the trivial semiring $\mathbb{1}$,
- moreover we assume that the image $[\![*]\!]$ of this interpretation is the top, denoted $\top$, of the internal semiring $\mathcal{S}_{\mathcal{L}}$.

Following these hypotheses, there should be a morphism $\varrho_{a,I} : a^{\top \geq I} :!A \to a^I$ for any $a \in \mathcal{L}$ and any $I \in \mathcal{S}$. Moreover, if we presume that this morphism is an isomorphism, then one can recover all the bounded exponential situation. For example, the contraction and digging are given by

$$\mathtt{c}'_{a,I,J} := \quad a^{I+J} \xrightarrow{\varrho^{-1}_{a,I+J}} !a \xrightarrow{\mathtt{c}_a} !a \otimes !a \xrightarrow{\varrho_{a,I} \otimes \varrho_{a,J}} a^I \otimes a^J$$

$$\mathtt{p}'_{a,I,J} := \quad a^{I \cdot J} \xrightarrow{\varrho^{-1}_{a,I \cdot J}} !a \xrightarrow{\mathtt{p}_a} !!a \xrightarrow{!\varrho_{a,I}} !(a^I) \xrightarrow{\varrho_{a^I,J}} a^{I^J}.$$

In order to relieve those strong hypothesis, we do not ask for ϱ to be inversible, but to be a natural epimorphism. We will see that this general recipe can be applied in several different situations with interesting and various results.

Definition 3.1.1.13. *A* stratification of a linear category $\mathcal{L}$ *is given by:*

- *an ordered semiring $\mathcal{S}$ (seen as a bimonoidal category);*
- *a bifunctor $(\_)^{\cdot} : \mathcal{S} \times \mathcal{L} \to \mathcal{L}$;*
- *a natural transformation $\varrho_{I,a} : !a \longrightarrow a^I$*
- *such that $\varrho_{I,a}$ is an epimorphism (Def. A.1.0.11) for every I and a, which means that for any morphism $\phi, \psi : a^I \to b$, if $\varrho; \phi = \varrho; \psi$ then $\phi = \psi$,*

- *such that there exists the morphisms:*[6]

$$\mathtt{p}'_{I,J,a} : a^{I\cdot J} \to a^{J^I}, \qquad \mathtt{d}'_a : a^1 \to a,$$
$$\mathtt{c}'_{I,J,a} : a^{I+J} \to a^I \otimes a^J, \qquad \mathtt{w}'_a : a^0 \to \mathbb{1},$$
$$\mathtt{m}'_{I,a,b} : a^I \otimes b^I \to (a \otimes b)^I, \qquad \mathtt{m}'_{I,\mathbb{1}} : \mathbb{1} \to \mathbb{1}^I,$$

that complete the diagrams of Figure 3.2.

Remark 3.1.1.14. *Notice that all the diagrams of Figure 3.2 simply state that each natural transformation e required for the linearity of $\mathcal{L}$ is transported along ϱ to its parameterized version e'.*

Notice, moreover, that when they exist $\mathtt{p}'$, $\mathtt{d}'$, $\mathtt{c}'$, $\mathtt{w}'$, $\mathtt{m}_{\mathbb{1}}$ and $\mathtt{m}$ are uniquely determined due to the universal property of epimorphisms and the diagrams of Figure 3.2. That is why we say that a stratification is a triple $(\mathcal{L}, (\_)^{\text{-}}, \varrho)$.

Lemma 3.1.1.15. *If $\varrho_{a,I}$ and $\varrho_{a,J}$ are epimorphisms, so is $\varrho_{a,I} \otimes \varrho_{a,J}$.*

Proof. We use the closure of $\otimes$. Indeed,

$$
\begin{array}{lcl}
 & (\varrho_{a,I} \otimes \varrho_{a,J}); \phi = (\varrho_{a,I} \otimes \varrho_{a,J}); \psi & \\
\Rightarrow & (\varrho_{a,I} \otimes id); ((id \otimes \varrho_{a,J}); \phi) = (\varrho_{a,I} \otimes id); ((id \otimes \varrho_{a,J}); \psi) & \text{rewriting} \\
\Rightarrow & \varrho_{a,I}; \Lambda((id \otimes \varrho_{a,J}); \phi) = \varrho_{a,I}; \Lambda((id \otimes \varrho_{a,J}); \psi) & \text{Curryfication} \\
\Rightarrow & \Lambda((id \otimes \varrho_{a,J}); \phi) = \Lambda((id \otimes \varrho_{a,J}); \psi) & \text{since } \varrho_{a,I} \text{ is epi} \\
\Rightarrow & (id \otimes \varrho_{a,J}); \phi = (id \otimes \varrho_{a,J}); \psi & \text{decurryfication} \\
\Rightarrow & \varrho_{a,J}; \Lambda(\phi) = \varrho_{a,J}; \Lambda(\psi) & \text{Curryfication} \\
\Rightarrow & \Lambda(\phi) = \Lambda(\psi) & \text{since } \varrho_{b,I} \text{ is epi} \\
\Rightarrow & \phi = \psi & \text{decurryfication}
\end{array}
$$

□

Theorem 3.1.1.16 (Stratification to bounded exponential). *A stratification $(\mathcal{S}, (\_)^{\text{-}}, \varrho)$ of a linear category yields a $\mathcal{S}$-bounded exponential situation, as defined in Definition 3.1.1.2, hence a model of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$.

Proof. The point is that all the structure defining a bounded exponential situation can be trivially obtained by translating back the structure of the linear category $\mathcal{L}$ along the epimorphism ϱ.

The naturality and the commutation of the diagrams associated with these transformations are obtained by using the usual diagrams enjoyed by a linear category and the diagrams of Figure 3.2 as well as the universal property of epimorphisms. For example, Figure 3.3 gives the commutation that the morphism $\mathtt{p}'_{a,I,J}$ should enjoy in order to give a positive action. The triangle **I** is naturality of ϱ over the associativity of the semiring multiplication, the square **IV** is the usual one of a linear category, **V**

[6] Which are not required to be natural.

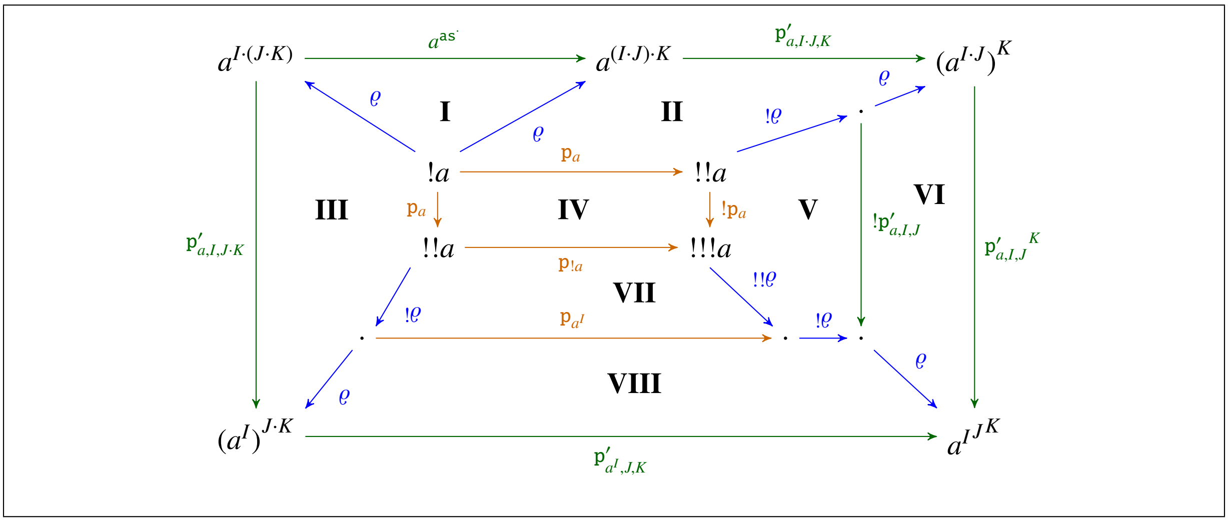
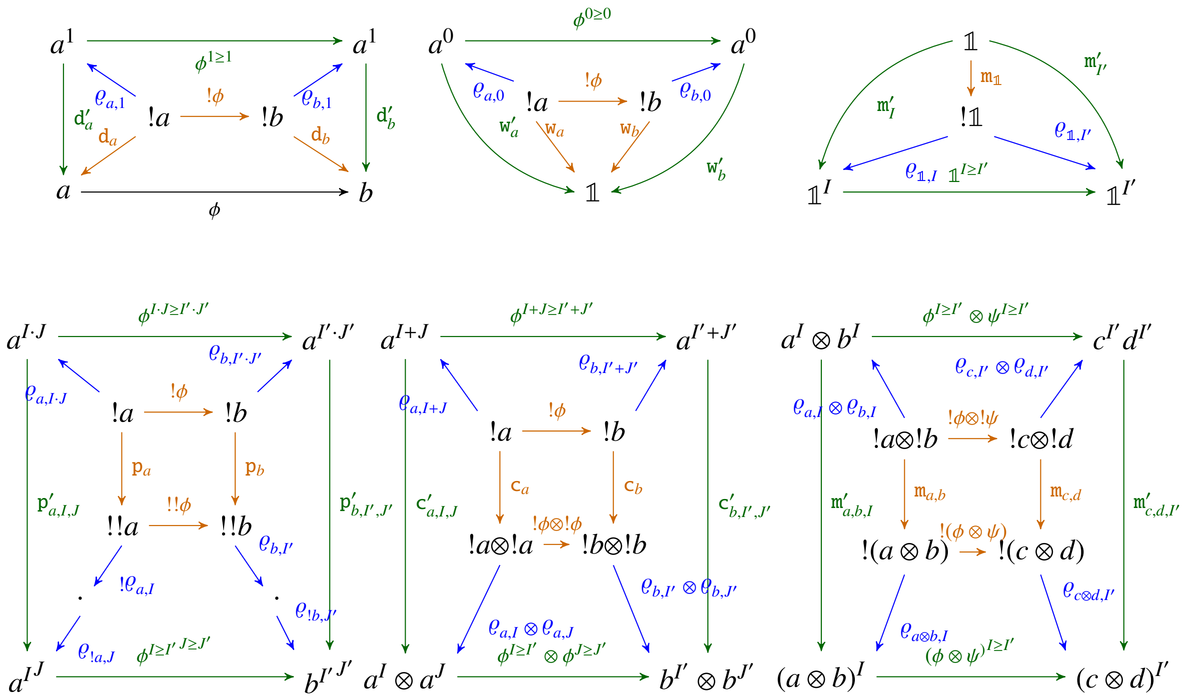


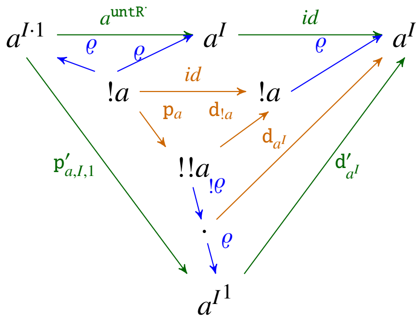
Figure 3.3.: An example of the proof of the commutation of the diagrams needed to have a bounded exponential situation.

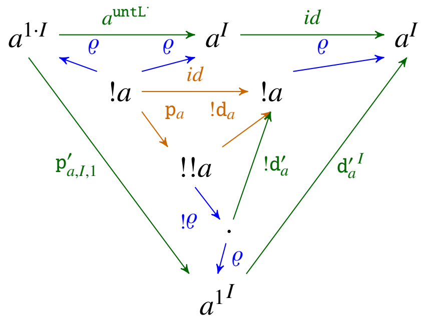
uses the promotion of the square on the first line of Figure 3.2, **VI** and **VII** are the naturality of, resp., ϱ and $\mathtt{p}$, and finally **II**, **III** and **VIII** are again squares of Figure 3.2. Notice that this is *a priori* not sufficient to obtain the commutation of the external cell due to the first ϱ that point on the wrong direction. However, we actually obtain that $\varrho; A^{\mathtt{as}^{\cdot}}; \mathtt{p}'; \mathtt{p}'^{K} = \varrho; \mathtt{p}'; \mathtt{p}'$ which results in the commutation of the external cell by the universal property of the epimorphism ϱ.

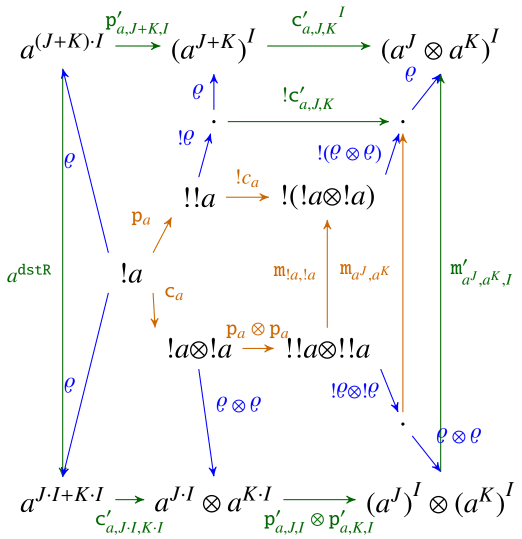
The naturality diagrams are obtained by using the universal property of epimorphism over the following diagrams:

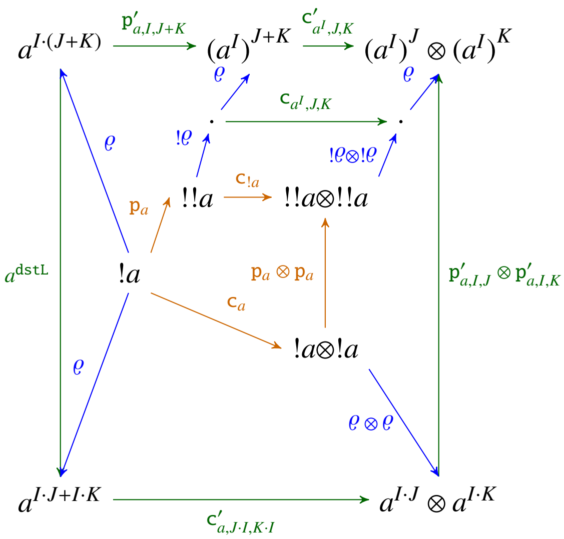


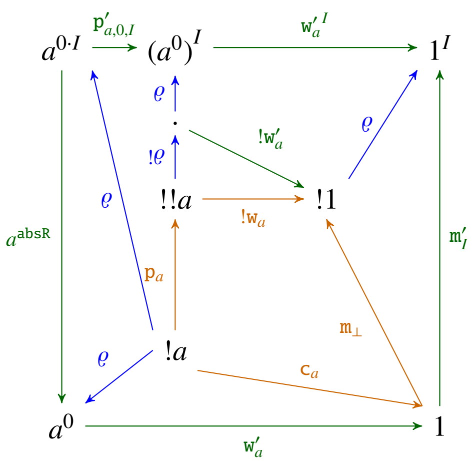
We show here the diagrams proving the most non trivial equation:

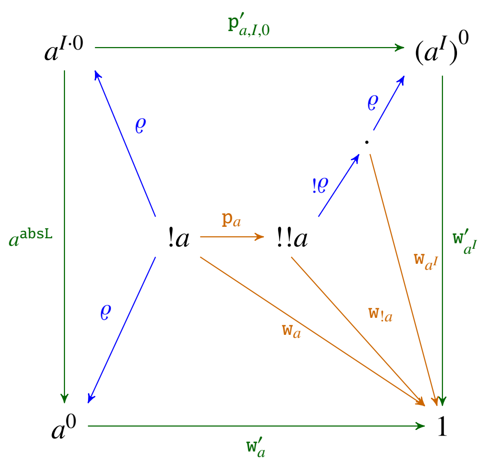

$a^{I\cdot 1}$
$a^{\mathtt{untR}}$
a^I
id
$!a$
$\mathtt{p}_a$
$\mathtt{d}_{!a}$
$\mathtt{d}_{a^I}$
$!!a$
$\mathtt{p}'_{a,I,1}$
$\mathtt{d}'_{a^I}$
a^{I^1}

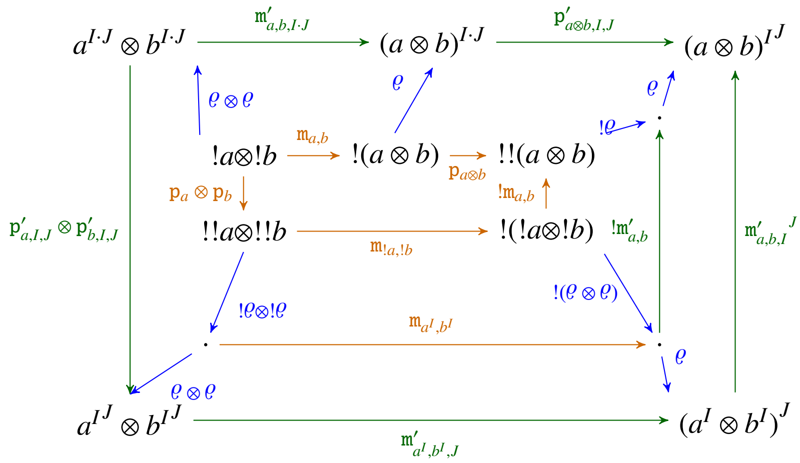

$a^{1\cdot I}$
$a^{\mathtt{untL}}$
a^I
id
$!a$
$\mathtt{p}_a$
$!\mathtt{d}_a$
$!!a$
$!\mathtt{d}'_a$
$\mathtt{p}'_{a,I,1}$
$\mathtt{d}'_a{}^I$
a^{1^I}


$a^{(J+K)\cdot I}$
$\mathtt{p}'_{a,J+K,I}$
$(a^{J+K})^I$
$\mathtt{c}'_{a,J,K}{}^I$
$(a^J \otimes a^K)^I$
$!\mathtt{c}'_{a,J,K}$
$!(\varrho \otimes \varrho)$
$!\mathtt{c}_a$
$!!a$
$!(!a \otimes !a)$
$a^{\mathtt{dstR}}$
$!a$
$\mathtt{c}_a$
$\mathtt{m}_{!a,!a}$
$\mathtt{m}_{a^J,a^K}$
$\mathtt{m}'_{a^J,a^K,I}$
$\mathtt{p}_a \otimes \mathtt{p}_a$
$!a \otimes !a$
$!!a \otimes !!a$
$!\varrho \otimes !\varrho$
$\varrho \otimes \varrho$
$a^{J\cdot I+K\cdot I}$
$\mathtt{c}'_{a,J\cdot I,K\cdot I}$
$a^{J\cdot I} \otimes a^{K\cdot I}$
$\mathtt{p}'_{a,J,I} \otimes \mathtt{p}'_{a,K,I}$
$(a^J)^I \otimes (a^K)^I$


$a^{I\cdot(J+K)}$
$\mathtt{p}'_{a,I,J+K}$
$(a^I)^{J+K}$
$\mathtt{c}'_{a^I,J,K}$
$(a^I)^J \otimes (a^I)^K$
$\mathtt{c}_{a^I,J,K}$
$!\varrho \otimes !\varrho$
$\mathtt{c}_{!a}$
$!!a$
$!!a \otimes !!a$
$a^{\mathtt{dstL}}$
$!a$
$\mathtt{p}_a \otimes \mathtt{p}_a$
$\mathtt{p}'_{a,I,J} \otimes \mathtt{p}'_{a,I,K}$
$\mathtt{c}_a$
$!a \otimes !a$
$\varrho \otimes \varrho$
$a^{I\cdot J+I\cdot K}$
$\mathtt{c}'_{a,J\cdot I,K\cdot I}$
$a^{I\cdot J} \otimes a^{I\cdot K}$


$a^{0\cdot I}$
$\mathtt{p}'_{a,0,I}$
$(a^0)^I$
$\mathtt{w}'_a{}^I$
1^I
$!\mathtt{w}'_a$
$!!a$
$!\mathtt{w}_a$
$!1$
$a^{\mathtt{absR}}$
$\mathtt{m}'_I$
$\mathtt{p}_a$
$\mathtt{m}_\perp$
$!a$
$\mathtt{c}_a$
a^0
$\mathtt{w}'_a$
1


$a^{I\cdot 0}$
$\mathtt{p}'_{a,I,0}$
$(a^I)^0$
$a^{\mathtt{absL}}$
$!a$
$\mathtt{p}_a$
$!!a$
$\mathtt{w}_{a^I}$
$\mathtt{w}'_{a^I}$
$\mathtt{w}_{!a}$
$\mathtt{w}_a$
a^0
$\mathtt{w}'_a$
1



□

Remark 3.1.1.17. *It is clear that they should be a higher order reformulation that would present this construction as an epimorphism between exponential actions.*

3.1.2. Concrete examples of stratifications

Stratification over the relational model

Recall (Def. A.3.3.3) that the category Rel, having sets as objects and relations as morphisms, forms a linear category $\text{Rel}^{\mathbb{N}}$ when endowed with the multiset comonad:

$$!a := \mathbb{N}_f\langle A\rangle \qquad\qquad !\phi := \{([\alpha_1, ..., \alpha_n], [\beta_1, ..., \beta_n]) \mid \forall i \leq n, (\alpha_i, \beta_i) \in \phi\}$$

We show how to associate with an ordered semiring $\mathcal{S}$ a stratification of the linear category $\text{Rel}^{\mathbb{N}}$, for any multiplicity semiring $\mathcal{R}$. The key-point is that any stratification over the exponential object $!\mathbb{1}$ actually extends to a stratification over any exponential object of $\text{Rel}^{\mathbb{N}}$. Since we are in a set-theoretical framework (the hom-sets are powersets) and $!\mathbb{1}$ is isomorphic to $\mathbb{N}$, stratifying $!\mathbb{1}$ corresponds to interpret the ordered semiring $\mathcal{S}$ into the powerset $\mathcal{P}(\mathbb{N})$. Definition 3.1.2.1 gives the conditions that such interpretation should enjoy in order to induce a stratification over the whole $\text{Rel}^{\mathbb{N}}$ (Th. 3.1.2.2).

Definition 3.1.2.1. *A* relational interpretation *for a lax-semiring $\mathcal{S}$ is an interpretation (Def. 3.1.1.8) $[\![-]\!] : \mathcal{S} \mapsto \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is the powerset lax-semiring of Example A.4.1.3. This means that $[\![-]\!]$ have to verify that (for all $I, J \in \mathcal{S}$):*

$$I \leq_{\mathcal{S}} J \;\Rightarrow\; [\![I]\!] \subseteq [\![J]\!], \qquad [\![I]\!] \oplus [\![J]\!] \subseteq [\![I +_{\mathcal{S}} J]\!], \qquad [\![I]\!] \odot [\![J]\!] \subseteq [\![I \cdot_{\mathcal{S}} J]\!],$$
$$\{0_{\mathbb{N}}\} \subseteq [\![0_{\mathcal{S}}]\!], \qquad \{1_{\mathbb{N}}\} \subseteq [\![1_{\mathcal{S}}]\!].$$

Theorem 3.1.2.2 (Semiring interpretation of stratification). *Any relational interpretation* $[\![-]\!]$ *of an ordered semiring* $\mathcal{S}$ *induces a stratification of the linear category* $\textsc{Rel}^{\mathbb{N}}$*, defined by:*

$$a^I := \left\{[\alpha_1, ..., \alpha_n] \in !a \mid n \in [\![I]\!]\right\}, \qquad f^{I \geq J} := \{([\alpha_1, ..., \alpha_n], [\beta_1, ..., \beta_n]) \in !f \mid n \in [\![J]\!]\},$$

$$\varrho_{I,a} := \{(u, u) \mid u \in a^I\}.$$

In particular,[7] $[\![-]\!]$ *extends to a sound interpretation of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ *into* $\textsc{Rel}^{\mathbb{N}}$.

Proof. One should prove that the couple $((-)^{-}, \varrho)$ enjoys the conditions of Definition 3.1.1.13 each $\varrho_{a,I}$ is a surjective relation and thus an epimorphism. Lemma 3.1.2.3 shows that $(\_)^{-}$ is a bifunctor. Lemma 3.1.2.4 shows the naturality of ϱ. And Lemma 3.1.2.5 shows the commutation of diagrams of Figure 3.2. □

In the following we denote $|||[\alpha_1, ..., \alpha_n]||| := n$ the norm of a multiset $[\alpha_1, ..., \alpha_n] \in !a$.

Lemma 3.1.2.3. *For any relational interpretation* $[\![-]\!]$, *of a lax-semiring* $\mathcal{S}$ *the exponential* $(\_)^{-}$ *forms a bifunctor:*

$$(\_)^{-} : \mathcal{S} \times \textsc{Rel}^{\mathbb{N}} \to \textsc{Rel}^{\mathbb{N}}$$

Proof. For simplicity, we rewrite the exponential on morphisms as:

$$f^{I \geq J} := \{(u, v) \in !f \mid ||v|| \in [\![J]\!]\},$$

- The identity is preserved:

$$\begin{aligned} {id_A}^{I \geq I} &= \{(u, v) \in !id_A \mid ||v|| \in [\![I]\!]\} \\ &= \{(u, v) \in id_{!A} \mid ||v|| \in [\![I]\!]\} \\ &= \{(u, u) \mid ||u|| \in [\![I]\!]\} \\ &= id_{A^I}. \end{aligned}$$

- The composition is preserved:

$$\begin{aligned} f^{I \geq J}; g^{J \geq K} &= \{(u, w) \mid \exists v,\ (u, v) \in !f,\ (v, w) \in !g,\ ||v|| \in [\![J]\!], ||w|| \in [\![K]\!]\} \\ &= \{(u, w) \mid \exists v,\ (u, v) \in !f,\ (v, w) \in !g,\ ||w|| \in [\![K]\!]\} \\ &= \{(u, w) \mid (u, w) \in !f; !g,\ ||w|| \in [\![K]\!]\} \\ &= (f; g)^{I \geq K}. \end{aligned}$$

□

Lemma 3.1.2.4. *For any relational interpretation* $[\![-]\!]$ *of a lax-semiring* $\mathcal{S}$*, the transformation* $\varrho_{I,a} = \{(u, u) \mid u \in a^{[\![I]\!]}\} : !a \longrightarrow a^I$ *is natural.*

[7] Assuming Conjecture 3.1.1.7.

Proof.

- Naturality of $\varrho_{I,a}$ in a:
  Let $f \in \textsc{Rel}(a,b)$, we must prove that for all I, $!f;\varrho_{I,b} = \varrho_{I,a}; f^{I \leq I}$.
  Let $(u,v) \in !f;\varrho_{I,b}$ then there exists w such that $(u,w) \in !f$ and $(w,v) \in \varrho_{I,b}$ thus $v = w$ and $\|v\| \in [\![I]\!]$; thus $\|u\| = \|v\| \in [\![I]\!]$ and $(u,u) \in \varrho_{I,a}$ what concludes since $(u,v) \in f^{id_I}$. Conversely let $(u,v) \in \varrho_{I,a}; f^{id_I}$ then there exists w such that $(u,w) \in \varrho_{I,a}$ and $(w,v) \in f^{id_I}$ thus $u = w$ and $\|w\| \in [\![I]\!]$; thus $\|v\| = \|w\| \in [\![I]\!]$ and $(v,v) \in \varrho_{I,b}$ what concludes since $(u,v) \in !f$.

- Naturality of $\varrho_{I,a}$ in I:
  For I, J, we must prove that for all a, $\varrho_{J,a} = \varrho_{I,a}; {id_a}^{I \geq J}$, what is trivial since

$$ {id_a}^{I \geq J} = \{(u,u) \mid \|u\| \in [\![J]\!]\} : a^I \Longrightarrow a^J. $$

□

Lemma 3.1.2.5. *The couple* $((-)^-, \varrho)$ *makes the diagrams of Figure 3.2 commute for:*

$$ \begin{array}{lll} \mathtt{d}'_a := \mathtt{d}_a \cap (a^1 \times a) & \mathtt{w}'_a := \mathtt{w}_a \cap (a^0 \times \mathbb{1}) & \mathtt{m}'_I := \mathtt{m}_{\mathbb{1}} \cap (\mathbb{1} \times \mathbb{1}^I) \\ \mathtt{p}'_{a,I,J} := \mathtt{p}_a \cap (a^{I \cdot J} \times (a^I)^J) & \mathtt{c}'_{a,I,J} := \mathtt{c}_a \cap (a^{I+J} \times (a^I \otimes a^J)) & \mathtt{m}'_{a,b,J} := \mathtt{m}_{a,b} \cap ((a^I \otimes b^I) \times (a \otimes b)^I) \end{array} $$

Proof.

- Diagram (ϱd) in Figure 3.2:

$$ \begin{aligned} \varrho_{a,1}; \mathtt{d}'_a &= \{([\alpha], \alpha) \mid \alpha \in a, 1_{\mathbb{N}} \in [\![1_{\mathcal{S}}]\!]\} \\ \mathtt{d}_a &= \{([\alpha], \alpha) \mid \alpha \in a, 1_{\mathbb{N}} \in [\![1_{\mathcal{S}}]\!]\} \end{aligned} $$

  those two sets are identical because $1_{\mathbb{N}} \in [\![1_{\mathcal{S}}]\!]$ by Item 5 of Definition 3.1.2.1.

- Diagram (ϱp) in Figure 3.2:

$$ \begin{aligned} \varrho_{a,I \cdot J}; \mathtt{p}'_{a,I,J} &= \{(u,U) \mid u(a) = \Sigma_{v \in !a} v(a) \cdot U(v), \|u\| \in [\![I \cdot J]\!], dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\} \\ &= \{(u,U) \in \mathtt{p}_a \mid \|\Sigma_{v \in !a} v \cdot U(v)\| \in [\![I \cdot J]\!], dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\} \\ &= \{(u,U) \in \mathtt{p}_a \mid \Sigma_{a \in a} \Sigma_{v \in !a} v(a) \cdot U(v) \in [\![I \cdot J]\!], dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\} \\ &= \{(u,U) \in \mathtt{p}_a \mid \Sigma_{v \in !a} \|v\| \cdot U(v) \in [\![I \cdot J]\!], dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\} \\ &= \{(u,U) \in \mathtt{p}_a \mid \Sigma_{v \in dom(U)} \|v\| \cdot U(v) \in [\![I \cdot J]\!], dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\} \\ \mathtt{p}_a; !\varrho_{a,I}; \varrho_{a^I,J} &= \{(u,U) \in \mathtt{p}_a \mid dom(U) \subseteq a^I, \|U\| \in [\![J]\!]\}. \end{aligned} $$

  those two sets are identical. Indeed, if $\mathrm{dom}(U) \subseteq a^I$ and $\|U\| \in [\![J]\!]$ then we can apply Item 3 of Definition 3.1.2.1:

$$ \begin{aligned} \Sigma_{v \in \mathrm{dom}(U)} \|v\| \cdot U(v) &\subseteq \{\Sigma_{v \in \mathrm{dom}(U)} p_v \cdot q_v \mid \Sigma_{v \in \mathrm{dom}(U)} q_v \in [\![J]\!],\ \forall v \in \mathrm{dom}(U), p_v \in [\![I]\!]\} \\ &\subseteq [\![I]\!] \odot [\![J]\!] \\ &\subseteq [\![I \cdot J]\!]. \end{aligned} $$

- Diagram (ϱw) in Figure 3.2:

$$ \begin{aligned} \varrho_{a,0}; \mathtt{w}'_a &= \{([a], a) \mid a \in a, \|[a]\| \in [\![1]\!]\} \\ \mathtt{w} &= \{([a], a) \mid a \in a\} \end{aligned} $$

  The two sets are the same since $\|[a]\| = 1_{\mathbb{N}} \in [\![1_{\mathcal{S}}]\!]$ by Item 4 of Definition 3.1.2.1.

- Diagram (ϱc) in Figure 3.2:

$$\begin{aligned}\varrho_{a,I,+J};\mathtt{c}'_{a,I,J} &= \{(u,(v,w)) \mid u = v{+}w, \|u\| \in [\![I{+}J]\!], \|v\| \in [\![I]\!], \|w\| \in [\![J]\!]\}\\ &= \{(v{+}w,(v,w)) \mid \|v\|{+}\|w\| \in [\![I{+}J]\!], \|v\| \in [\![I]\!], \|w\| \in [\![J]\!]\}\\ \mathtt{c}_a;\varrho_{I,a}\otimes\varrho_{J,a} &= \{(v{+}w,(v,w)) \mid \|v\| \in [\![I]\!], \|w\| \in [\![J]\!]\}\end{aligned}$$

The two sets are the same because the conditions on v and w imply that on $v+w$, since $[\![I]\!] \oplus [\![J]\!] \subseteq [\![I+J]\!]$.

- Diagram ($\varrho\mathtt{m}$) in Figure 3.2:

$$\begin{aligned}(\varrho_{a,I}\otimes\varrho_{b,I});\mathtt{m}'_{a,b} &= \{((u,v),w) \mid u(\alpha) = \Sigma_{\beta\in b}w(\alpha,\beta),\ v(\beta) = \Sigma_{\alpha\in a}w(\alpha,\beta),\ \|u\|,\|v\|,\|w\| \in [\![I]\!]\}\\ &= \{(((u,v),w) \in \mathtt{m}_{a,b} \mid \|\alpha\mapsto\Sigma_\beta w(\alpha,\beta)\| \in [\![I]\!], \|\beta\mapsto\Sigma_\alpha w(\alpha,\beta)\| \in [\![I]\!], \|w\| \in [\![I]\!]\}\\ &= \{((u,v),w) \in \mathtt{m}_{a,b} \mid \Sigma_\alpha\Sigma_\beta w(\alpha,\beta) \in [\![I]\!], \Sigma_\beta\Sigma_\alpha w(\alpha,\beta) \in [\![I]\!], \|w\| \in [\![I]\!]\}\\ &= \{((u,v),w) \in \mathtt{m}_{a,b} \mid \Sigma_{(\alpha,\beta)\in a\times b}w(\alpha,\beta) \in [\![I]\!], \|w\| \in [\![I]\!]\}\\ &= \{((u,v),w) \in \mathtt{m}_{a,b} \mid \|w\| \in [\![I]\!]\}\\ &= \mathtt{m}_{a,b};\varrho_{a\otimes b,I}\end{aligned}$$

- Recall the the remaining diagram $\varrho\mathtt{m}\mathbb{1}$ is always trivial (Rk. 3.1.1.14).

□

Example 3.1.2.6. *Let us apply Theorem 3.1.2.2 to the ordered semirings discussed in Section 1.3.2.*

There is only one possible interpretation of the trivial semiring into the multiplicity semiring $\mathbb{N}$*, associating the unique element* $*$ *with the whole set* $\mathbb{N}$*. In fact, Definition 3.1.2.1 requires that* $[\![*]\!]$ *contains* $0, 1$ *and that it is closed under addition. This interpretation gives the usual multi-set based model of linear logic.*

The interpretation of a Boolean-based ordered semiring into $\mathbb{N}$ *depends on the order between* $\mathtt{tt}$ *and* ff*. In the case* $\mathit{ff} \leq \mathtt{tt}$*, we can set either* $[\![\mathtt{tt}]\!] = \mathbb{N}$ *and* $[\![\mathit{ff}]\!] = \{0\}$*, or* $[\![\mathtt{tt}]\!] = \mathbb{N} = [\![\mathit{ff}]\!]$*.*[8] *The latter collapses the two modalities to the usual multiset comonad, while the former interprets the formula* A^{ff} *by the singleton of the empty multiset, representing the type of unused resources. In the case* ff *and* $\mathtt{tt}$ *are incomparable in* $\mathcal{S}$*, then we can set* $[\![\mathtt{tt}]\!] = \mathbb{N} - \{0\}$ *and* $[\![\mathit{ff}]\!] = \{0\}$*, strictly distinguishing between used resources (type* $A^{\mathtt{tt}}$*) and unused resources (type* A^{ff}*).*

A way to rephrase Theorem 3.1.2.2 is that the bounded logic $\mathrm{B}_{\mathcal{P}(\mathbb{N})}\mathrm{LL}$ over powerset lax-semiring $\mathcal{P}(\mathbb{N})$ can be interpreted in $\textsc{Rel}^{\mathbb{N}}$. Indeed, all the other interpretations can be recovered from this one using Proposition 3.1.1.10.

In fact, the powerset lax-semiring $\mathcal{P}(\mathbb{N})$ also seems to be the finest of the lax-semiring that are relationally interpreted without collapse ($I \neq J \Rightarrow [\![A^I]\!] \neq [\![A^J]\!]$). That is why we expect $\mathcal{P}(\mathbb{N})$ to be a sort of "internal semiring" of $\textsc{Rel}^{\mathbb{N}}$. In section 3.2, we will see that it is indeed the case for a suitable notion of internal lax-semiring (Def. 3.2.2.4).

[8] There are some other uninteresting possibilities.

Coherent and Scott models

The stratification of Rel that we have studied in the previous section may appear degenerated. Indeed, the epiness of ϱ is too easily obtained in Rel that has every morphisms you need. We will now present some stratifications of other linear categories, namelly ScottL, $\text{Coh}^{\mathbb{B}}$ and $\text{Coh}^{\mathbb{N}}$ (Prop. A.3.4.6 and Def. A.3.3.7).

Proposition 3.1.2.7. *The linear category* $\text{Coh}^{\mathbb{N}}$ *is the category of coherence spaces (Def. A.3.3.6) endowed with the multiset exponential (Def. A.3.3.7):*

$$!a := (C_m(a), \{(\phi, \psi) \mid \forall\alpha \in \phi, \forall\beta \in \psi, \alpha \frown \beta\});$$
$$!\phi := \left\{([\alpha_1, ..., \alpha_n], [\beta_1, ..., \beta_n]) \;\middle|\; n \in \mathbb{N},\ \forall i \leq n, (\alpha_i, \beta_i) \in \phi\right\}$$

The following describes a stratification of $\text{Coh}^{\mathbb{N}}$ *by* $\mathcal{P}(\mathbb{N})$ *(for* $\phi \in \text{Coh}[a, b]$ *and* $I \subseteq J \subseteq \mathbb{N}$*):*

$$a^I := \{[\alpha_1 \cdots, \alpha_n] \in |!_{\mathbb{N}}a| \mid n \in I\}, \quad u \frown_{a^I} u' := u \frown_{!_{\mathbb{N}}a} u', \quad \phi^{I \supseteq J} := \{(u, v) \in !\phi \mid v \in b^J\}$$

with the codifferential natural transformation:

$$\varrho_{I,a} := \{(u, u) \mid u \in |a^I|\}, \qquad \textit{for } I \in \mathcal{P}(\mathbb{N}).$$

Proof. This is the same stratification as $\text{Rel}^{\mathbb{N}}$. In fact, since $\text{Coh}^{\mathbb{N}}$ is a sub linear category of $\text{Rel}^{\mathbb{N}}$, the only things to prove are:

- that $\phi^{I\supseteq J} \in \text{Coh}[a^I, b^J]$:
  - since the morphism $!\phi$ conserves the size of the multisets we have $\phi^{I\supseteq J} \subseteq (a^J \times b^J)$, and since $a^I \subseteq a^J$ whenever $I \supseteq J$, we have $\phi^{I\supseteq J} \subseteq (a^I \times b^J)$
  - $\phi^{I\supseteq J}$ is a clique of $!a \multimap !b$ since it is included in the clique $!\phi$, thus it is a clique of $a^I \multimap b^J$,
- that the $\varrho_{I,a}$ are morphisms, which is clear since it is a subclique of the identity morphism,
- that $\mathtt{d}'$, $\mathtt{p}'$, $\mathtt{w}'$, $\mathtt{c}'$ and $\mathtt{m}'$ are actualy transformations of $\text{Coh}^{\mathbb{N}}$: this is immediate since each of them are subcliques of their unparameterized version.

□

Proposition 3.1.2.8. *The linear category* $\text{Coh}^{\mathbb{B}}$ *is the category of coherence spaces (Def. A.3.3.6) endowed with the set exponential (Def. A.3.3.7):*

$$!a := (C(a), \{(\phi, \psi) \mid \forall\alpha \in \phi, \forall\beta \in \psi, \alpha \frown \beta\});$$
$$!\phi := \left\{(\{\alpha_1, ..., \alpha_n\}, \{\beta_1, ..., \beta_n\}) \in C(a) \times C(b) \;\middle|\; n \in \mathbb{N},\ \forall i \leq n, (\alpha_i, \beta_i) \in \phi\right\}$$

We define in Example A.4.1.3 the four element lax-semiring $\Diamond$ *which elements are* $|\Diamond| := \{0, 1, \bot, \top\}$ *where* $1{+}1 = 1{+}\top = 1$ *and where* $\bot$ *is a universal absorber (*$\bot{+}I{=}\bot{\cdot}I{=}I{\cdot}\bot{=}\bot$

excepts for $\bot \cdot 0 = 0$). Let $[\![\_]\!] : \Diamond \to \mathcal{P}(\mathbb{N})$ defined by:

$$[\![0]\!] := \{0\} \qquad [\![1]\!] := \mathbb{N} - \{0\} \qquad [\![\bot]\!] := \emptyset \qquad [\![\top]\!] := \mathbb{N}$$

The following describes a stratification of $\textsc{Coh}^{\mathbb{B}}$ *by* $\Diamond$ *(for $\phi \in \textsc{Coh}[a,b]$ and $I \geq J \subseteq \mathbb{N}$):*

$$a^I := \{\{\alpha_1 \cdots, \alpha_n\} \in |!_{\mathbb{B}}a| \mid n \in [\![I]\!]\}, \quad u \frown_{a^I} u' := u \frown_{!_{\mathbb{B}}a} u', \quad \phi^{I \supseteq J} := \{(u,v) \in !\phi \mid v \in b^J\}$$

with the codifferential natural transformation:

$$\varrho_{I,a} := \{(u,u) \mid u \in |a^I|\}, \qquad for\ I \in \Diamond.$$

Proof. First remark that the following holds (by case analysis over J):

$$I \geq J \quad \text{implies that} \quad a^I \supseteq a^J \tag{3.1}$$
$$(v,w) \in !\psi \quad \text{implies that} \quad v \in b^J \quad \Leftrightarrow \quad w \in c^J. \tag{3.2}$$

- We have $\phi^{I \geq J} \in \textsc{Coh}[a^I, b^J]$:
  - We get $\phi^{I \geq J} \subseteq (a^I \times b^J)$, indeed by Implication (3.2) we have $\phi^{I \geq J} \subseteq (a^J \times b^J)$ and we can conclude using Implication (3.1) over $I \geq J$,
  - moreover, $\phi^{I \geq J}$ is a clique of $!a \multimap !b$ since it is included in the clique $!\phi$, an thus it is a clique of $a^I \multimap b^J$.
- The functor is stable by identity and composition:

$$\begin{aligned}
id_a^{I \geq I} &= \{(u,v) \in !id_a \mid v \in a^I\} \\
&= \{(u,u) \in id_{!a} \mid v \in a^I\} \\
&= id_{a^I} \\
\phi^{I \geq J}; \psi^{J \geq K} &= \{(u,w) \mid \exists v,\ (u,v) \in !\phi,\ (v,w) \in !\psi,\ v \in b^J,\ w \in c^K\} \\
&= \{(u,w) \mid \exists v,\ (u,v) \in !\phi,\ (v,w) \in !\psi,\ v \in b^J,\ v \in c^J\ w \in c^K\} && \text{by Eq. (3.2)} \\
&= \{(u,w) \mid \exists v,\ (u,v) \in !\phi,\ (v,w) \in !\psi,\ w \in c^K\} && \text{by Eq. (3.1)} \\
&= (\phi;\psi)^{I \geq J}
\end{aligned}$$

- For each $a \in \textsc{Coh}$ and $I \in \Diamond$, $\varrho_{I,a} \in \textsc{Coh}[!a, a^I]$: already proved since $\varrho_{I,a} = id_a^{\top \geq I}$.
- The transformation ϱ is natural: idem.
- For each $a \in \textsc{Coh}$ and $I \in \Diamond$, $\varrho_{I,a}$ is epi: this is immediat since it is a surjective relation.
- If we set $\mathrm{d}'_a := \{(\{\alpha\}, \alpha) \mid \alpha \in a\}$ we get a morphism in $\textsc{Coh}[a^1, a]$ that verifies $\varrho_{a,1}; \mathrm{d}'_a = \mathrm{d}_a$:
  - $\mathrm{d}'_a \subseteq a^1 \times a$ is immediate,
  - that d'_a is a clique comes from it being a clique in $!a \multimap a$,
  - $\varrho_{a,1}; \mathrm{d}'_a = \mathrm{d}_a$ is immediate.
- If we set $\mathrm{p}'_{a,I,J} := \left\{ (\bigcup_{i \leq n} u_i, \{u_1, ..., u_n\}) \,\middle|\, n \in [\![J]\!], \forall i, u_i \in a^I,\ \forall i,j,\ u_i \frown_{a^I} u_j \right\}$ we get a morphism in $\textsc{Coh}[a^{I \cdot J}, a^{I^J}]$ that verifies $\varrho_{a,I \cdot J}; \mathrm{p}'_{a,I,J} = \mathrm{p}_a; !\varrho_{a,I}; \varrho_{a^I,J}$:

– $\mathtt{p}'_{a,I,J} \subseteq a^{I\cdot J} \times (a^{I^J})$ is an easy case analysis over

$$(n \in [\![J]\!],\ \forall i \leq n,\ u_i \in a^I) \quad \Rightarrow \quad \bigcup_{i\leq n} u_i \in a^{I\cdot J}, \tag{3.3}$$

– $\mathtt{p}'_{a,I,J}$ is a clique since it is a subset of $\mathtt{p}_a$,

– $\varrho_{a,I\cdot J}; \mathtt{p}'_{a,I,J} = \mathtt{p}_a; !\varrho_{a,I}; \varrho_{a^I,J}$ is immediate by Equation. (3.3).

- If we set $\mathtt{w}'_a := \{(\emptyset, *)\}$ we get a morphism in Coh$[a^0, \mathbb{1}]$ that verifies $\varrho_{a,0}; \mathtt{w}'_a = \mathtt{w}_a$:
  – $\mathtt{w}_a \subseteq a^0 \times \mathbb{1}$ is immediate,
  – that $\mathtt{w}'_a$ is a clique comes from it being a clique in $!a \multimap \mathbb{1}$,
  – $\varrho_{a,0}; \mathtt{w}'_a = \mathtt{w}_a$ is immediate.

- If we set $\mathtt{c}'_{a,I,J} := \left\{(u\cup v, (u,v)) \;\middle|\; u \in a^I,\ v \in a^J,\ u \frown_{!a} v\right\}$ we get a morphism in Coh$[a^{I+J}, a^I \otimes C^J]$ that verifies $\varrho_{a,I+J}; \mathtt{c}'_{a,I,J} = \mathtt{c}_a; (\varrho_{a,I} \otimes \varrho_{a,J})$:

– $\mathtt{c}_{a,I,J} \subseteq a^{I+J} \times (a^I \otimes a^J)$ is an easy case analisis over

$$(u \in a^I,\ v \in a^J) \quad \Rightarrow \quad (u \cup v) \in a^{I+J}, \tag{3.4}$$

– $\mathtt{c}'_{a,I,J}$ is a clique since it is a subset of $\mathtt{c}_a$,

– $\varrho_{a,I+J}; \mathtt{c}'_{a,I,J} = \mathtt{c}_a; (\varrho_{a,I} \otimes \varrho_{a,J})$ is immediate by Equation. (3.4).

- If we set $\mathtt{m}'_{a,b,I} := \left\{((u,v), u \times v) \;\middle|\; u \in a^I,\ v \in b^I\right\}$ we get a morphism in Coh$[a^I \otimes b^J, (a \otimes b)^I]$ that verifies $(\varrho_{a,I} \otimes \varrho_{b,I}); \mathtt{m}'_{a,b,I} = \mathtt{m}_{a,b}; (\varrho_{a\otimes b,I})$:

– $\mathtt{m}'_{a,b,I} \subseteq (a^I \otimes b^J) \times (a \otimes b)^I$ is an easy case analisis over

$$(u \in a^I,\ v \in b^I) \quad \Rightarrow \quad (u \times v) \in (a \otimes b)^I, \tag{3.5}$$

– $\mathtt{m}'_{a,b,I}$ is a clique since it is a subset of $\mathtt{m}_{a,b}$,

– $(\varrho_{a,I} \otimes \varrho_{b,I}); \mathtt{m}'_{a,b,I} = \mathtt{m}_{a,b}; (\varrho_{a\otimes b,I})$ is immediate by Equation. (3.5).

□

Remark 3.1.2.9. *The lax-semiring $\Diamond$ can be seen as a lax-semiring over $\mathcal{P}(\mathbb{B})$.*

Proposition 3.1.2.10. *Recall that the category* ScottL *of posets and linear functions between initial segments (Def.A.3.4.3), is a linear category when endowed with the antichain exponential (Prom. A.3.4.6):*

$$!a := \mathcal{A}_f(a); \qquad\qquad !\phi(U) := \downarrow\{v \mid \exists u \in U, \phi(\downarrow u) = \downarrow v\}$$

Recall that the bottomed Boolean lax-semiring $\mathbb{B}_\perp$ (Ex. A.4.1.3) is the three objects fully ordered lax-semiring $tt > ff > \perp$ where $\perp$ is a universal absorber ($\perp + I = \perp\cdot I = I\cdot\perp = \perp$ excepts for $\perp\cdot ff = ff$).

The following describes a stratification of ScottL *by $\mathbb{B}_\perp$:*

$$a^\perp := (\{\}, id) \qquad a^{ff} := (\{\emptyset\}, id) \qquad a^{tt} := !a \qquad \phi^{I\geq J}(U) := !\phi(U) \cap a^J$$

with the codifferential natural transformation:

$$\varrho_{a,I}(U) := U \cap a^I$$

Proof. Here is a little lemma for any $I \in \mathbb{B}_\perp$ and and $\phi \in$ ScottL$[a,b]$:

$$!\phi(U \cap a^I) \quad = \quad !\phi(U) \cap b^I. \tag{3.6}$$

For $I = \perp$ and for $I = tt$ it is trivial, for $I = ff$ it comes from the linearity of ϕ (so that $\phi(\emptyset) = \emptyset$).

- If $\phi \in$ ScottL$[a,b]$ then $\phi^{I\geq J}$ is linear since $!\phi$ is linear and its domains and codomain range over the initial segments of a^I and b^J since the domain and codomain of $!\phi$ range over the initial segments of $!a$ and $!b$ and since a^I and b^J are initial segments of $!a$ and $!b$ respectivelly.

- for any $a \in$ ScottL and any $I \in \mathbb{B}_\perp$,

$$id_a^{I\geq I} \; = !id_a \cap a^I = id_{a^I}$$

- for any $\phi : a \to b$, any $\psi : b \to c$ and any $I \geq J \geq K \in \mathbb{B}_\perp$:

$$\begin{aligned}
\phi^{I\geq J}; \psi^{J\geq K}(U) &= !\psi(!\phi(U) \cap b^J) \cap c^K \\
&= !\psi(!\phi(U)) \cap (c^J \cap c^K) && \text{by Eq.3.6} \\
&= !(\phi;\psi)(U) \cap (c^J \cap c^K) && \text{funct. of !} \\
&= !(\phi;\psi)(U) \cap c^K && \text{since } J \geq K \\
&= (\phi;\psi)^{I\geq K}(U).
\end{aligned}$$

- For any $\phi : a \to b$, and any $I \geq J \in \mathbb{B}_\perp$, we have

$$\begin{aligned}
!\phi; \varrho_{b,J}(U) &:= !\phi(U) \cap b^J \\
\varrho_{a,I}; \phi^{I\geq J}(U) &:= !\phi(U \cap a^I) \cap b^J \\
&:= !\phi(U) \cap b^I \cap b^J && \text{by Eq.3.6} \\
&:= !\phi(U) \cap b^J && \text{since} I \geq J,
\end{aligned}$$

  which prove the naturality of ϱ.

- For any $\phi, \psi : a^I \to b$, if $\varrho_{a,I}; \phi = \varrho_{a,I}; \psi$, then $\phi(U \cap a^I) = \psi(U \cap a^I)$ for any $U \in \mathcal{I}(a)$ so that $\phi = \psi$ since $\mathcal{I}(a^I) = \mathcal{I}(a) \cap a^I$; this conclud that $\varrho_{a,I}$ is an epimorphism.

- Diagram ϱd is verified for $\mathsf{d}'_a := \mathsf{d}_a$ since $\varrho_{a,tt} = id_{!a}$.

- Diagram ϱp is verified for $\mathsf{p}_{a,I,J}(U) := \mathsf{p}_a(U) \cap a^{I^J}$. Indeed, we have:

$$\begin{aligned}
(\mathsf{p}_a; !\varrho_{a,I}; \varrho_{a^I,J})(U) &= \downarrow_{\mathcal{A}_f(a)}\{\{u\} \mid u \in U\} \cap a^{I^J} \\
(\varrho_{a,I\cdot J}; \mathsf{p}_{a,I,J})(U) &= \downarrow_{\mathcal{A}_f(a)}\{\{u\} \mid u \in (U \cap a^{I\cdot J})\} \cap a^{I^J} \\
&= \downarrow_{\mathcal{A}_f(a)}\{\{u\} \mid u \in U\} \cap \mathcal{A}_f(a^{I\cdot J}) \cap a^{I^J},
\end{aligned}$$

  remains to check that $\mathcal{A}_f(a^{I\cdot J}) \supseteq a^{I^J}$ for any $I, J \in \mathbb{B}_\perp$ which is clear.

- Diagram $\varrho\mathtt{w}$ is verified by $\mathtt{w}'_a := \binom{\emptyset \mapsto \emptyset}{\{\emptyset\} \mapsto *}$. Indeed, we trivialy have $(\varrho_{a,0}; \mathtt{w}'_a)(\emptyset) = \emptyset = \mathtt{w}_a(\emptyset)$ and if $U \neq \emptyset$, we have:

$$\begin{aligned}(\varrho_{a,0}; \mathtt{w}'_a)(U) &= \mathtt{w}'_a(U \cap a^0)\\ &= \mathtt{w}'_a(\{\emptyset\})\\ &= \mathtt{w}_a(U)\end{aligned}$$

since $U \cap a^0 = U \cap \{\emptyset\}$ with U that is non empty and downward close and with $\emptyset$ wich is a bottom in $\mathcal{A}_f(a)$.

- Diagram $\varrho\mathsf{c}$ is verified by $\mathsf{c}'_{a,I,J}(U) := \{(u,v) \in a^I \times a^J \mid u, v \leq w \in U\}$ for $U \in a^{I+J}$. Indeed, we have (for any $U \in !a$):

$$\begin{aligned}(\mathsf{c}_a; (\varrho_{a,I} \otimes \varrho_{a,J})(U) &= \{(u,v) \mid u, v \leq w \in U\} \cap (a^I \times a^J)\\ &= \{(u,v) \in a^I \times a^J \mid u, v \leq w \in U\}\\ (\varrho_{a,I+J}; \mathsf{c}'_{a,I,J})(U) &= \{(u,v) \in a^I \times a^J \mid u, v \leq w \in U \cap a^{I+J}\}\end{aligned}$$

If $I = \bot$ or $J = \bot$ then both resolve to the emptyset, if $I = \mathit{tt}$ or $J = \mathit{tt}$ then $a^{I+J} = !a$ making the two terms equals and if $I = J = \mathit{ff}$ then $u = v = w = \emptyset$.

- Diagram $\varrho\mathtt{m}$ is verified by $\mathtt{m}'_{a,b,I}(U) := \downarrow\{u \times v \mid (u,v) \in U\}\}$. Indeed, we have:

$$\begin{aligned}(\mathtt{m}_{a,b}; \varrho_{a\otimes b,I})(U) &= \downarrow\{u \times v \in (a \otimes b)^I \mid (u,v) \in U\}\}\\ ((\varrho_{a,I} \otimes \varrho_{b,I}); \mathtt{m}'_{a,b,I})(U) &= \downarrow\{u \times v \mid (u,v) \in (U \cap a^I \times b^I)\}\end{aligned}$$

it is then easy to check that $u \times v \in (a \otimes b)^I$ iff $u \in a^I$ and $v \in b^I$.

- As usual, Diagram $\varrho\mathtt{m}\mathbb{1}$ is trivially obtained for $\mathtt{m}'_I = \mathtt{m}_{\mathbb{1}}; \varrho_{\mathbb{1},I}$.

□

3.1.3. A parametric generalisation with multiplicity exponential

In the previous section, we have seen several models and what seems to be their internal lax-semiring. However, there are too few examples to oversee a generality. In this section we will use a parameterized class of models called $\textsc{Rel}^{\mathcal{R}}$. They correspond to the relational model endowed with different, non-free, exponential comonads $!_{\mathcal{R}}$, where $\mathcal{R}$ ranges over a specific subclass of semirings called *multiplicity semirings* (Def. 3.1.3.1).

We will see that the "internal lax-semiring" depends on $\mathcal{R}$. In fact, it corresponds to $\mathcal{P}(\mathcal{R})$, the powerset lax-semiring over $\mathcal{R}$ (rather than $\mathbb{N}$) of Definition 3.1.3.4. Bypassing, we will show that this case is so large that any logic $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ can be interpreted in some $\textsc{Rel}^{\mathcal{R}}$ so that the quantitative information does not collapse (if $I \neq J$ the $[\![A^I]\!] \neq [\![A^J]\!]$).

Multiplicity semirings

It has been pointed out by Hyland et al. [HNPR06] that several linear categories such as Rel can be seen as Kleisli categories over some variant of the non-deterministic monad. This is a reminiscent of the granularity of these models that can decompose programs into an "aggregation' of their different possible behaviors.

In the case of REL, this fact could not be clearer, indeed, REL is the Kleisli of the non-deterministic monad (the powerset monad) in SET. It particular, this means that any exponential comonad have to somehow distribute over this powerset monad.

These were some informal intuitions, however, they correspond to a concrete notion of *multiplicity semiring* introduced by Carraro, Ehrhard and Salibra [CES10]. This notion was introduced to show that decomposing the exponential of the model Rel of linear logic with such a semiring still yields to a model of linear logic.[9]

Definition 3.1.3.1 ([CES10]). *A* multiplicity semiring *is a semiring* $\mathcal{R} = (|\mathcal{R}|, \cdot, 1, +, 0)$ *such that* $(p, q, r$ *will vary over* $\mathcal{R}$*):*

(MS1) $\mathcal{R}$ *is* positive*:* $p+q = 0$ *implies* $p = q = 0$*;*

(MS2) $\mathcal{R}$ *is* discrete*:* $p+q = 1$ *implies* $p = 0$ *or* $q = 0$*;*

(MS3) $\mathcal{R}$ *is* additively splitting*:* $p_1 + p_2 = q_1 + q_2$ *implies* $\exists r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}$*, such that*

$$p_i = r_{i,1} + r_{i,2}\ , \qquad\qquad q_i = r_{1,i} + r_{2,i}\ ;$$

(MS4) $\mathcal{R}$ *is* multiplicatively splitting*:* $q_1+q_2 = r \cdot p$ *implies there is* $l \in \mathbb{N}$ *such that for all* $j \leq l$*, we can find* $r_j, p_{1,j}, p_{2,j}$ *such that*

$$\begin{aligned} r &= r_1 + \cdots + r_l, \\ p &= p_{1,j} + p_{2,j} && \text{for all } j \leq l, \\ q_i &= r_1 \cdot p_{i,1} + \cdots + r_k \cdot p_{i,l}. \end{aligned}$$

The notion of multiplicity semiring given by Definition 3.1.3.1 is a slight generalization of the one in [CES10], because the multiplicative splitting has been slightly relaxed. It is straightforward to check that all proofs in [CES10] still hold, in particular we have Theorem 3.1.3.3.

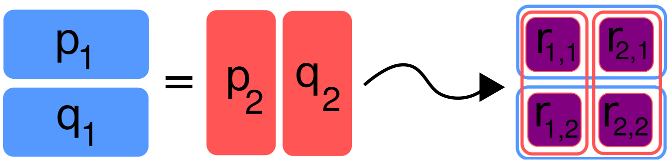
First, we present some intuitions for these rules:

(MS1): This equation states that there is no negative at all, *i.e.* no pairs of resources that cancel each other. Technically, this rule will be needed for the naturality of the weakening in the proof of Theorem 3.1.3.3.

(MS2) This equation says that $1_{\mathcal{R}}$ is a non-breakable atom. Technically, this rule will be needed for the naturality of the dereliction in the proof of Theorem 3.1.3.3.

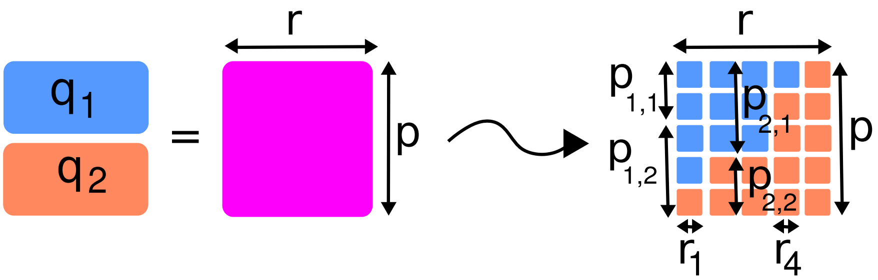
(MS3) This equation says that if two different summations result in the same quantity, then there is a finer decomposition factorizing them:

[9] Those were used to create relational models for pure λ-calculus that are not sensible (Def 1.2.1.7)



Technically, this rule will be needed for the naturality of the contraction in the proof of Theorem 3.1.3.3.

(MS4) This equation says that if a sum and a product result in the same quantity, then there is a finite decomposition of this result that factorizes both (here $l = 5$):



Technically, this rule will be needed for the naturality of the digging in the proof of Theorem 3.1.3.3.

Example 3.1.3.2. *The semiring of natural numbers $\mathbb{N}$ is the prototypical example of multiplicity semiring, while the Boolean semiring (as well as any cyclic semiring) is a non-example because the discreteness condition fails. In fact, one can show that any multiplicity must contain $\mathbb{N}$ as a sub-semiring generated by* 0 *and* 1*. The completed natural numbers $\bar{\mathbb{N}}$ of Example A.4.1.3 forms a multiplicity semiring, as well as the polynomial semiring of Example A.4.2.3.*

The category REL of sets and relation can be turned into a linear category using the multiset comonad as exponential modality (see Appendix A.3.3). The following is a generalization by Carraro et al. [CES10]. In place of multisets, the exponential modality is composed of the free $\mathcal{R}$-semimodules (Def. A.4.1.4), for any multiplicity semiring $\mathcal{R}$.

Theorem 3.1.3.3 ($\mathrm{REL}^{\mathcal{R}}$ **[CES10]).** *In the monoidal category* REL *of Example A.1.0.4 (Prop. A.3.4.4), any multiplicity semiring $\mathcal{R}$ defines an exponential comonad making it a linear category with:*

- *the exponential functor is defined by (for $r \in \mathrm{REL}(a,b)$):*

$$
\begin{aligned}
!_{\mathcal{R}} a &:= \mathcal{R}_f\langle a\rangle,\\
!_{\mathcal{R}} r &:= \{(u,v) \in \mathrm{REL}(!_{\mathcal{R}} a, !_{\mathcal{R}} b) \mid \exists \sigma \in \mathcal{R}_f\langle r\rangle,\ u(\alpha) = \sum_{\beta\in b} \sigma(\alpha,\beta),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v(\beta) = \sum_{\alpha\in a} \sigma(\alpha,\beta)\},
\end{aligned}
$$

- *the dereliction is $\mathsf{d}_a := \{(\delta_\alpha, \alpha) \mid \alpha \in a\} : !_{\mathcal{R}} a \to a$, where $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\alpha') = 0$ for every $\alpha \neq \alpha'$,*

- *the digging is* $\mathtt{p}_a := \{(m, M) \mid \forall\alpha \in a, m(\alpha) = \Sigma_{n\in !_{\mathcal{R}}a} n(\alpha)M(n)\} : !_{\mathcal{R}}a \to !_{\mathcal{R}}!_{\mathcal{R}}a$*,*
- *the contraction is* $\mathtt{c}_a := \{(u, (v_1, v_2)) \mid \forall\alpha \in a, u(\alpha) = v_1(\alpha) + v_2(\alpha)\} : !_{\mathcal{R}}a \to !_{\mathcal{R}}a \otimes !_{\mathcal{R}}a$*,*
- *the weakening is* $\mathtt{w}_a = \{(0, *)\} : !_{\mathcal{R}}a \to \mathbb{1}$*, where* 0 *denotes the constant zero function in* $\mathcal{R}_f\langle a\rangle$*,*
- *the 0-ary promotion is* $\mathtt{m}_{\mathbb{1}} = \{(*, u) \mid u \in !_{\mathcal{R}}\mathbb{1}\} : \mathbb{1} \to !_{\mathcal{R}}\mathbb{1}$*,*
- *and the binary promotion is* $\mathtt{m}_{a,b} := \{((u_1, u_2), v) \mid u_1(\alpha) = \Sigma_\beta v(\alpha,\beta),\ u_2(\beta) = \Sigma_\alpha v(\alpha,\beta)\} : (!_{\mathcal{R}}a \otimes !_{\mathcal{R}}b) \to !_{\mathcal{R}}(a \otimes b)$*.*

We denote by $\mathrm{REL}^{\mathcal{R}}$ *the linear category induced by this exponential comonad.*

The powerset lax-semiring of a multiplicity semiring

The notion of internal semiring for multiplicity exponentials should extend the notion of internal semiring for $\mathrm{REL}^{\mathbb{N}}$ (the powerset lax semiring $\mathcal{P}(\mathbb{N})$ over natural numbers). It is only natural to expect the internal semiring of $\mathrm{REL}^{\mathcal{R}}$ to be some $\mathcal{P}(\mathcal{R})$.

We will see in this section that if $\mathcal{R}$ is a multiplicity semiring, then its powerset can be endowed with a lax-semiring structure. In fact, we conjecture that the conditions are equivalent.

Definition 3.1.3.4. *Given a multiplicity semiring* $\mathcal{R}$*, we define the* powerset lax-semiring *of* $\mathcal{R}$ *as* $\mathcal{P}(\mathcal{R})$ *endowed with the structure* $(\odot, \{1_{\mathcal{R}}\}, \oplus, \{0_{\mathcal{R}}\}, \subseteq)$ *where the operations are defined by (*I, J, K *vary over* $\mathcal{P}(\mathcal{R})$*):*

$$I \oplus J := \{p +_{\mathcal{R}} q \mid p \in I, q \in J\},$$
$$I \odot J := \Big\{\sum_{i=1}^{k} p_i \cdot_{\mathcal{R}} q_i \mid k \leq 0, \sum_{i=1}^{k} q_i \in J, \forall i \leq k, p_i \in I\Big\}.$$

Lemma 3.1.3.5. *For* $I, J \in \mathcal{P}(\mathcal{R})$ *and* $k \in \mathbb{N}$*, any sequence* $(r_i)_{i\leq k}$ *of elements of* $I \oplus J$ *can be decomposed so that* $r_i = \sum_{j=1}^{k'} p_j \cdot q_{i,j}$ *with* $k' \in \mathbb{N}$*,* $\sum_{j=1}^{k'} q_{i,j} \in J$ *and for all* $j \leq k'$*,* $p_j \in I$*. In particular* p_j *and* k' *do not depend on* i *(they are shared by each* r_i *of the sequence).*

In fact the converse is true, so that the following equality holds:

$$(I \oplus J)^k \quad = \quad \Big\{(\sum_{j=1}^{k'} p_j \cdot q_{i,j})_{i\leq k} \mid k' \in \mathbb{N},\ (\forall i \leq k, \sum_{j=1}^{k'} q_{i,j} \in J),\ (\forall j \leq k', p_j \in I)\Big\},$$

where $(I \oplus J)^k$ *is a notation for* $\underbrace{(I \oplus J) \otimes \cdots \otimes (I \oplus J)}_{k\ times}$*.*

Proof. By definition:

$$
\begin{aligned}
(I \oplus J)^k &= \Big\{ (\sum_{j=1}^{k'_i} p_{i,j} \cdot q_{i,j})_{i\le k} \mid (k'_i) \in \mathbb{N}^k,\ (\forall i, \sum_{j=1}^{k'_i} q_{i,j} \in I),\ (\forall i, j, p_{i,j} \in J) \Big\} \\
&\supseteq \Big\{ (\sum_{j=1}^{k'} p_j \cdot q_{i,j})_{i\le k} \mid k' \in \mathbb{N},\ (\forall i, \sum_{j=1}^{k'} q_{i,j} \in J),\ (\forall i, j, p_j \in I) \Big\}.
\end{aligned}
$$

The reverse inclusion is obtained by specifying that k'_i and $p_{i,j}$ does not depend on i:

- We set $k' := \sum_{i\le k} k'_i$,
- for all $i \le k$, and all $\jmath \le k'_i$, we set $q'_{i,j+\Sigma_{i'<i} k'_{i'}} := q'_{i,j}$
- for all $i \le k$, and any other j we set $q'_{i,j} := 0_{\mathcal{R}}$,
- and for all $j \le k'$, there is i and $j' \le k_i$ such that $j = j' + \Sigma_{i'<i} k'_{i'}$ so that $p'_j := p_{i,j'}$.

It results that

$$
\begin{aligned}
(I \odot J)^k &= \Big\{ (\sum_{j=1}^{k'_i} p_{i,j} \cdot q_{i,j})_{i\le k} \mid (k'_i) \in \mathbb{N}^k,\ (\forall i, \sum_{j=1}^{k'_i} q_{i,j} \in J),\ (\forall i, j, p_{i,j} \in I) \Big\} \\
&\subseteq \Big\{ (\sum_{j=1}^{k'} p'_j \cdot q'_{i,j})_{i\le k} \mid k' \in \mathbb{N},\ (\forall i, \sum_{j=1}^{k'} q'_{i,j} \in J),\ (\forall i, j, p'_j \in I) \Big\}.
\end{aligned}
$$

□

Proposition 3.1.3.6. *Given a multiplicity semiring $\mathcal{R}$, the powerset lax-semiring $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$ is a lax-semiring.*

Proof. The proof is very similar to the proof of Proposition A.4.2.6 for $\mathcal{P}(\mathbb{N})$, only much more complicated and with an explicit usage of properties over $\mathcal{R}$ (in particular we use each axiom of multiplicity).

- $\oplus$ is associative and commutative: immediate by associativity and commutativity of $\mathcal{R}$.
- $\{0_{\mathcal{R}}\}$ is neutral for $\oplus$: immediate by neutrality of $0_{\mathcal{R}}$ for $+_{\mathcal{R}}$.
- $\{1_{\mathcal{R}}\}$ is left-neutral for $\odot$:

$$
\begin{aligned}
\{1_{\mathcal{R}}\} \odot J &= \Big\{ \sum_{i=1}^{k} p_i \cdot q_i \mid k \le 0, \sum_{i=1}^{k} q_i \in J, \forall i \le k, p_i \in \{1_{\mathcal{R}}\} \Big\} \\
&= \Big\{ \sum_{i=1}^{k} 1_{\mathcal{R}} \cdot q_i \mid k \le 0, \sum_{i=1}^{k} q_i \in J \} \\
&= \Big\{ \sum_{i=1}^{k} q_i \mid k \le 0, \sum_{i=1}^{k} q_i \in J \} \\
&= J.
\end{aligned}
$$

- $\{1_{\mathcal{R}}\}$ is right-neutral for $\odot$:

$$
\begin{aligned}
I \odot \{1_{\mathcal{R}}\} &= \Big\{ \sum_{i=1}^{k} p_i{\cdot}q_i \mid k \leq 0, \sum_{i=1}^{k} q_i \in \{1_{\mathcal{R}}\}, \forall i \leq k, p_i \in I \Big\} \\
&= \Big\{ \sum_{i=1}^{k} p_i{\cdot}q_i \mid k \leq 0, (\exists i_0 \leq k, q_{i_0} = 1_{\mathcal{R}}, \forall i \neq i_0, q_i = 0_{\mathcal{R}}),\ p_i \in I \Big\} && (MS2) \\
&= I.
\end{aligned}
$$

- $\odot$ left-distributes over $\oplus$:

$$
\begin{aligned}
I \odot (J \oplus K) &= \Big\{ \sum_{i=1}^{k} p_i{\cdot}r'_i \mid k \leq 0, \sum_{i=1}^{k} r'_i \in (J \oplus K), \forall i \leq k, p_i \in I \Big\} \\
&= \Big\{ \sum_{i=1}^{k} p_i{\cdot}r'_i \mid k \leq 0, \sum_{i=1}^{k} r'_i = q{+}r, \\
&\qquad\qquad q \in J,\ r \in K,\ \forall i \leq k, p_i \in I \Big\} \\
&= \Big\{ \sum_{i=1}^{k} p_i{\cdot}(q_i{+}r_i) \mid k \leq 0, \sum_{i=1}^{k} q_i \in J,\ \sum_{i=1}^{k} r_i \in K,\ \forall i \leq k, p_i \in I \Big\} && (MS3) \\
&= \Big\{ \sum_{i=1}^{k} (p_i{\cdot}q_i) + \sum_{i=1}^{k} (p_i{\cdot}r_i) \mid k \leq 0, \sum_{i=1}^{k} q_i \in J,\ \sum_{i=1}^{k} r_i \in K,\ \forall i \leq k, p_i \in I \\
&= \Big\{ \sum_{i=1}^{k} (p_i{\cdot}q_i) + \sum_{j=1}^{k'} (p'_j{\cdot}r_j) \mid k \leq 0, k' \leq 0, \sum_{i=1}^{k} q_i \in J,\ \sum_{j=1}^{k'} r_j \in K, \\
&\qquad\qquad \forall i \leq k, p_i \in I,\ \forall j \leq k', p'_j \in I\} \Big\} \\
&= (I \odot J) \oplus (I \odot K)
\end{aligned}
$$

- $\{0_{\mathcal{R}}\}$ is right absorbing for $\odot$:

$$
\begin{aligned}
I \odot \{0_{\mathcal{R}}\} &= \Big\{ \sum_{i=1}^{k} p_i{\cdot}r'_i \mid k \leq 0, \sum_{i=1}^{k} r'_i \in \{0_{\mathcal{R}}\}, \forall i \leq k, p_i \in I \Big\} \\
&= \Big\{ \sum_{i=1}^{k} p_i{\cdot}0_{\mathcal{R}} \mid k \leq 0, \sum_{i=1}^{k} q_i \in J, \Big\} && MS1 \\
&= \{0_{\mathcal{R}}\} && \text{r. abs. in } \mathcal{R}
\end{aligned}
$$

- $\odot$ is associative:

$$
\begin{aligned}
(I \odot J) \odot K &= \Big\{ \sum_{i=1}^{k} p'_i \cdot r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \forall i \leq k, p'_i \in (I \odot J) \Big\} \\
&= \Big\{ \sum_{i=1}^{k} \Big(\sum_{j=1}^{k'} p_j \cdot q_{i,j} \Big) \cdot r_i \mid k, k' \in \mathbb{N}, \sum_{i=1}^{k} r_i \in K, && \text{Lm. 3.1.3.5} \\
&\qquad \forall i \leq k, \sum_{j=1}^{k_i} q_{i,j} \in J,\ \forall j \leq k, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} \sum_{i=1}^{k'} p_j \cdot (q_{i,j} \cdot r_i) \mid k, k' \in \mathbb{N}, \\
&\qquad \sum_{i=1}^{k'} r_i \in K,\ \forall i, \sum_{j=1}^{k} q_{i,j} \in J,\ \forall j, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} \sum_{i=1}^{k'} \sum_{t}^{k''_i} p_j \cdot (q_{i,j,t} \cdot r_{i,t}) \mid k, k' \in \mathbb{N}, (k''_i)_i \in \mathbb{N}^{k'}, && i \rightsquigarrow (i,t) \\
&\qquad \sum_{i=1}^{k'} \sum_{t=1}^{k''_i} r_{i,t} \in K,\ \forall i, t, \sum_{j=1}^{k} q_{i,j,t} \in J,\ \forall j, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} p_j \cdot (\sum_{i=1}^{k'} \sum_{t}^{k''_i} q_{i,j,t} \cdot r_{i,t}) \mid k, k' \in \mathbb{N}, (k_i)_i \in \mathbb{N}^{k'}, && \text{dist. in } \mathcal{R} \\
&\qquad \sum_{i=1}^{k'} \sum_{t=1}^{k''_i} r_{i,t} \in K,\ \forall i, t, \sum_{j=1}^{k} q_{i,j,t} \in J,\ \forall j, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} p_j \cdot (\sum_{i=1}^{k'} \sum_{t}^{k''_i} q_{i,j,t} \cdot r_{i,t}) \mid k, k' \in \mathbb{N}, (k_i)_i \in \mathbb{N}^{k'}, \\
&\qquad \forall i, \sum_{t=1}^{k''_i} r_{i,t} = r_i,\ \forall i, t, \sum_{j=1}^{k} q_{i,j,t} = q_i,\ \sum_{i=1}^{k'} r_i \in K, \\
&\qquad \forall i \leq k', q_i \in J,\ \forall j \leq k, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} p_j \cdot (\sum_{i=1}^{k'} q'_{i,j}) \mid k, k' \in \mathbb{N}, \forall i, \sum_{j=1}^{k} q'_{i,j} = q_i \cdot r_i,\ \sum_{i=1}^{k'} r_i \in K, && (MS4) \\
&\qquad \forall i \leq k', q_i \in J,\ \forall j \leq k, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} p_j \cdot q'_j \mid k, k' \in \mathbb{N}, \sum_{j=1}^{k} q'_j = \sum_{i=1}^{k'} q_i \cdot r_i,\ \sum_{i=1}^{k'} r_i \in K, && (MS3) \\
&\qquad \forall i \leq k', q_i \in J,\ \forall j \leq k, p_j \in I \Big\} \\
&= \Big\{ \sum_{j=1}^{k} p_j \cdot q'_j \mid k \in \mathbb{N}, \sum_{j=1}^{k} q'_j \in (J \odot K), \forall j \leq k, p_j \in I \Big\} \\
&= I \odot (J \odot K)
\end{aligned}
$$

- $\odot$ right-distribute over $\oplus$ in the lax way:

$$
\begin{aligned}
(I \oplus J) \odot K &:= \Big\{\sum_{i=1}^{k} p'_i{\cdot}r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \forall i \leq k, p'_i \in (I \oplus J)\Big\} \\
&:= \Big\{\sum_{i=1}^{k} (p_i{+}q_i){\cdot}r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \forall i \leq k, p_i \in I, q_i \in J\Big\} \\
&:= \Big\{(\sum_{i=1}^{k} p_i{\cdot}r_i){+}(\sum_{i=1}^{k} q_i{\cdot}r_i) \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \forall i \leq k, p_i \in I, q_i \in J\Big\} \\
&\subseteq \Big\{(\sum_{i=1}^{k} p_i{\cdot}r_i){+}(\sum_{i=1}^{k'} q_i{\cdot}r'_i) \mid k \leq 0, k' \leq 0, \sum_{i=1}^{k} r_i \in K, \sum_{i=1}^{k'} r'_i \in K, \forall i \leq k, p_i \in I, q_i \in J\Big\} \\
&= \Big\{p'{+}q' \mid p' \in (I \odot K), q' \in (J \odot K)\Big\} \\
&= (I \odot K) \oplus (J \odot K)
\end{aligned}
$$

- $\{0_{\mathcal{R}}\}$ is left absorbing for $\odot$ in the lax way:

$$
\begin{aligned}
\{0_{\mathcal{R}}\} \odot K &:= \Big\{\sum_{i=1}^{k} p'_i{\cdot}r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \forall i \leq k, p'_i \in \{0_{\mathcal{R}}\}\Big\} \\
&:= \Big\{\sum_{i=1}^{k} 0_{\mathcal{R}}{\cdot}r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \Big\} \\
&:= \Big\{\sum_{i=1}^{k} 0_{\mathcal{R}}{\cdot}r_i \mid k \leq 0, \sum_{i=1}^{k} r_i \in K, \Big\} \\
&\subseteq \{0_{\mathcal{R}}\}
\end{aligned}
$$

□

Stratification over $\textsc{Rel}^{\mathcal{R}}$

The stratification is then very similar to the stratification over $\textsc{Rel}^{\mathbb{N}}$.

Definition 3.1.3.7. *A multiplicity semiring* interprets *a lax-semiring $\mathcal{S}$ if its powerset lax-semiring $\mathcal{P}(\mathcal{R})$ interprets $\mathcal{S}$ (Def. 3.1.2.1). A $\mathcal{R}$-*relational interpretation *of a lax-semiring $\mathcal{S}$ is an interpretation $[\![-]\!] : \mathcal{S} \mapsto \mathcal{P}(\mathcal{R})$ of $\mathcal{S}$ into $\mathcal{P}(\mathcal{R})$.*

Theorem 3.1.3.8 (Semiring interpretation of stratification). *Any interpretation $[\![-]\!]$ of an ordered semiring $\mathcal{S}$ into a multiplicity semiring $\mathcal{R}$ induces a stratification of the linear category* $\textsc{Rel}^{\mathcal{R}}$*, defined by:*

$$
a^I := \Big\{u \in !_{\mathcal{R}}a \mid \sum_{x \in a} u(x) \in [\![I]\!]\Big\}, \qquad f^{I \geq J} := \{(u,v) \in !_{\mathcal{R}}f \mid u \in a^I, v \in b^J\},
$$

$$
\partial_{I,a} := \varrho_{I,a} := \{(u,u) \mid u \in a^I\}.
$$

In particular,[10] $[\![-]\!]$ $[\![-]\!]$ *extends to a sound interpretation of* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ *into* $\textsc{Rel}^{\mathcal{R}}$.

Proof. The proof is rigorously similar to the proof of Theorem 3.1.2.2 when you accept Lemma 3.1.3.9. ☐

In the following we denote $\|u\| = \sum_{\alpha \in a} u(\alpha)$ for $u \in !_{\mathcal{R}} a$.

Lemma 3.1.3.9. *For any* $f : !_{\mathcal{R}} a \to !_{\mathcal{R}} b$, *if* $(u, v) \in f$, *then* $\|u\| = \|v\|$.
In particular, the functoriality can be rewritten:

$$f^{I \geq J} :=_{\mathcal{R}} \{(u, v) \in !f \mid \|v\| \in [\![J]\!]\}.$$

Proof. If $(u, v) \in f$ then there is $\sigma \in \mathcal{R}_f\langle f \rangle$ such that $u(\alpha) = \sum_\beta \sigma(\alpha, \beta)$ and $v(\beta) = \Sigma_\alpha \sigma(\alpha, \beta)$ thus

$$\|u\| = \sum_\alpha u(\alpha) = \sum_\alpha \sum_\beta \sigma(\alpha, \beta) = \sum_\beta \sum_\alpha \sigma(\alpha, \beta) = \sum_\beta v(\beta) = \|\beta\|$$

One can rewrite the functoriality

$$f^{I \geq J} :=_{\mathcal{R}} \{(u, v) \in !f \mid \|u\| \in [\![I]\!],\ \|v\| \in [\![J]\!]\},$$

but since $I \geq J$, we get $[\![I]\!] \supseteq [\![J]\!]$ and

$$f^{I \geq J} :=_{\mathcal{R}} \{(u, v) \in !f \mid \|v\| \in [\![J]\!]\}.$$

☐

The free multiplicity semirings

It is not clear, *a priori*, whether any lax-semiring $\mathcal{S}$ can be interpreted into some $\mathcal{P}(\mathcal{R})$ for a multiplicity semiring $\mathcal{R}$ (turning it into an interpretation of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ into $\textsc{Rel}^{\mathcal{R}}$). We will show that this is the case by exhibiting the "free" multiplicity semiring $\mathbb{N}_f\langle \mathcal{S}_\cdot \rangle$ induced by the multiplicative monoid $\mathcal{S}_\cdot$ of $\mathcal{S}$.

Recall from Definition A.4.2.1 and Proposition A.4.2.2 that, given a monoid $\mathbb{M}$ and a semiring $\mathcal{R}$, the set $\mathcal{R}_f\langle \mathbb{M} \rangle$ of finitely supported functions from $\mathbb{M}$ to $\mathcal{R}$ forms a semiring with:

$$0_{\mathcal{R}_f\langle \mathbb{M} \rangle} := [\,], \qquad (\mu +_{\mathcal{R}_f\langle \mathbb{M} \rangle} \nu)(g) := \mu(g) +_{\mathcal{R}} \nu(g),$$
$$1_{\mathcal{R}_f\langle \mathbb{M} \rangle} := [1_{\mathbb{M}}], \qquad (\mu \cdot_{\mathcal{R}_f\langle \mathbb{M} \rangle} \nu)(g) := \sum_{\substack{g', g'' \in \mathbb{M} \\ g' \cdot_{\mathbb{M}} g'' = g}} \mu(g') \cdot_{\mathcal{R}} \nu(g''),$$

we recall, moreover, that [] stands for the zero constant ($I \mapsto 0_{\mathcal{R}}$) and $[I]$ stands for the function mapping I into $\mathbb{1}_{\mathcal{R}}$ and any $J \neq I$ into $0_{\mathcal{R}}$.

We will see that when $\mathcal{R} = \mathbb{N}$, *i.e.*, when $\mathbb{N}_f\langle \mathbb{M} \rangle$ represents finite multisets over $\mathbb{M}$, the semiring $\mathbb{N}_f\langle \mathbb{M} \rangle$ is a multiplicity semiring (Prop. 3.1.3.11). Further on, we will see that when the monoid is the multiplicative monoid $\mathcal{S}_\cdot$ of a ordered semiring $\mathcal{S}$, the multiplicity semiring $\mathbb{N}_f\langle \mathcal{S}_\cdot \rangle$ interprets $\mathcal{S}$.

[10] Assuming Conjecture 3.1.1.7.

Lemma 3.1.3.10. *If $\mathcal{R}$ is a multiplicity semiring and $\mathbb{M}$ a monoid, $\mathcal{R}_f\langle\mathbb{M}\rangle$ respects (MS1), (MS2) and (MS3).*

Proof. (MS1) We suppose that $\mu +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \nu = 0_{\mathcal{R}_f\langle\mathbb{M}\rangle}$, *i.e.*, $\mu(g)+_{\mathcal{R}}\nu(g) = 0_{\mathcal{R}}$ for all g.
Then $\mu(g) = \nu(g) = 0_{\mathcal{R}}$ by (MS1) in $\mathcal{R}$.
Thus $\mu = \nu = 0_{\mathcal{R}_f\langle\mathbb{M}\rangle}$.

(MS2) We suppose that $\mu +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \nu = 1_{\mathcal{R}_f\langle\mathbb{M}\rangle}$, *i.e.*, $\mu(1_{\mathbb{M}})+_{\mathcal{R}}\nu(1_{\mathbb{M}}) = 1_{\mathcal{R}}$ and $\mu(g)+\nu_{\mathcal{R}}(g) = 0_{\mathcal{R}}$ for all $g \neq 1_{\mathbb{M}}$.
Then $\mu(1_{\mathbb{M}}) = 0_{\mathcal{R}}$ (or $\nu(1_{\mathbb{M}}) = 0_{\mathcal{R}}$) by (MS2) in $\mathcal{R}$ and for all $g \neq 1_{\mathbb{M}}$, $\mu(g) = \nu(g) = 0_{\mathcal{R}}$ by (MS1) in $\mathcal{R}$.
Thus $\mu = 0_{\mathcal{R}_f\langle\mathbb{M}\rangle}$ (or $\nu = 0_{\mathcal{R}_f\langle\mathbb{M}\rangle}$).

(MS3) We suppose that $\mu_1 +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \mu_2 = \nu_1 +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \nu_2$, *i.e.*, $\mu_1(g)+_{\mathcal{R}}\nu_1(g) = \mu_2(g)+_{\mathcal{R}}\nu_2(g)$ for all g.
Then by (MS3) in $\mathcal{R}$ we have $(k^g_{i,j})_{g\in\mathbb{M},1\leq i,j,\leq 2}$ such that $\mu_i(g) = k^g_{i,1} +_{\mathcal{R}} k^g_{i,2}$ and $\nu_j(g) = k^g_{1,j} +_{\mathcal{R}} k^g_{2,j}$.
Thus, if we denote $\kappa_{i,j} : (g \mapsto k^g_{i,j})$ for all i, j, we indeed have $\mu_i = \kappa_{i,1} +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \kappa_{i,2}$ and $\nu_j = \kappa_{1,j} +_{\mathcal{R}_f\langle\mathbb{M}\rangle} \kappa_{2,j}$.

□

Proposition 3.1.3.11. *The semiring $\mathbb{N}_f\langle\mathbb{M}\rangle$ is a multiplicity semiring.*

Proof. Because of Lemma 3.1.3.10, we just have to prove (MS4).
We suppose that $\nu_1 +_{\mathbb{N}_f\langle\mathbb{M}\rangle} \nu_2 = \kappa \cdot_{\mathbb{N}_f\langle\mathbb{M}\rangle} \mu$ with (we suppose that N_1 and N_2 are disjoint)

$$\nu_1 = [g_{n_1} \mid n_1 \in N_1], \qquad \nu_2 = [g_{n_2} \mid n_2 \in N_2],$$
$$\kappa = [f_k \mid k \in K] \qquad \mu = [h_m \mid m \in M]$$

There is then a bijection $\phi : (N_1 \cup N_2) \leftrightarrow K \times M$.
We can denote $\kappa_k = [f_k]$ and $\mu_{i,k} = [h_{\pi_2(\phi(n))} \mid n \in N_i, \pi_1(\phi(n)) = k]$ for any $k \in K$. Then:

- $\Sigma_{k\in K}\kappa_k = [f_k \mid k \in K] = \kappa$,
- and for $k \in K$, $\mu_{1,k} + \mu_{2,k} = [h_{\pi_2(\phi(n))} \mid \pi_1(\sigma(n)) = k] = \mu$ since $g_n = f_{\pi_1(\phi(n))}{\cdot}h_{\pi_2(\phi(n))}$,
- and we have $\Sigma_k\kappa_k{\cdot}\mu_{i,k} = [f_k{\cdot}h_{\pi_2(\phi(n))} \mid k \in K, n \in N_i, \pi_1(\phi(n)) = k] = [g_n | n \in N_i] = \nu_i$.

□

Remark 3.1.3.12. *In fact, we conjecture that the semiring $\mathcal{R}_f\langle\mathbb{M}\rangle$ is a multiplicity semiring for any multiplicity semiring $\mathcal{R}$ and any monoid $\mathbb{M}$, not just for $\mathcal{R} = \mathbb{N}$. We were able to prove Proposition 3.1.3.11 also for $\mathcal{R} = \bar{\mathbb{N}}$, but the proof is long and tiresome (based on a case disjunction between natural numbers and ω). For the general we are clueless, but we know that Theorem 3.1.3.3 applies on $\mathcal{R}_f\langle\mathbb{M}\rangle$, with $\mathcal{R}$ a multiplicity semiring, even if $\mathcal{R}_f\langle\mathbb{M}\rangle$ is not a multiplicity semiring.*

If $\mathcal{S}$ is not a multiplicity semiring, one can interpret it into the *"free" multiplicity semiring* $\mathbb{N}_f\langle\mathcal{S}_\cdot\rangle$ induced by the multiplicative monoid $\mathcal{S}_\cdot$ of $\mathcal{S}$ (recall Proposition 3.1.3.11):

Proposition 3.1.3.13. *For any ordered semiring $\mathcal{S}$, the following is an interpretation of $\mathcal{S}$ into $\mathbb{N}_f\langle\mathcal{S}.\rangle$:*[11]

$$[\![I]\!] = \left\{[J_1, ..., J_n] \mid \sum_{i\leq n} J_i \leq_{\mathcal{S}} I\right\}. \tag{3.7}$$

Proof.

- If $p \leq_S q$ then $[\![p]\!] = \{[q_1, ..., q_n] \mid \sum_{i\leq n} q_i \leq_S p\} \subseteq \{[q_1, ..., q_n] \mid \sum_{i\leq n} q_i \leq_S q\} = [\![q]\!]$. Conversely, if $[\![p]\!] \subseteq [\![q]\!]$, then $[p] \in [\![q]\!]$ so that $p \leq q$.

- The addition is preserved:

$$\begin{aligned}[\![p]\!] \oplus [\![q]\!] &= \{[p_1, ..., p_n, q_1, ..., q_m] \mid \sum_{i\leq n} p_i \leq_S p, \sum_{i\leq m} q_i \leq_S q\} \\ &\subseteq \{[q_1, ..., q_n] \mid \sum_{i\leq n} q_i \leq_S p + q\} \\ &= [\![p +_{\mathcal{S}} q]\!]\end{aligned}$$

- The 0 is preserved:

$$\begin{aligned}\{0_{\mathbb{N}_f\langle\mathcal{S}.\rangle}\} &= \{[]\} \\ &\subseteq \{[q_1, ..., q_n] \mid \sum_{i\leq n} q_i \leq_S 0_{\mathcal{S}}\} \\ &= [\![0_{\mathcal{S}}]\!]\end{aligned}$$

- The 1 is preserved:

$$\begin{aligned}\{1_{\mathbb{N}_f\langle\mathcal{S}.\rangle}\} &= \{[1_{\mathcal{S}}]\} \\ &\subseteq \{[q_1, ..., q_n] \mid \sum_{i\leq n} q_i \leq_S 1_{\mathcal{S}}\} \\ &= [\![1_{\mathcal{S}}]\!]\end{aligned}$$

[11] Recall that elements of $\mathbb{N}_f\langle\mathcal{S}.\rangle$ are finite multisets

- The multiplication is preserved:

$$
\begin{aligned}
[\![p]\!] \odot [\![q]\!] &= \left\{ \sum_{i=1}^{h} I_i \cdot J_i \;\middle|\; h \geq 0,\ \sum_{i=1}^{h} J_i \in [\![q]\!],\ \forall i \leq h, I_i \in [\![p]\!] \right\} \\
&= \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0, && \text{unfold } I, J \\
&\quad [q_{i,j} \mid i \leq h, j \leq j_i] \in [\![q]\!],\ \forall i \geq h,\ [p_{i,k} \mid k \leq k_i] \in [\![p]\!] \Big\} \\
&= \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0, && \text{def } [\![\_]\!] \\
&\quad \sum_{i \leq h} \sum_{j \leq j_i} q_{i,j} \leq_{\mathcal{S}} q,\ \forall i \geq h,\ \sum_{k \leq k_i} p_{i,k} \leq_{\mathcal{S}} p \Big\} \\
&\subseteq \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0, \text{mon. mult.} \\
&\quad p \cdot (\sum_{i \leq h} \sum_{j \leq j_i} q_{i,j}) \leq_{\mathcal{S}} p \cdot q,\ \forall i \geq h,\ \sum_{k \leq k_i} p_{i,k} \leq_{\mathcal{S}} p \Big\} \\
&= \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0, && \text{l. dist.} \\
&\quad \sum_{i \leq h} (p \cdot (\sum_{j \leq j_i} q_{i,j})) \leq_{\mathcal{S}} p \cdot q,\ \forall i \geq h,\ \sum_{k \leq k_i} p_{i,k} \leq_{\mathcal{S}} p \Big\} \\
&\subseteq \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0,\ \sum_{i \leq h} ((\sum_{k \leq k_i} p_{i,k}) \cdot (\sum_{j \leq j_i} q_{i,j})) \leq_{\mathcal{S}} p \cdot q \Big\} \\
&= \Big\{ [p_{i,k} \cdot q_{i,j} \mid i \leq h, j \leq j_i, k \leq k_i] \;\Big|\; h \geq 0,\ \sum_{i \leq h} \sum_{j \leq j_i} \sum_{k \leq k_i} p_{i,k} \cdot q_{i,j} \leq_{\mathcal{S}} p \cdot q \Big\} && \text{r.\&l. dist.} \\
&\subseteq \Big\{ [r_{i'} \mid i' \leq h'] \;\Big|\; h' \geq 0,\ \sum_{i' \leq h'} r_{i'} \leq_{\mathcal{S}} p \cdot q \Big\} \\
&= [\![p \cdot q]\!]
\end{aligned}
$$

□

3.2. A dependent $B_{\mathcal{S}}LL$?

In Section 3.1, we have seen that for $B_{\mathcal{S}}LL$ to be modeled by $\textsc{Rel}^{\mathcal{R}}$, the powerset lax-semiring $\mathcal{P}(\mathcal{R})$ must interpret the semiring $\mathcal{S}$. This shows that the relational models for semirings are powersets over multiplicity semirings. However, the semiring structure of $\mathcal{P}(\mathcal{R})$ is especially unusual: where does it come from? and is it intrinsically related to the relational category?

In this section we will track the "semantical semiring", whose existence was conjectured in the introduction, for any linear category (Def. 3.2.1.2). For this we will first oidify[12] the notion of linear category to a 2-categorical framework in Section 3.2.1, or more exactly to an order-enriched framework. This allows us to categorically define the *internal semiring* (Th. 3.2.2.2 and Th.3.2.2.4) of any such order-enriched linear category in Section 3.2.2. In Section 3.2.3, we will show how to generically transform any order-enriched linear category (with an additional light condition) into a bounded exponential situation regarding its internal semiring (Th.3.2.3.3).

While carrying this study on the emergence of semiring structures, we oversaw what seems to be a model for some dependent version of $B_{\mathcal{S}}LL$. Indeed, the most interesting BLL-like logics and calculi such as the original BLL, Gaboardi *et al*'s DFuzz [GHH+13b] or Dal Lago and Gaboardi's D_{ℓ}PCF [dLG11] carry a notion of dependency. Their parameters can depend on resource-variables which can be bounded and instantiated along the type derivation. This allows, for example, to distinguish the resource usage of the two branches of a conditional, or, combined with a fixpoint, to give a resulting parameter that depends on the number of evaluations of some loop.

Sections 3.2.4 and 3.2.5 present our first results in the quest for modeling BLL dependency.

3.2.1. An order-enriched linear category

In Remark 3.1.1.3, we have enlightened an emerging relation between the axioms of semiring[13] and the required coherence diagrams in a bounded exponential situation. The surprising point of this remark is that coherence diagrams are 2-dimensional cells rather than 1-dimensional cells represented by the morphisms $a^{\mathtt{com}^+}$, $a^{\mathtt{as}^+}$...

Based on this intuition, we will see that the semantical semiring $\mathcal{R}$ can be tracked in the higher order dimension of the considered linear category $\mathcal{L}$. This means that the objects of $\mathcal{R}$ will be morphisms in $\mathcal{L}$ and the order relation will be described by 2-cells.

For this we need to have non-trivial 2-cells (otherwise we will see that the order is trivial) and to define a 2-categorical version of a linear category. However, extending the notion to a full 2-categorical framework is extremely heavy and non justified as we will not give any example that uses the rich structure of 2-categories. Thus we will stay inside the specific case of order-enriched categories (Def. A.1.0.25). Such a restriction is similar to the restriction of bimonoidal categories into semirings.

Remark 3.2.1.1. *We believe that all our work can be generalized to the full 2-categorical framework (Def. A.1.0.21). However, this generalization requires a suitable notion of linear 2-category which is*

[12] The oidification consists in considering the objects of the initial category as morphisms in the targeted 2-category.

[13] or equivalently the natural transformations of the bimonoidal category

cumbersome to express.

Order-degenerated linear 2-categories are basically linear categories with an order on morphisms that is coherent with the monoidal and exponential functors.

Definition 3.2.1.2. *An* order-enriched linear category *is an order-enriched category (Def. A.1.0.25) with the axiomatisation of a linear category (Sec. A.3.2) where:*

- *each functor is generalized to a 2-functor,*
- *each natural transformation is generalized to a natural 2-transformation,*
- *each required coherence diagram remains the same.*

Proposition 3.2.1.3. *Any linear category induces an order-enriched linear category with discrete hom-posets (*i.e.,*the order over $\mathcal{L}[a,b]$ is discrete).*

Conversely, any ordered-enriched linear category induces a linear category with the 0-cells as objects and the 1-cells as morphisms.

Example 3.2.1.4.

- *The order-enriched category* $\text{Rel}^{\mathcal{R}}$ *endowed with inclusions as 2-cells is linear:*
  - *its 0-cells are the sets,*
  - *its hom-poset* $\text{Rel}^{\mathcal{R}}_1[a,b]$ *are the posets of relations from a to b with the inclusion order,*
  - *its horizontal composition is the composition of relation (preserving ordering),*
  - *the monoidal and exponential functors preserves the ordering, making them 2-functors,*
  - *each of* $\mathtt{d},\mathtt{p},\mathtt{w},\mathtt{c},\mathtt{m}$*, being a natural transformation for the 1-Category* $\text{Rel}^{\mathcal{R}}$*, remains a natural 2-transformation for the order-degenerated* $\text{Rel}^{\mathcal{R}}$ *(see Def. A.1.0.27).*
- *The order-enriched category* ScottL *is linear:*
  - *its 0-cells are the posets,*
  - *its hom-posets* $\text{ScottL}[a,b]$ *are the posets of linear functions from the initial segments of a to the initial segments of b endowed with the pointwise order.*
- *The order-enriched categories* $\text{Coh}^{\mathbb{B}}$ *and* $\text{Coh}^{\mathbb{N}}$ *(Def. A.3.3.7) are linear when endowed with inclusion order.*

3.2.2. The left-semiring $\mathcal{L}[!\mathbb{1},\mathbb{1}]$

In this section we will recover the "internal semiring" from the hom-poset $\mathcal{L}[!\mathbb{1},\mathbb{1}]$ of any order-enriched linear category $\mathcal{L}$.

For this, we will describe three different structures: the internal left-semiring $\mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$, the internal lax-semiring $\mathcal{S}^{\mathtt{lax}}_{\mathcal{L}}$ and the internal strict semiring $\mathcal{S}^{\mathtt{str}}_{\mathcal{L}}$. The internal left-semiring of Definition 3.2.2.1 is the simplest and more natural, but lacks any notion of right-distributivity (or left-absorption). Definitions 3.2.2.3 and 3.2.2.5 overcome this difficulties by restricting $\mathcal{L}[!\mathbb{1},\mathbb{1}]$ respectively to the co-lax monoid morphisms and to the strict monoid morphisms (depending on whether one is looking for the lax or strict semiring).

Definition 3.2.2.1. *Given an order-enriched linear category $\mathcal{L}$, we call* internal left-semiring *of $\mathcal{L}$, denoted $\mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$, the poset $\mathcal{L}(!\mathbb{1},\mathbb{1})$ endowed with the following structure:*

- *the functor $+ : \mathcal{S}^{\mathtt{left}}_{\mathcal{L}} \times \mathcal{S}^{\mathtt{left}}_{\mathcal{L}} \to \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$ is basically the monoidal functor pre-composed by the contraction, concretely it is defined by (for any $I, J \in \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$)*

$$I{+}J := \qquad !\mathbb{1} \xrightarrow{\mathtt{c}_{\mathbb{1}}} !\mathbb{1}\otimes!\mathbb{1} \xrightarrow{I\otimes J} \mathbb{1}\otimes\mathbb{1} \xrightarrow{\lambda} \mathbb{1}$$

  where $\lambda_{\mathbb{1}} : \mathbb{1}\otimes\mathbb{1} \to \mathbb{1}$ is the monoidal unity, and extends to the morphisms by (for any $\mu,\nu \in \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}[I,J]$)

$$\mu{+}\nu := \qquad !\mathbb{1} \xrightarrow{\mathtt{c}_{\mathbb{1}}} !\mathbb{1}\otimes!\mathbb{1} \underset{J\otimes J'}{\overset{I\otimes I'}{\rightrightarrows}}\ {\Downarrow}\mu\otimes\nu\ \ \mathbb{1}\otimes\mathbb{1} \xrightarrow{\lambda} \mathbb{1}$$

- *the functor $\cdot : \mathcal{S}^{\mathtt{left}}_{\mathcal{L}} \times \mathcal{S}^{\mathtt{left}}_{\mathcal{L}} \to \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$ is basically the horizontal composition pre-composed with a dereliction, concretely it is defined by (for any $I, J \in \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$)*

$$I{\cdot}J := \qquad !\mathbb{1} \xrightarrow{\mathtt{p}_{\mathbb{1}}} !!\mathbb{1} \xrightarrow{!I} !\mathbb{1} \xrightarrow{J} \mathbb{1}$$

  and extends to the morphisms by (for any $\mu,\nu \in \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}[I,J]$):

$$\mu{\cdot}\nu := \qquad !\mathbb{1} \xrightarrow{\mathtt{p}_{\mathbb{1}}} !!\mathbb{1} \underset{!J}{\overset{!I}{\rightrightarrows}}\ {\Downarrow}!\mu\ \ !\mathbb{1} \underset{J'}{\overset{I'}{\rightrightarrows}}\ {\Downarrow}!\nu\ \ \mathbb{1}$$

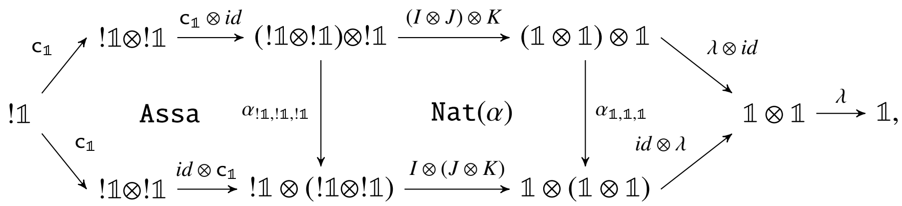
- *the functor $0 : \mathbb{1} \to \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$ defined by $0 := \mathtt{w}_{\mathbb{1}}$,*
- *the functor $1 : \mathbb{1} \to \mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$ defined by $1 := \mathtt{d}_{\mathbb{1}}$.*

Theorem 3.2.2.2 (The internal left-semiring). *Given an order-enriched linear category $\mathcal{L}$, the internal left-semiring $\mathcal{S}^{\mathtt{left}}_{\mathcal{L}}$ is a left-bimonoidal category.*

Proof.
- Functoriality of +:
  directly obtained by the functoriality of the monoidal product.

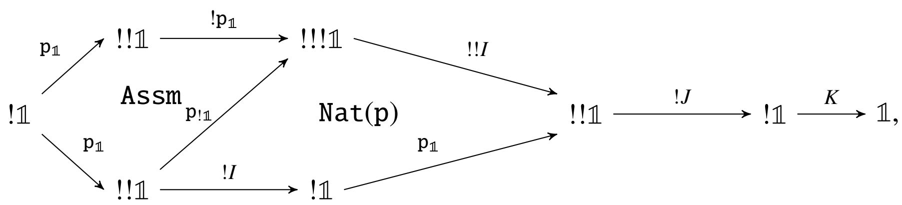
- Functoriality of the product $\cdot$:
  directly obtained by the functoriality of the vertical composition.

- $\mathtt{as}^+ : (I{+}J){+}K \longleftrightarrow I{+}(J{+}K)$ is defined by:

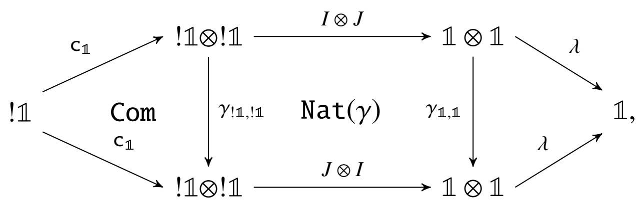


  where the rightmost 2-cell correspond to the associativity diagram of the monoidal structure of $\mathcal{L}$,

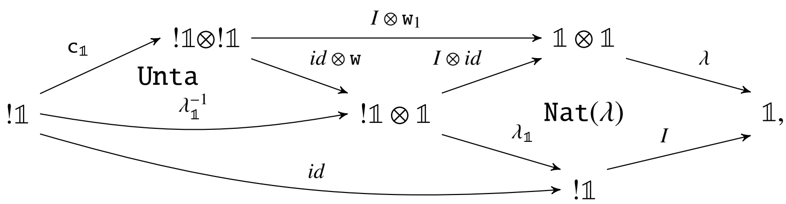
- $\mathtt{as}^{\cdot} : (I{\cdot}J){\cdot}K \longleftrightarrow I{\cdot}(J{\cdot}K)$ is defined by:



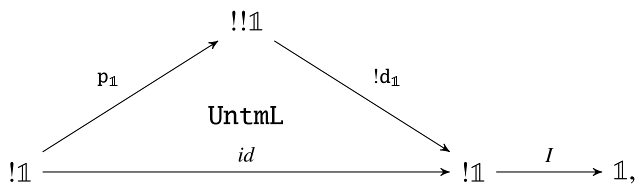
- $\mathtt{com}^+ : I{+}J \longleftrightarrow J{+}I$ is defined by:



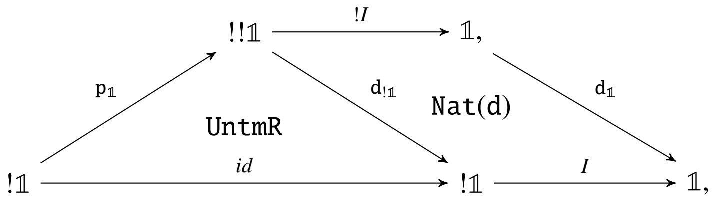
- $\mathtt{unt}^+ : I{+}0 \longleftrightarrow I$ is defined by:



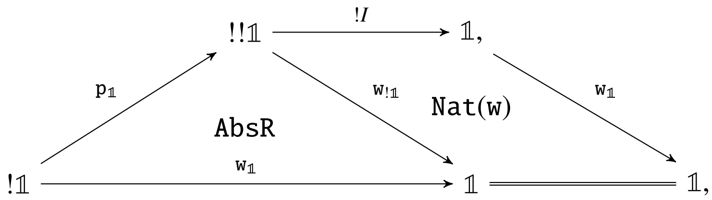
- $\mathtt{untL}^{\cdot} : 1{\cdot}I \longleftrightarrow I$ is defined by:



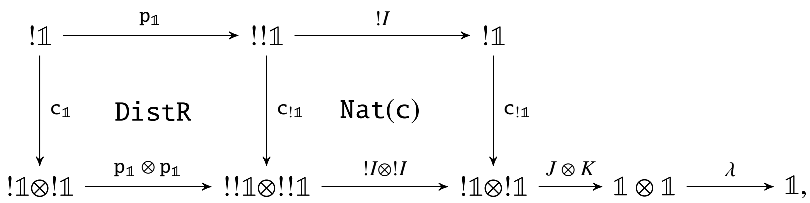
- untR$^{\cdot}$: $I{\cdot}1 \longleftrightarrow I$ is defined by:



- absR : $I{\cdot}0 \longleftrightarrow 0$ is defined by:



- and dstL : $I{\cdot}(J{+}K) \longleftrightarrow (I{\cdot}J){+}(I{\cdot}K)$ is defined by:

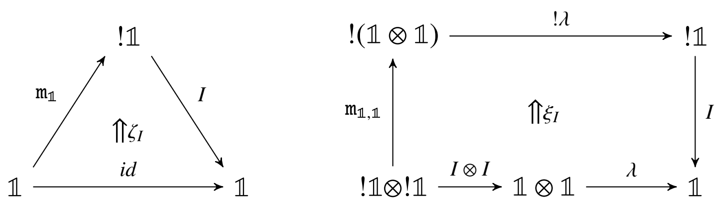


- each of these isomorphisms are equality (due to the order-degeneration), so that their naturality is immediate.

□

As we have seen in Remark 3.1.1.5, the left absorption and the right distributivity have a special statute. The internal left-semiring will not have them (*a priori* not even in a lax way). However, we can quite naturally "force them" which is the object of the following theorem.

Definition 3.2.2.3. *Let $\mathcal{L}$ be an order-enriched linear category. The* internal lax-semiring *of $\mathcal{L}$, denoted $\mathcal{S}_{\mathcal{L}}^{\mathtt{lax}}$, is the category of colax monoid morphisms between $(!\mathbb{1}, (\mathtt{m}_{\mathbb{1},\mathbb{1}}; !\lambda), \mathtt{m}_{\mathbb{1}})$ and $(\mathbb{1}, \lambda, id_{\mathbb{1}})$. Spelled out, this means that $\mathcal{S}_{\mathcal{L}}^{\mathtt{lax}}$ is the full subcategory of $\mathcal{S}_{\mathcal{L}}^{\mathtt{left}}$ whose objects are triples (I, ζ_I, ξ_I) such that $I \in \mathcal{L}[!\mathbb{1}, \mathbb{1}]$ and ζ_I, ξ_I are 2-morphisms respecting:*[14]



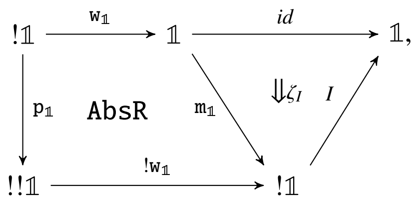
The corresponding morphisms are the 2-morphisms in $\mathcal{L}[!\mathbb{1}, \mathbb{1}]$.

The pair (ζ_I, ξ_I) *of 2-morphisms will be called the* distributive structure .

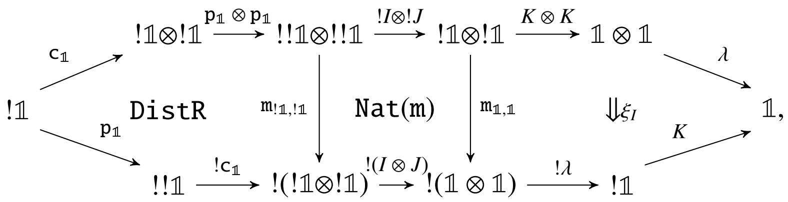
Theorem 3.2.2.4 (The internal lax semiring). *Let* $\mathcal{L}$ *be an order-enriched linear category. The internal lax semiring* $\mathcal{S}_{\mathcal{L}}^{\mathtt{lax}}$ *of* $\mathcal{L}$ *is a lax bimonoidal category when endowed with the same structure as the internal left-semiring.*

Proof. We have to prove the existence of the natural transformations `absR` and `dstL` and to extends the functors $+$, 0, $\cdot$ and 1 so that they transport the distributive structure.

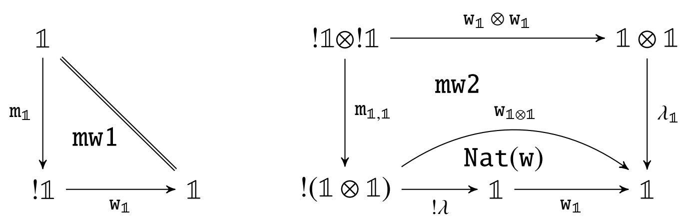
- `absR` : $0 \longrightarrow 0{\cdot}I$



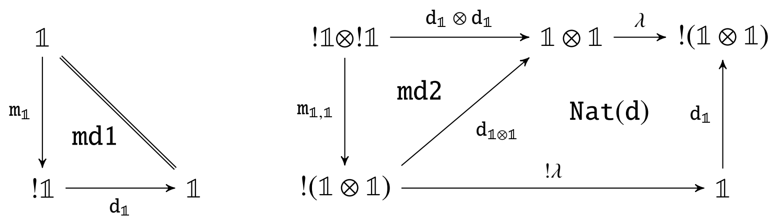
- `dstL` : $(I{\cdot}K) + (J{\cdot}K) \longrightarrow (I{+}J){\cdot}K$



- The functor 0 transports the distributive structure:

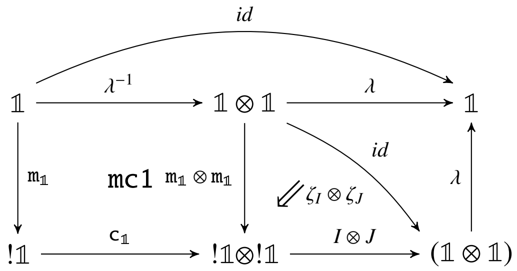


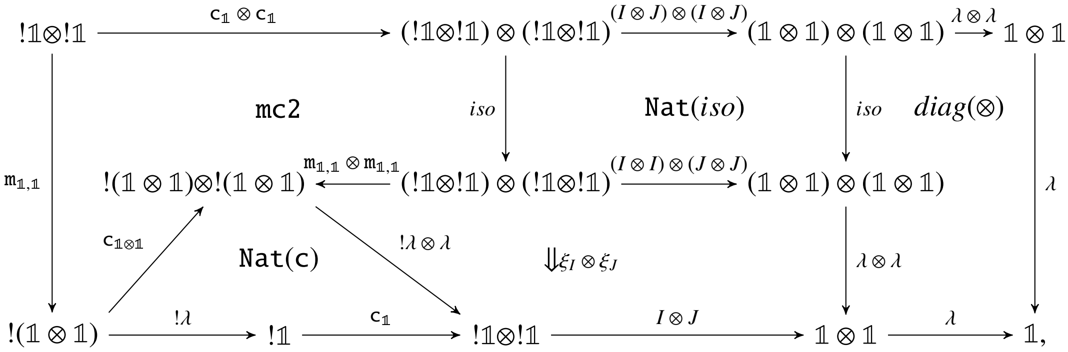
- The functor 1 transports the distributive structure:



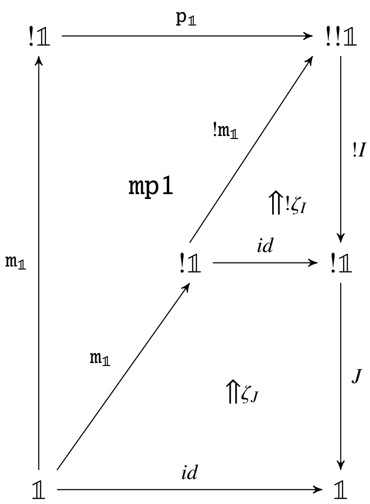
- The functor + transports the distributive structure:

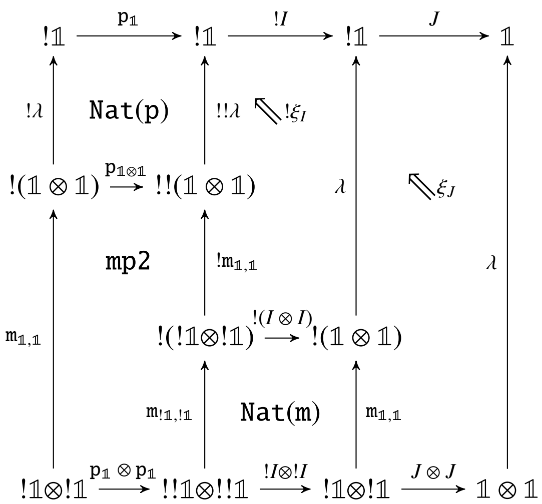
[14]Recall that, in order-degenerated 2-categories, for any $I \in \mathcal{L}[!\mathbb{1}, \mathbb{1}]$ there is almost one suitable ζ_I and one ξ_I.





- The functor · transports the distributive structure:





□

Definition 3.2.2.5. *Let $\mathcal{L}$ be an order-enriched linear category. The* internal strict-semiring *of $\mathcal{L}$, denoted $\mathcal{S}^{\mathtt{str}}_{\mathcal{L}}$, is the restriction of the internal lax-semiring of $\mathcal{L}$ to the strict monoid morphisms (so that ζ and ξ are identities).*

Theorem 3.2.2.6 (internal strict-semiring). *The internal strict-semiring $\mathcal{S}^{\mathtt{str}}_{\mathcal{L}}$ of an order-enriched linear category $\mathcal{L}$ is a bimonoidal category.*

Proof. The proof is similar to the proof of Theorem 3.2.2.4. □

Remark 3.2.2.7. *The internal strict-semiring is a full sub-semiring (or full sub bimonoidal category) of the internal lax semiring, in particular the first can be interpreted (Def.3.1.1.8) into the second by the inclusion functor.*

Example 3.2.2.8. *Here are the semirings that one gets by applying Theorems 3.2.2.2, 3.2.2.4 and 3.2.2.6 to different order-enriched linear categories (the resulting semirings are defined in Example A.4.1.3 or Definition 3.1.3.4).*

- *given a multiplicity semiring $\mathcal{R}$ (Def. 3.1.3.1), we can apply these theorems to the relational semantics* $\textsc{Rel}^{\mathcal{R}}$ *(Th. 3.1.3.3) and get:*

$$\mathcal{S}^{\texttt{left}}_{\textsc{Rel}^{\mathcal{R}}} \simeq \mathcal{P}(\mathcal{R}) \qquad \mathcal{S}^{\texttt{lax}}_{\textsc{Rel}^{\mathcal{R}}} \simeq \mathcal{P}(\mathcal{R}) \qquad \mathcal{S}^{\texttt{str}}_{\textsc{Rel}^{\mathcal{R}}} \simeq \mathcal{R}.$$

  Indeed, the internal left-semiring corresponds to the relations between $!_{\mathcal{R}}\mathbb{1} \simeq \mathcal{R}$ and the singleton $\mathbb{1}$ which give the powerset $\mathcal{P}(\mathcal{R})$, it is then straightforward to check that the sum and product correspond to $\oplus$ and $\otimes$ of Definition 3.1.3.4. For the lax and strict semirings, one can verify that:

$$\begin{array}{cc} \mathtt{m}_{\mathbb{1}}; \emptyset = \emptyset & \mathtt{m}_{\mathbb{1}}; I = id_{\mathbb{1}} \;\; \textit{if } I \neq \emptyset \\ \mathtt{m}_{\mathbb{1},\mathbb{1}}; !\lambda; I = \{((p,p),*) \mid p \in I\} & (I \otimes I); \lambda = \{((p,q),*) \mid p,q \in I\}, \end{array}$$

  so that the lax distribution structure always exists and the strict exists whenever I is a singleton.

- *in* $\textsc{ScottL}$ *(Def A.3.4.3 and Ex. 3.2.1.4), we get:*

$$\mathcal{S}^{\texttt{left}}_{\textsc{ScottL}} \simeq \mathbb{B}^{op}_{\bot} \qquad \mathcal{S}^{\texttt{lax}}_{\textsc{ScottL}} \simeq \mathbb{B}^{op}_{\bot} \qquad \mathcal{S}^{\texttt{str}}_{\textsc{ScottL}} \simeq \mathbb{B}^{op},$$

  where $\mathbb{B}^{op}_{\bot}$ is the Boolean semiring with revers order and with a bottom element (resulting in a lax-semiring).

  Indeed, $\textsc{ScottL}[!\mathbb{1}, \mathbb{1}]$ *is formed by linear functions from $\mathcal{I}(\mathcal{A}_f(\{*\}))$, which is the totally ordered set with three elements $\{\{*\}\} \geq \{*\} \geq \emptyset$, and $\mathcal{I}(\{*\})$, which is the totally ordered set with two elements $\{\emptyset\} \geq \emptyset$; by linearity the bottom is map to the bottom and it remains 3 possible linear mappings of $\{\{\emptyset\}\} \geq \{\emptyset\}$ into $\{\emptyset\} \geq \emptyset$, which correspond to the three elements of $\mathbb{B}_{\bot}$; it is the straightforward to verify that the order, sum and product correspond.*

  For the lax and strict semirings, one can verify that:

$$\begin{array}{cc} \mathtt{m}_{\mathbb{1}}; \bot = (p \mapsto \emptyset) & \mathtt{m}_{\mathbb{1}}; I = id_{\mathbb{1}} \;\;\; \textit{if } I \neq \bot \\ \mathtt{m}_{\mathbb{1},\mathbb{1}}; !\lambda; I = (I \otimes I); \lambda & \end{array}$$

  so that the lax distribution structure always exists and the strict one exists whenever $I \neq \bot$.

- *In* $\textsc{Coh}^{\mathbb{B}}$ *(Def. A.3.3.7 and Ex 3.2.1.4), we get:*

$$\mathcal{S}^{\texttt{left}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{B}_f \qquad \mathcal{S}^{\texttt{lax}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{B}_f \qquad \mathcal{S}^{\texttt{str}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{B}_d,$$

  where $\mathbb{B}_d$ is the discrete Boolean semiring and $\mathbb{B}_f$ is the Sierpinski lax-semiring (the discrete Boolean semiring with a bottom element).

- *In* $\textsc{Coh}^{\mathbb{N}}$ *(Def. A.3.3.7 and Ex 3.2.1.4), we get:*

$$\mathcal{S}^{\texttt{left}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{N}_f \qquad \mathcal{S}^{\texttt{lax}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{N}_f \qquad \mathcal{S}^{\texttt{str}}_{\textsc{Coh}^{\mathbb{B}}} \simeq \mathbb{N}.$$

Remark 3.2.2.9. *Example 3.2.2.8 gives us an alternative proof of Proposition 3.1.3.6. Indeed, we obtained that $\mathcal{P}(\mathcal{R})$ being the internal lax-semiring of* $\text{R}\textsc{el}^{\mathcal{R}}$, *it is in particular a lax-semiring.*

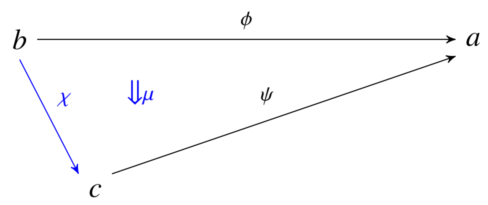
Remark 3.2.2.10. *The internal lax-semiring seems to correspond to our first naive idea of internal semiring in Section 3.1. However, in the case of coherent spaces (both* $\text{C}\textsc{oh}^{\mathbb{B}}$ *and* $\text{C}\textsc{oh}^{\mathbb{N}}$*) and* ScottL, *the results differ with those in Section 3.1.2.These differences will need further considerations.*

3.2.3. The bounded exponential situation

In Section 3.2.2, we have extracted a semiring (in fact three) naturally emerging from any order-enriched linear category. It remains to show that we can refine any order-enriched linear category $\mathcal{L}$ to get a bounded exponential situation relatively to $\text{B}_{\mathcal{S}_{\mathcal{L}}^{\mathtt{lax}}}\text{LL}$.[15]In the following we will only consider the lax internal semiring $\mathcal{S}_{\mathcal{L}}^{\mathtt{lax}}$ as we consider it to be the most important one. This transformation consists in taking the *lax-sliced category* around $\mathbb{1}$.

Definition 3.2.3.1. *Given any 2-category C and any object $a \in C_0$, we call the* lax-sliced category *of C over a, the category $C_{/a}$:*

- *which objects are the couples (b, ϕ) with $b \in C_0$ and $\phi \in C_1[b, a]$.*
- *which morphisms from (b, ϕ) to (c, ψ) are couples (χ, μ) with $\chi \in C_1[b, c]$ and $\mu \in C_2[\phi, (\chi; \psi)]$,* i.e. *so that:*



For simplicity, we denote the objects $(b, \phi) \in C_{/a}$ by just ϕ when there is no ambiguity.

Proposition 3.2.3.2. *For any order-enriched category C, if C is symmetric monoidal (with $\otimes$ a 2-functor), so is $C_{/\mathbb{1}}$ with the monoidal product given by :*

$$\phi \otimes_s \psi \quad := \quad a \otimes b \xrightarrow{\phi \otimes \psi} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda} \mathbb{1}$$

$$\mathbb{1}_s \quad := \quad \mathbb{1} \xrightarrow{id_{\mathbb{1}}} \mathbb{1}.$$

and

$$(\chi,\mu) \otimes_s (\chi',\mu') := \begin{array}{ccccc} a \otimes b & \xrightarrow{\phi \otimes \phi'} & & & \\ \downarrow{\scriptstyle \chi \otimes \chi'} & \Downarrow{\scriptstyle \mu \otimes \mu'} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda} & \mathbb{1} \\ a' \otimes b' & \xrightarrow{\psi \otimes \psi'} & & & \end{array}$$

Proof.

- The functoriality of $\otimes_s$ results from the 2-functoriality of $\otimes$.
- The associativity is given by:

$$\alpha'_{\phi,\psi,\chi} := \begin{array}{ccccccc} (a \otimes b) \otimes c & \xrightarrow{(\phi \otimes \psi) \otimes \chi} & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}} \otimes id_{\mathbb{1}}} & & & \\ \updownarrow{\scriptstyle \alpha_{a,b,c}} & \mathtt{Nat}(\alpha) & \updownarrow{\scriptstyle \alpha_{\mathbb{1},\mathbb{1},\mathbb{1}}} & \mathtt{neutTens} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda} & \mathbb{1} \\ a \otimes (b \otimes c) & \xrightarrow{\phi \otimes (\psi \otimes \chi)} & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) & \xrightarrow{id_{\mathbb{1}} \otimes \lambda_{\mathbb{1}}} & & & \end{array}$$

- The left neutrality is given by:

$$\lambda'_{\phi} := \begin{array}{cccc} \mathbb{1} \otimes a & \xrightarrow{id_{\mathbb{1}} \otimes \phi} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}}} \\ \updownarrow{\scriptstyle \lambda_a} & \mathtt{Nat}(\lambda) & & \mathbb{1} \\ a & & \xrightarrow{\phi} & \end{array}$$

- The commutativity is given by:

$$\gamma'_{\phi,\psi} := \begin{array}{ccccc} a \otimes b & \xrightarrow{\phi \otimes \psi} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}}} & \\ \updownarrow{\scriptstyle \gamma_{a,b}} & \mathtt{Nat}(\gamma) & \updownarrow{\scriptstyle \gamma_{\mathbb{1},\mathbb{1}}} & \mathtt{untTens} & \mathbb{1} \\ b \otimes a & \xrightarrow{\psi \otimes \phi} & \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\lambda_{\mathbb{1}}} & \end{array}$$

- The naturality of α', λ' and γ' is immediate as they are formed by natural 1-cells and 2-cells.
- The coherence diagrams are all obtained since they consists in a cone closed by the corresponding diagram in $\mathcal{L}$ and automatically filled since $\mathcal{L}$ is order-degenerated.

□

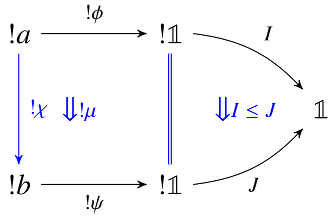
Theorem 3.2.3.3 (Slicing categories form bounded exponential situations). *Let $\mathcal{L}$ be an order-enriched linear category. Then if the monoidal product of $\mathcal{L}_{/\mathbb{1}}$ is closed, it forms a bounded exponential situation (Def. 3.1.1.2) for both* $\mathrm{B}_{\mathcal{S}^{\mathrm{lax}}_{\mathcal{L}}}\mathrm{LL}$ *and* $\mathrm{B}_{\mathcal{S}^{\mathrm{str}}_{\mathcal{L}}}\mathrm{LL}$.

Proof.

- The bi-functor $(-)^{-}$ is defined by:

$$(a,\phi)^I := \left(!a\,,\ \ !a \xrightarrow{!\phi} !\mathbb{1} \xrightarrow{I} \mathbb{1}\right)$$

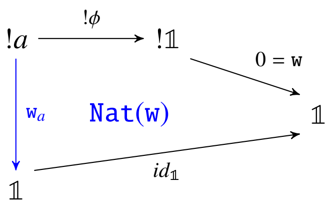
$(\chi,\mu)^{I\leq J}$:=



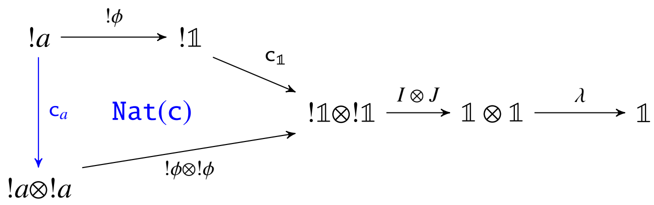
The functoriality is immediate form the 2-functoriality of ! in $\mathcal{L}$.

- $\mathtt{w}'$, $\mathtt{c}'$, $\mathtt{d}'$ and $\mathtt{p}$ are defined as the couples formed by the transformation in $\mathcal{L}$ and the corresponding naturality diagram

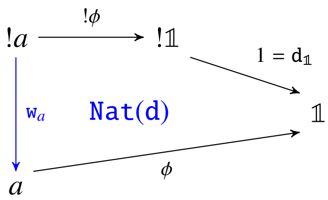
$\mathtt{w}'_{(a,\phi)}$:=



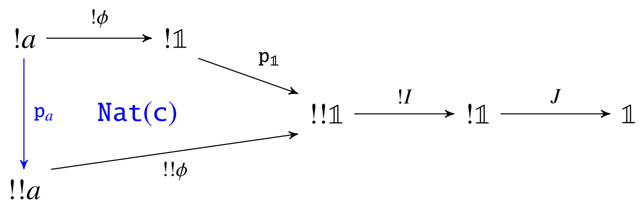
$\mathtt{c}'_{(a,\phi),I,J}$:=



$\mathtt{w}'_{(a,\phi)}$:=

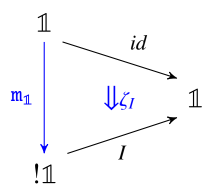


$\mathtt{p}'_{(a,\phi),I,J}$:=



- $\mathtt{m}'_I$ is defined by:

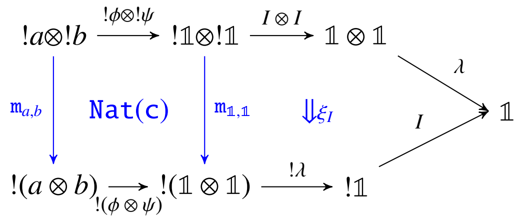
$\mathtt{m}'_I$:=



where ζ_I is the first 2-morphism of the distributive structure.

- $\mathtt{m}'_{(a,\phi),(b,\psi),I}$ is defined by:

$\mathtt{p}'_{(a,\phi),(b,\psi),I} :=$



where ξ_I is the second 2-morphism of the distributive structure.

- Naturalities and coherence diagrams are obtained automatically: they correspond to cones that are completed by the equivalent diagram in $\mathcal{L}$ and filled automatically by the order-degeneration of $\mathcal{L}$.

□

Remark 3.2.3.4. *Recall (Prop. 3.1.1.10) that for $\mathcal{S}$ that is interpreted in $\mathcal{S}^{\mathtt{lax}}_{\mathcal{L}}$ (Def. 3.1.1.8), its corresponding logic* $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ *is interpreted also in $\mathcal{L}_{/\mathbb{1}}$ (just by composing the interpretations).*

Example 3.2.3.5. *We are applying Theorem 3.2.3.3 to* $\textsc{Rel}^{\mathcal{R}}$ *(for $\mathcal{R}$ a multiplicity semiring, see Definition 3.1.3.1 and Theorem 3.1.3.3).*

The sliced category $\textsc{Rel}_{/\mathbb{1}}$ *is as follows:*

- $\textsc{Rel}_{/\mathbb{1}}$ *has for objects the couples $a = (|a|, C(a))$ with $C(a) \subseteq |a| \in$* $\textsc{Set}$*; notice that the coherence $C(a)$ corresponds to the co-domain of the morphism $\phi \in$* $\textsc{Rel}(|a|, \mathbb{1})$*, we will call $|a|$ the domain and $C(a)$ the coherence,*

- $\textsc{Rel}_{/\mathbb{1}}$ *has for morphisms from a to b the relations between the domains that co-preserve the coherence,* i.e.*, the relations $\phi \in \mathcal{P}(|a|, |b|)$ such that*

$$\forall (\alpha, \beta) \in \phi, \quad \beta \in C(b) \;\Rightarrow\; \alpha C(a),$$

- $\textsc{Rel}_{/\mathbb{1}}$ *has for tensorial product and unit the functors:*

$$|a \otimes_s b| := |a| \times |b| \qquad C(a \otimes_s b) := C(a) \times C(b)$$
$$\phi \otimes_s \psi := \phi \otimes_{\mathcal{R}} \psi \qquad \mathbb{1}_s := (\mathbb{1}, \mathbb{1}),$$

- $\textsc{Rel}_{/\mathbb{1}}$ *has for linear arrow the functor*

$$|a \otimes_s b| := (C(a) \times C(b)) \cup (|a| \times (|b| - C(b)))$$
$$C(a \multimap_s b) := C(a) \times C(b)$$
$$\phi \multimap_s \psi := \{((\alpha, \beta), (\alpha', \beta')) \mid (\alpha', \alpha) \in \phi,\ (\beta, \beta') \in \psi,\ (\beta \in C(b) \Rightarrow \alpha \in C(a))\},$$

- $\textsc{Rel}_{/\mathbb{1}}$ *has an exponential functor (for $I \in \mathcal{P}(\mathcal{R})$):*

$$|a^I| := \mathcal{R}_f\langle |a| \rangle, \qquad C(a^I) := \{u \in \mathcal{R}_f\langle C(a) \rangle \mid \|u\| \in I\}$$
$$\phi^{I \geq J} := !_{\mathcal{R}}\phi$$

where $\|u\| := \sum_{\alpha \in C(a)} u(\alpha)$*,*

- $\text{Rel}_{/\mathbb{1}}$ *has for parametrized contraction* $\mathtt{c}'_{a,I,J} := \mathtt{c}_a$*, so that the parameters* I*,* J *only serves to set the domain an co-domain, and similarly for* $\mathtt{w}', \mathtt{p}', \mathtt{d}', \mathtt{m}'$*.*

Remark that this model can be map into the model defined by stratification in Theorem 3.1.3.8. The mapping is the functor that forget the domain structure and project every relation into their restriction to the coherence.

Remark 3.2.3.6. *The assumption of Theorem 3.2.3.3 (requireing that the monoidal product of* $\mathcal{L}_{/\mathbb{1}}$ *is closed) seems a strong assumption but is verified by most of our models of interest.*

The fact that we change the definition of linear category, not asking for products and coproducts is important here since we obtain products but not necessarily co-products. However, those co-products also seem to be present in most of our models of interest.

3.2.4. Toward a dependent version

Along Sections 3.2.2 and 3.2.3, the tensorial unit $\mathbb{1}$ is extremely central; indeed, we were studying, respectively, morphisms from $!\mathbb{1}$ to $\mathbb{1}$ and sliced categories around $\mathbb{1}$. What about taking objects other that $\mathbb{1}$? The goal of this section is to perform such a generalization and show that we get some flavor of dependency.

As we have already seen, the logical power of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ is generally limited and most of the applications based on quantitative exponential (like BLL [GSS92b], DFuzz [GHH+13a] or D_ℓPCF [dLG11]) are dependent extensions. In these extensions, you can write formulas which resources can share some dependency like:

$$A^x \multimap B^x \multimap C$$

that type programs represent using their two arguments the same number of times.

The algebraic structure on exponential (syntactical generalizations of “semirings”) are composed of expressions dependent on some resource variables. The exponents can be, for example, binders $A^{x \in I}$ with x that can appear free in the exponentials of A. Moreover, these versions contain higher order rewriting premises so that derivation rules can perform substitutions over resources variables inside a formula and cut elimination procedural can perform substitutions inside a whole proof tree. For example the following sequent rule [DLH09] performs a substitution over x in the type A:

$$\frac{A\{1/x\}, \Gamma \vdash B \qquad 1 \sqsubseteq p}{!_{x \leq p} A, \Gamma \vdash B}$$

This way to extend $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$ has been defined for several cases but lacks a general formalization. In this section we will see that our semantics naturally extends with dependency and then we will deduce a logical extension.

From an abstract point of view, dependency has two main characteristic:

1. the elements of the “semiring” are dependent on some resource context and exponentials can modify this context (they are binders),
2. derivation and cut-elimination can perform global rewriting that change the resource context (over formulas and proofs respectivelly).

In our semantics we will see that this characteristic are translated into:

1. the elements of the "semiring" morphisms over some "resource context",
2. a second emerging structure will be able to go through terms and act on the "semiring" elements.

Our approach being semantic directed, we will not expend more on what should be a dependent syntax/semantics. We rather begin by a natural generalization for the position of $\mathbb{1}$. In this section we are replacing $\mathbb{1}$ by a whole sub-category of objects called co-classical fragment. The objects u of the category can be co-erased with some $\epsilon_u : \mathbb{1} \longrightarrow u$ and co-duplicated with some $\rho_u : u \otimes u \longrightarrow u$.

Definition 3.2.4.1. *Given an order-enriched linear category $\mathcal{L}$, we call* co-classical fragment *of $\mathcal{L}$ a subcategory $\mathcal{U}$ of the category of monoids over $\mathcal{L}$. Spelled out, $\mathcal{U}$ is the category*

- *which objects are triples $u = (|u|, \epsilon_u, \rho_u)$ where:*
  - *$|u|$ is an object (0-cell) of $\mathcal{L}$,*
  - *$\epsilon_u : \mathbb{1} \longrightarrow u$ is a morphism in $\mathcal{L}$,*
  - *$\rho_u : u \otimes u \longrightarrow u$ is a morphism in $\mathcal{L}$,*
  - *such that the following diagrams commutes (they are 2-isomorphisms)*

$$
\begin{array}{ccc}
|u| \otimes |u| & \xrightarrow{\rho_u} & |u| \\
{\scriptstyle \epsilon_u \otimes id_{|u|}} \uparrow & & \Vert \\
\mathbb{1} \otimes |u| & \xrightarrow{\lambda_{|u|}} & |u|
\end{array}
\qquad
\begin{array}{ccc}
|u| \otimes |u| & \xrightarrow{\rho_u} & |u| \\
\downarrow {\scriptstyle \gamma_{|u|,|u|}} & & \Vert \\
|u| \otimes |u| & \xrightarrow{\rho_u} & |u|
\end{array}
$$

$$
\begin{array}{ccccc}
(|u| \otimes |u|) \otimes |u| & \xrightarrow{\rho_u \otimes id_{|u|}} & |u| \otimes |u| & \xrightarrow{\rho_{u,u}} & |u| \\
{\scriptstyle \alpha_{|u|,|u|,|u|}} \uparrow & & & & \Vert \\
|u| \otimes (|u| \otimes |u|) & \xrightarrow{id_{|u|} \otimes \rho_u} & |u| \otimes |u| & \xrightarrow{\rho_{u,u}} & |u|.
\end{array}
$$

- *the morphisms $\iota \in \mathcal{U}[u, v]$ are morphisms $\iota \in \mathcal{L}[|u|, |v|]$ so that the following diagrams commute:*

$$
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\epsilon_u} & |u| \\
\Vert & & \downarrow {\scriptstyle \iota} \\
\mathbb{1} & \xrightarrow{\epsilon_v} & |v|
\end{array}
\qquad
\begin{array}{ccc}
|u| \otimes |u| & \xrightarrow{\rho_u} & |u| \\
{\scriptstyle \iota \otimes \iota} \downarrow & & \downarrow {\scriptstyle \iota} \\
|v| \otimes |v| & \xrightarrow{\rho_v} & |v|
\end{array}
$$

so that $\epsilon_u : \mathbb{1} \longrightarrow |u|$ and $\rho_u : |u| \otimes |u| \longrightarrow |u|$ are natural in $\mathcal{U}$.

Remark 3.2.4.2. *We will consider, in our examples, only co-classical fragments $\mathcal{U}$ that are subcategories of $\mathcal{L}$,* i.e., *so that $|u| \neq |v|$ whenever $u \neq v$. Thus, in the following, we will use the notation u for $|u|$.*

Example 3.2.4.3.

- *For any order-enriched linear category $\mathcal{L}$, the one-point category $\{\mathbb{1}\}$ is the* trivial co-classical fragment *of $\mathcal{L}$, with $\rho_{\mathbb{1}} = \lambda_{\mathbb{1}}$ and with $\epsilon_{\mathbb{1}} = id_{\mathbb{1}}$.*

- *A good candidate the subcategory of $\mathcal{L}$ generated by $\mathbb{1}$, $!$ and $\otimes$, with ρ and ϵ defined by induction*

$$\rho_{\mathbb{1}} := \lambda_{\mathbb{1}}, \qquad \rho_{!u} := \mathtt{m}_{u,u}; !\rho_u, \qquad \rho_{u\otimes v} := \alpha^{-1}_{u\otimes v,u,v}; \gamma_{u\otimes v,u}; (\alpha^{-1}_{u,u,v} \otimes id_v); \alpha_{u\otimes u,v,v}; (\rho_u \otimes \rho_v)$$
$$\epsilon_{\mathbb{1}} := id_{\mathbb{1}}, \qquad \epsilon_{!u} := \mathtt{m}_{\mathbb{1}}; !\epsilon_u, \qquad \epsilon_{u\otimes v} := \lambda^{-1}; (\epsilon_u \otimes \epsilon_v)$$

  this fragment endowed with all possible morphisms will be called the free co-classical fragment *and denoted $\mathcal{U}_F$.*

- *In the specific case of $\textsc{Rel}^{\mathcal{R}}$, one can show (by an easy induction) that for the free co-classical fragment:*
  - *ϵ_u is full relation between $\mathbb{1}$ and u,*
  - *ρ_u is the reversed copy-cat,* i.e., *the relation $\{((\alpha,\alpha),\alpha) \mid \alpha \in u\}$.*

  Thus the morphisms of the free co-classical fragment are the right-to-left functions. Indeed, the naturality of ϵ forces the relation to be surjective and the naturality of ρ forces the relation to be injective.

- *In fact, in $\textsc{Rel}^{\mathcal{R}}$, $\textsc{Set}^{op}$ is a co-classical fragment, with ϵ being the full relations and ρ_u the reversed copy-cat functions.*

Definition 3.2.4.4. *Let $\mathcal{L}$ be an order-enriched linear category and let $\mathcal{U}$ be a co-classical fragment over $\mathcal{L}$.*

We call $\mathcal{U}$-dependent internal semiring *over $\mathcal{L}$, denoted $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}$, the order-enriched category:*

- *which objects are objects of $\mathcal{U}$: $(\mathcal{S}^{\mathcal{U}}_{\mathcal{L}})_0 := \mathcal{U}_0$,*

- *which hom-poset $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u,v]$ is the category of co-lax monoid morphisms between $(!u, \mathtt{m}_{\mathbb{1},\mathbb{1}}; !\rho_u, \mathtt{m}_{\mathbb{1}}; !\epsilon_u)$ and $(\mathbb{1}, \rho_v, \epsilon_v)$ in $\mathcal{L}$:*
  - *its objects are triples (I, ζ_I, ξ_I) where $I : !u \to v$ is a morphisms in $\mathcal{L}$ and ζ_I, ξ_I are completing the diagrams:*

$$\begin{array}{ccc} !\mathbb{1} & \xrightarrow{\;!\epsilon_u\;} & !u \\ {\scriptstyle \mathtt{m}_{\mathbb{1}}}\uparrow & \Uparrow\zeta_I & \downarrow{\scriptstyle I} \\ \mathbb{1} & \xrightarrow{\;\epsilon_v\;} & v \end{array} \qquad \begin{array}{ccccc} !(u\otimes u) & & \xrightarrow{\;!\rho_u\;} & & !u \\ {\scriptstyle \mathtt{m}_{u,u}}\uparrow & & \Uparrow\xi_I & & \downarrow{\scriptstyle I} \\ !u\otimes !u & \xrightarrow{\;I\otimes I\;} & v\otimes v & \xrightarrow{\;\rho_v\;} & v \end{array}$$

  - *$(I, \zeta_I, \xi_I) \geq (J, \zeta_J, \xi_J)$, whenever $I \geq J$ in $\mathcal{L}$.*

- *the composition in $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}$ is the composition in the Kleisli $\mathcal{L}_!$:*

$$I{\cdot}J \quad := \quad !u \xrightarrow{\mathrm{p}_u} !!u \xrightarrow{!I} !v \xrightarrow{J} w$$

- *the identity in $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}$ is the identity in the Kleisli $\mathcal{L}_!$:*

$$1_u \quad := \quad !u \xrightarrow{\mathrm{d}_u} u,$$

Definition 3.2.4.5. *We call* dependent semiring *the given of*

- *a semigroup-enriched category $\mathcal{S}$ called semiringoid and given by:*
  - *a class $\mathcal{S}_0$ of objects denoted u, v...,*
  - *for each object $u, v \in \mathcal{S}_0$, an ordered semigroup $\mathcal{S}[u, v]$ (equivalently an order-degenerated monoidal category) which elements are denoted I, J..., which sum is denoted $+ : \mathcal{S}[u, v] \times \mathcal{S}[u, v] \to \mathcal{S}[u, v]$ and which neutral element is denoted $0_{u,v}$,*
  - *a composition functor $\cdot : \mathcal{S}[u, v] \times \mathcal{S}[v, w] \to \mathcal{S}[v, w]$ for each object $u, v, w \in \mathcal{S}_0$,*
  - *a unity $1_u \in \mathcal{S}[u, u]$ for each $u \in \mathcal{S}_0$*
  - *such that:*
    * $\mathtt{absR}_{I,u}$: $I{\cdot}0_{\mathcal{S}[\mathrm{cod}(I),u]} = 0_{\mathcal{S}[\mathrm{dom}(I),u]}$,
    * $\mathtt{dstL}_{I,J,K}$: $I{\cdot}(J{+}K) = (I{\cdot}J) + (I{\cdot}K)$,
    * $\mathtt{absL}_{I,u}$: $0_{\mathcal{S}[u,\mathrm{cod}(I)]} = 0_{\mathcal{S}[u,\mathrm{dom}(I)]}I$,
    * $\mathtt{dstR}_{I,J,K}$: $(I{\cdot}K) + (J{\cdot}K) = (I + J){\cdot}K$,
    * $\mathtt{as}^{\cdot}_{I,J,K}$: $(I; J); K = I; (J; K)$,
    * $\mathtt{untL}^{\cdot}_{I,u}$: $1_u; I = I$,
    * $\mathtt{untR}^{\cdot}_{I,u}$: $I; 1_u = I$.
- *a category $\mathcal{U}$ called actor which objects are denoted by u, v... and which morphisms are denoted by ι, ι'...,*
- *such that the objects of the ringoid and the actor are the same:*

$$\mathcal{S}_0 = \mathcal{U}_0,$$

- *for each object $u, v, w \in \mathcal{U}$, two functors called left and right actions:*

$$\_\ltimes\_ : \mathcal{U}[u, v] \times \mathcal{S}_{v,w} \to \mathcal{S}_{u,w}\_\rtimes\_ : \mathcal{S}_{v,w} \times \mathcal{U}[u, v] \to \mathcal{S}_{u,w}$$

  that forms an external product:

– *interior neutrality of the unit:* $\mathtt{unt}^{\mathtt{int}}_{\iota} : \iota \ltimes 1_{\mathrm{cod}(\iota)} = 1_{\mathrm{dom}(\iota)} \rtimes \iota$,
– *right exterior neutrality of the unit* $\mathtt{untL}^{\mathtt{ext}}_{I} : id_{\mathrm{dom}(I)} \ltimes I = I$,
– *left exterior neutrality of the unit* $\mathtt{untR}^{\mathtt{ext}}_{I} : I \ltimes id_{\mathrm{cod}(I)} = I$,
– *left interior associativity* $\mathtt{asL}^{\mathtt{int}}_{\iota,J,K} : (J{\cdot}K) \rtimes \iota = I{\cdot}(K \rtimes \iota)$,
– *right interior associativity* $\mathtt{asR}^{\mathtt{int}}_{\iota,J,K} : \iota \ltimes (J{\cdot}K) = (\iota \ltimes I){\cdot}K$,
– *left exterior associativity* $\mathtt{asL}^{\mathtt{ext}}_{\iota,\iota',J} : (J \rtimes \iota') \rtimes \iota = I \rtimes (\iota; \iota)$,
– *right exterior associativity* $\mathtt{asR}^{\mathtt{ext}}_{\iota,\iota',K} : \iota \ltimes (\iota' \ltimes K) = (\iota; \iota') \ltimes K$,
– *right interior absorption* $\mathtt{absR}^{\mathtt{ext}}_{\iota,u} : \iota \ltimes 0_{\mathcal{S}[\mathrm{cod}(\iota),u]} = 0_{\mathcal{S}[\mathrm{dom}(\iota),u]}$,
– *left interior absorption* $\mathtt{absL}^{\mathtt{ext}}_{\iota,u} : 0_{\mathcal{S}[u,\mathrm{dom}(\iota)]} \rtimes \iota = 0_{\mathcal{S}[u,\mathrm{cod}(\iota)]}$,
– *left interior distribution* $\mathtt{dstL}^{\mathtt{ext}}_{\iota,J,K} : \iota \ltimes (J{+}K) = (\iota \ltimes J) + (\iota \ltimes K)$,
– *right interior distribution* $\mathtt{dstR}^{\mathtt{ext}}_{\iota,J,K} : (J{+}K) \rtimes \iota = (I \rtimes \iota) + (K \rtimes \iota)$,

For simplicity, we will occasionally denote 1 *for any* 1_u *and* 0 *for any* $0_{\mathcal{S}[u;\mathrm{cod}(I)]}$ *and we will denote* $\cdot$ *for* $\ltimes$ *and* $\rtimes$.

Remark 3.2.4.6. *This is the oidification of an algebraic structure composed of a monoid acting on a semiring. Intuitively, the semiring is similar to the bounding semiring: it will actively treat weakening and contractions, while the monoid is the duplicable information that can flow through the formula to rely dependencies.*

In term of models, we will see that (intuitively) the semiringoid[16] *corresponds to the Kleisli over* $\mathcal{U}$. *When* $\mathcal{U}$ *has only one element* $\mathbb{1}$, *then* $\mathcal{S} \simeq \mathcal{L}[!\mathbb{1}, \mathbb{1}]$ *and* $\mathcal{U} \simeq [\mathbb{1}, \mathbb{1}]$ *(which may be non trivial if* $\mathbb{1}$ *is not initial).*

Example 3.2.4.7. *For any monoid* $\mathbb{M}$ *and any semiring* $\mathcal{S}$, $\mathbb{M}$ *is acting on the* $\mathcal{S}$*-linear semiring* $\mathcal{S}_f\langle\mathbb{M}\rangle$ *over* $\mathbb{M}$ *(Prop. A.4.2.2) via the action* $f{\cdot}m := (I \mapsto f(I){\cdot}_{\mathbb{M}}m)$ *(and* $m{\cdot}f$ *defined symmetrically).*

Given a field K:

- *the category of vector spaces and linear functions forms a ringoid*[17]
- *the category of vector spaces and unitary operators can act on this ringoid (using the composition) forming a dependent semiring.*

Theorem 3.2.4.8 ($\mathcal{U}$-dependent internal semiring). *Let* $\mathcal{L}$ *be an order-enriched linear category and let* $\mathcal{U}$ *a co-classical fragment over* $\mathcal{L}$.

Then the $\mathcal{U}$-dependent internal semiring $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}$ *forms a dependent semiring with* $\mathcal{U}$ *acting on* $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}$ *via the composition in* $\mathcal{L}$*:*

$$I \rtimes \iota := I; \iota \qquad \text{and} \qquad \iota \ltimes I := !\iota; I.$$

Proof. • for each $u, v \in \mathcal{U}$, the category $\mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u, v]$ is monoidal:

[16] oidification of a semiring
[17] see Footnote 16

– the functor $0_{u,v} : \mathbb{1} \to \mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u,v]$ is defined by

$$0_{u,v} \quad := \qquad !u \xrightarrow{\mathtt{w}_u} \mathbb{1} \xrightarrow{\epsilon_v} v$$

– the functor $+_{u,v} : \mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u,v] \times \mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u,v] \to \mathcal{S}^{\mathcal{U}}_{\mathcal{L}}[u,v]$ is defined by:

$$I +_{u,v} J \quad := \qquad !u \xrightarrow{\mathtt{c}_u} !u \otimes !u \xrightarrow{I \otimes J} v \otimes v \xrightarrow{\rho_v} v$$

– their functoriality as well as the natural transformations are defined according to the proof of Theorem 3.2.2.4.

- $\mathcal{S}$ forms a semiringoid: their functoriality as well as the natural transformations are defined according to the proof of Theorem 3.2.2.4.
- the functors $\ltimes, \rtimes$ form an external product:
  – $\mathtt{unt}^{\mathtt{int}}_{\iota} : \iota \ltimes 1_{\mathrm{cod}(\iota)} \Longleftrightarrow 1_{\mathrm{dom}(\iota)} \rtimes \iota$ is the naturality of the dereliction,
  – the exterior unities use the associativity of the identity (in any 2-category),
  – the associative 2-isomorphisms use the associativity of the composition,
  – $\mathtt{absR}^{\mathtt{ext}}_{\iota,u} : \iota \ltimes 0_{\mathcal{S}[\mathrm{cod}(\iota),u]} \Longleftrightarrow 0_{\mathcal{S}[\mathrm{dom}(\iota),u]}$ is the naturality of the weakening,
  – $\mathtt{absL}^{\mathtt{ext}}_{\iota,u} : 0_{\mathcal{S}[u,\mathrm{dom}(\iota)]} \rtimes \iota \Longleftrightarrow 0_{\mathcal{S}[u,\mathrm{cod}(\iota)]}$ is the naturality of ϵ,
  – $\mathtt{dstR}^{\mathtt{ext}}_{\iota,J,K} : \iota \ltimes (J+K) \Longleftrightarrow (\iota \ltimes J) + (\iota \ltimes K)$ is the naturality of the contraction,
  – $\mathtt{dstL}^{\mathtt{ext}}_{\iota,J,K} : (J+K) \rtimes \iota \Longleftrightarrow (I \rtimes \iota) + (K \rtimes \iota)$ is the naturality of $\mathtt{c}$,

□

Following Theorem 3.2.3.3, one can turn this dependent semiring into a full fledged model. Of course, the result will not be a bounded exponential situation, but a dependent version. However, such a generalization would require a full formalization that represent a lot of work and lacks of interest in this stage of the study. Indeed, such a *ad hoc* formalization would not carry any interest without a full syntax and a better understanding of our objects.

Nonetheless, we can skip the step of the full fledged model and directly describe the corresponding syntax (that we respect to be invariant wrt cut elimination). This is the purpose of Section 3.2.5.

3.2.5. A dependent logic?

We have seen, at semantical level, how dependent semirings extend the notion of semiring in a dependent way. In this section, we present the corresponding generalization at logical level.

> **Definition 3.2.5.1.** *Given a dependent semiring* $(\mathcal{U}, \mathcal{S})$ *(Def. 3.2.4.5), we call* bounded linear logic with dependent $(\mathcal{U}, \mathcal{S})$-exponentials $\mathrm{B}^{\mathrm{d}}_{(\mathcal{U},\mathcal{S})}\mathrm{LL}$ *the logic where:*
>
> - *the* formulas *are defined by the grammar (where J is a morphism of $\mathcal{S}$):*
>   *(formulas)* $\quad A, B, C := \alpha \mid A \otimes B \mid A \multimap B \mid A^J$,
> - *the* sequent calculus *is given in Figure 3.4. There is two kind of sequents:*
>   – *in a sequent $A \Vdash u$, one verify that the resources used inside the formula A co-depend on $u \in \mathcal{S}_0$,*

$$\frac{}{\alpha \Vdash u} \qquad \frac{A \Vdash u \qquad B \Vdash u}{A \otimes B \Vdash u} \qquad \frac{A \Vdash u \qquad B \Vdash u}{A \multimap B \Vdash u} \qquad \frac{A \Vdash u \qquad I : u \to v}{A^I \Vdash v}$$

$$\frac{A \Vdash u}{A \vdash_u A}\,\mathtt{Ax} \qquad \frac{\Gamma, A, B \vdash_u C}{\Gamma, A \otimes B \vdash_u C}\,\otimes\mathrm{L} \qquad \frac{\Gamma \vdash_u A \qquad \Delta \vdash_u B}{\Gamma, \Delta \vdash_u A \otimes B}\,\otimes\mathrm{R}$$

$$\frac{\Gamma \vdash_u A \qquad \Delta, A \vdash_u B}{\Gamma, \Delta \vdash_u B}\,\mathtt{Cut} \qquad \frac{\Gamma \vdash_u A \qquad \Delta, B \vdash_u C}{\Gamma, \Delta, A \multimap B \vdash_u C}\multimap \mathrm{L} \qquad \frac{\Gamma, A \vdash_u B}{\Gamma \vdash_u A \multimap B}\multimap \mathrm{R}$$

$$\frac{\Gamma \vdash_u B}{\Gamma, A^0 \vdash_u B}\,\mathtt{Weak} \qquad \frac{\Gamma, A^\iota \vdash_u B}{\Gamma, A^{1\cdot\iota} \vdash_u B}\,\mathtt{Der}$$

$$\frac{\Gamma, A^I, A^J \vdash_u B}{\Gamma, A^{I+J} \vdash_u B}\,\mathtt{Contr} \qquad \frac{\Gamma, A^I \vdash_u B \qquad J \geq I}{\Gamma, A^J \vdash_u B}\,\mathcal{S}\mathtt{weak}$$

$$\frac{A_1{}^{I_1}, \cdots, A_n{}^{I_n} \vdash_u B \qquad J : u \to v}{A_1{}^{I_1\cdot J}, \cdots A_n{}^{I_n\cdot J} \vdash_v B^J}\,\mathtt{Prom}$$

Figure 3.4.: The sequent calculus of $\mathrm{B}^{\mathrm{d}}_{\mathcal{S}}\mathrm{LL}$.

- *in a sequent $\Gamma \vdash_u A$, the context Γ is supposed to be a multiset of formulas (no implicit contraction rule is admitted) and the target environment u is an object of $\mathcal{U}$, moreover we denote A^ι (where $\iota \in \mathcal{U}[u, v]$) for the substitution:*

$$\alpha^\iota := \alpha \qquad (A \otimes B)^\iota := A^\iota \otimes B^\iota \qquad (A \multimap B)^\iota := A^\iota \multimap B^\iota \qquad (A^I)^\iota := A^{I \rtimes \iota}$$

- *the* cut-elimination procedure *is defined by the usual rules of multiplicative linear logic plus the rules of Figure 3.6. Given a derivation Π and $\iota \in \mathcal{U}[u, v]$, we denote Π^ι the substitution described by Figure 3.5*

Example 3.2.5.2. *As we have seen in 3.2.4.3, the category of set and right-to left functions* SET^{op} *is a co-classical fragment of* REL. *In the example we develop the bounded linear logic with dependent* $(\mathrm{SET}^{op}, \mathcal{S}^{\mathrm{SET}^{op}}_{\mathrm{REL}^{\mathcal{R}}})$*-exponentials.*

Recall that the morphisms $I \in \mathcal{S}^{\mathrm{SET}^{op}}_{\mathrm{REL}}[a, b]$ are the relations from $!a$ to b. Only, for the sake of simplicity, we will first only consider the right-to-left functions that we denote by a right-to-left arrow. Thus $\lambda x.[x]$ is the identity and $I{\cdot}J := \lambda x. \sum_{y \in b} J(x)(y){\cdot}I(y) : \mathbb{N}_f\langle a\rangle \leftarrow c$ for $I : \mathbb{N}_f\langle a\rangle \leftarrow b$ and $J : \mathbb{N}_f\langle b\rangle \leftarrow a$. The external composition is defined by composition $I \rtimes \iota := \lambda x.I(\iota(x)) : \mathbb{N}_f\langle a\rangle \leftarrow c$ for $\iota : b \leftarrow c$.

In the resulting system we can prove that $(A^{\lambda x.x} \multimap B^{\lambda x.x})^{[2,3]} \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)$ where 2 and 3 are notations for $(* \mapsto [*, *]), (* \mapsto [*, *, *]) :!\mathbb{1} \leftarrow \mathbb{1}$ and where $[2, 3]$ is a notation for $(* \mapsto [[*, *], [*, *, *]]) :!!\mathbb{1} \leftarrow \mathbb{1}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A^2 \multimap B^2 \vdash_{\mathbb{1}} A^2 \multimap B^2}\,\mathtt{Ax} \qquad \dfrac{}{A^3 \multimap B^3 \vdash_{\mathbb{1}} A^3 \multimap B^3}\,\mathtt{Ax}}{(A^2 \multimap B^2), (A^3 \multimap B^3) \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\,\otimes\mathrm{R}}{(A^2 \multimap A^2), (A^{\lambda x.x} \multimap B^{\lambda x.x})^{[3]} \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\,\mathtt{Der}}{(A^{\lambda x.x} \multimap B^{\lambda x.x})^{[2]}, (A^{\lambda x.x} \multimap B^{\lambda x.x})^{[3]} \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\,\mathtt{Der}}{(A^{\lambda x.x} \multimap B^{\lambda x.x})^{[2,3]} \vdash_{\mathbb{1}} (A^2 \multimap B^2) \otimes (A^3 \multimap B^3)}\,\mathtt{Contr}$$

$$\left(\frac{\dfrac{\Pi_1}{A \Vdash \mathrm{dom}(\iota)} \qquad \dfrac{\Pi_2}{B \Vdash \mathrm{dom}(\iota)}}{A \otimes B \Vdash \mathrm{dom}(\iota)}\right)^{\iota} := \frac{\dfrac{\Pi_1^{\iota}}{A^{\iota} \Vdash \mathrm{cod}(\iota)} \qquad \dfrac{\Pi_2^{\iota}}{B^{\iota} \Vdash \mathrm{cod}(\iota)}}{A^{\iota} \otimes B^{\iota} \Vdash \mathrm{cod}(\iota)}$$

$$\left(\frac{\dfrac{\Pi_1}{A \Vdash \mathrm{dom}(\iota)} \qquad \dfrac{\Pi_2}{B \Vdash \mathrm{dom}(\iota)}}{A \multimap B \Vdash \mathrm{dom}(\iota)}\right)^{\iota} := \frac{\dfrac{\Pi_1^{\iota}}{A^{\iota} \Vdash \mathrm{cod}(\iota)} \qquad \dfrac{\Pi_2^{\iota}}{B^{\iota} \Vdash \mathrm{cod}(\iota)}}{A^{\iota} \multimap B^{\iota} \Vdash \mathrm{cod}(\iota)}$$

$$\left(\frac{\dfrac{\Pi}{A \Vdash u} \qquad I : u \to \mathrm{dom}(\iota)}{A^{I} \Vdash \mathrm{dom}(\iota)}\right)^{\iota} := \frac{\dfrac{\Pi}{A \Vdash u} \qquad I\cdot\iota : u \to \mathrm{cod}(\iota)}{A^{I\cdot\iota} \Vdash \mathrm{cod}(\iota)}$$

$$\left(\frac{}{\alpha \Vdash \mathrm{dom}(\iota)}\right)^{\iota} := \frac{}{\alpha \Vdash \mathrm{cod}(\iota)} \qquad \left(\frac{\dfrac{\Pi}{A \Vdash \mathrm{dom}(\iota)}}{A \vdash_{\mathrm{dom}(\iota)} A}\,\texttt{Ax}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{A^{\iota} \Vdash \mathrm{cod}(\iota)}}{A^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota}}\,\texttt{Ax}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma, A, B \vdash_{\mathrm{dom}(\iota)} C}}{\Gamma, A \otimes B \vdash_{\mathrm{dom}(\iota)} C}\,\otimes\mathrm{L}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma^{\iota}, A^{\iota}, B^{\iota} \vdash_{\mathrm{cod}(\iota)} C^{\iota}}}{\Gamma^{\iota}, A^{\iota} \otimes B^{\iota} \vdash_{\mathrm{cod}(\iota)} C^{\iota}}\,\otimes\mathrm{L}$$

$$\left(\frac{\dfrac{\Pi_1}{\Gamma \vdash_{\mathrm{dom}(\iota)} A} \qquad \dfrac{\Pi_2}{\Delta \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma, \Delta \vdash_{\mathrm{dom}(\iota)} A \otimes B}\,\otimes\mathrm{R}\right)^{\iota} := \frac{\dfrac{\Pi_1^{\iota}}{\Gamma^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota}} \qquad \dfrac{\Pi_2^{\iota}}{\Delta^{\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota}, \Delta^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota} \otimes B^{\iota}}\,\otimes\mathrm{R}$$

$$\left(\frac{\dfrac{\Pi_1}{\Gamma \vdash_{\mathrm{dom}(\iota)} A} \qquad \dfrac{\Pi_2}{\Delta, A \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma, \Delta \vdash_{\mathrm{dom}(\iota)} B}\,\texttt{Cut}\right)^{\iota} := \frac{\dfrac{\Pi_1^{\iota}}{\Gamma^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota}} \qquad \dfrac{\Pi_2^{\iota}}{\Delta^{\iota}, A^{\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota}, \Delta^{\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}\,\texttt{Cut}$$

$$\left(\frac{\dfrac{\Pi_1}{\Gamma \vdash_{\mathrm{dom}(\iota)} A} \qquad \dfrac{\Pi_2}{\Delta, B \vdash_{\mathrm{dom}(\iota)} C}}{\Gamma, \Delta, A \multimap B \vdash_{\mathrm{dom}(\iota)} C}\,\multimap\mathrm{L}\right)^{\iota} := \frac{\dfrac{\Pi_1^{\iota}}{\Gamma^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota}} \qquad \dfrac{\Pi_2^{\iota}}{\Delta^{\iota}, B^{\iota} \vdash_{\mathrm{cod}(\iota)} C^{\iota}}}{\Gamma^{\iota}, \Delta^{\iota}, A^{\iota} \multimap B^{\iota} \vdash_{\mathrm{cod}(\iota)} C^{\iota}}\,\multimap\mathrm{L}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma, A \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma \vdash_{\mathrm{dom}(\iota)} A \multimap B}\,\multimap\mathrm{R}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma^{\iota}, A^{\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota} \vdash_{\mathrm{cod}(\iota)} A^{\iota} \multimap B^{\iota}}\,\multimap\mathrm{R}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma, A^{0} \vdash_{\mathrm{dom}(\iota)} B}\,\texttt{Weak}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota}, A^{0} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}\,\texttt{Weak}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma, A^{\iota'} \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma, A^{1\cdot\iota'} \vdash_{\mathrm{dom}(\iota)} B}\,\texttt{Der}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma^{\iota}, A^{\iota';\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota}, A^{1\cdot\iota\cdot\iota'} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}\,\texttt{Der}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma, A^{I}, A^{J} \vdash_{\mathrm{dom}(\iota)} B}}{\Gamma, A^{I+J} \vdash_{\mathrm{dom}(\iota)} B}\,\texttt{Contr}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma^{\iota}, A^{I\cdot\iota}, A^{J\cdot\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}}{\Gamma^{\iota}, A^{(I\cdot\iota)+(J\cdot\iota)} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}\,\texttt{Contr}$$

$$\left(\frac{\dfrac{\Pi}{\Gamma, A^{I} \vdash_{\mathrm{dom}(\iota)} B} \qquad J \geq I}{\Gamma, A^{J} \vdash_{\mathrm{dom}(\iota)} B}\,\mathcal{S}\texttt{weak}\right)^{\iota} := \frac{\dfrac{\Pi^{\iota}}{\Gamma^{\iota}, A^{I\cdot\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}} \qquad J\cdot\iota \geq I\cdot\iota}{\Gamma^{\iota}, A^{J\cdot\iota} \vdash_{\mathrm{cod}(\iota)} B^{\iota}}\,\mathcal{S}\texttt{weak}$$

$$\left(\frac{\dfrac{\Pi}{{A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash_{\mathrm{dom}(\iota)} B} \quad J : u \to \mathrm{dom}(\iota)}{{A_1}^{I_1\cdot J}, \cdots {A_n}^{I_n\cdot J} \vdash_{\mathrm{dom}(\iota)} B^{J}}\,\texttt{Prom}\right)^{\iota} := \frac{\dfrac{\Pi}{{A_1}^{I_1}, \cdots, {A_n}^{I_n} \vdash_{\mathrm{dom}(\iota)} B} \quad J\cdot\iota : u \to \mathrm{cod}(\iota)}{{A_1}^{I_1\cdot J\cdot\iota}, \cdots {A_n}^{I_n\cdot J\cdot\iota} \vdash_{\mathrm{cod}(\iota)} B^{J\cdot\iota}}\,\texttt{Prom}$$

Figure 3.5.: The substitution Π^{ι}

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad 0 : u \to v}{\Delta^0 \vdash_v B^0}\,\texttt{Prom} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma \vdash_v C}}{\Gamma, B^0 \vdash_v C}\,\texttt{Weak}}{\Delta^0, \Gamma \vdash_v C}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{\dfrac{\Pi_2}{\Gamma \vdash_v C}}{\cdots}\,\texttt{Weak}}{{\Delta_1}^0, \Gamma \vdash_v C}\,\texttt{Weak}$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad 1\rtimes\iota : u \to v}{\Delta^{1\rtimes\iota} \vdash B^{1\rtimes\iota}}\,\texttt{Prom} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, B^\iota \vdash_v C}}{\Gamma, B^{1\rtimes\iota} \vdash_v C}\,\texttt{Der}}{\Delta^\iota, \Gamma \vdash C}\,\texttt{Cut} \quad\longrightarrow\quad \dfrac{\dfrac{{\Pi_1}^\iota}{\Delta^\iota \vdash_u B^\iota} \qquad \dfrac{\Pi_2}{\Gamma, B^\iota \vdash_v C}}{\Delta^\iota, \Gamma \vdash_v C}\,\texttt{Cut}$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad K + J : u \to v}{\Delta^{K+J} \vdash_v B^{K+J}}\,\texttt{Prom} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, B^K, B^J \vdash_v C}}{\Gamma, B^{K+J} \vdash_v C}\,\texttt{Contr}}{\Delta^{K+J}, \Gamma \vdash_v C}\,\texttt{Cut} \quad\longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{Delta \vdash B} \qquad K : u \to v}{\Delta^K \vdash_v B^K}\,\texttt{Prom} \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash B} \qquad J : u \to v}{\Delta^J \vdash_v B^J}\,\texttt{Prom} \qquad \dfrac{\Pi_2}{\Gamma, B^K, B^J \vdash_v C}}{\Gamma, B^K, \Delta^J \vdash_v C}\,\texttt{Cut}}{\Delta^K, \Delta^J, \Gamma \vdash_v C}\,\texttt{Cut}}{\dfrac{\cdots}{\Delta^{K+J}, \Gamma \vdash_v C}\,\texttt{Contr}}\,\texttt{Contr}$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad K{\cdot}J : u \to v}{\Delta^{K\cdot J} \vdash_v B^{K\cdot J}}\,\texttt{Prom} \qquad \dfrac{\dfrac{\Pi_2}{\Sigma, B^K \vdash_w C} \qquad J : w \to v}{\Sigma^J, B^{K\cdot J} \vdash_v C^J}\,\texttt{Prom}}{\Delta^{K\cdot J}, \Sigma^J \vdash_v C^J}\,\texttt{Cut} \quad\longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad K : u \to w}{\Delta^K \vdash_w B^K}\,\texttt{Prom} \qquad \dfrac{\Pi_2}{\Sigma, B^K \vdash_w C}}{\Delta^K, \Sigma \vdash_w C}\,\texttt{Cut} \qquad J : w \to v}{\Delta^{K\cdot J}, \Sigma^J \vdash_v C^J}\,\texttt{Prom}$$

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad J : w \to v}{\Delta^K \vdash_v B^K}\,\texttt{Prom} \qquad \dfrac{\dfrac{\Pi_2}{\Gamma, B^K \vdash_v C} \qquad J \geq K}{\Gamma, B^J \vdash_v C}\,\mathcal{S}\texttt{wL}}{\Delta^J, \Gamma \vdash_v C}\,\texttt{Cut} \quad\longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta \vdash_u B} \qquad K : w \to v}{\Delta^K \vdash_u B^K}\,\texttt{Prom} \qquad \dfrac{\Pi_2}{\Gamma, B^K \vdash_u C}}{\Delta^K, \Gamma \vdash_u C}\,\texttt{Cut} \qquad \dfrac{J \geq K}{I_n{\cdot}J \geq I_n{\cdot}K}}{\dfrac{\cdots}{\Delta^J, \Gamma \vdash_u C}\,\mathcal{S}\texttt{wL}}\,\mathcal{S}\texttt{wL}$$

Figure 3.6.: Cut-elimination rules (for the exponentials only). Given a sequent $\Delta = {A_1}^{I_1}, \ldots, {A_n}^{I_n}$ and a parameter J, we denote by Δ^J, the sequent ${A_1}^{J\cdot I_1}, \ldots, {A_n}^{J\cdot I_n}$. Notice in particular that $\Delta^0 = {A_1}^0, \ldots, {A_n}^0$, that $\Delta^1 = \Delta$ and that $\Delta^{1\rtimes\iota} = \Delta^\iota$.

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A. Appendix

A.1. A little dictionary for category theory

We have seen that, in this thesis, category theory is a language for expressing abstractions and mathematical structures. How do you learn the most efficiency a new natural language? By reading linearly a dictionary? By looking a textbook with self-maid examples? If all of this can help and is even necessary at some point, the ideal exercise is the full immersion and the learning with actual concrete cases. The same applies for the category theory: looking only for definitions and basic examples is artificial, boring and pointless. The best approach, for our personal experience, is to go way back and forth between

- the reading of short, unintelligible descriptions siding with redundant more concrete ones to forge an intuition,
- the apparently pointless reading of soulless definitions.

This thesis is constructed along this philosophy, with a categorical appendix containing technical, difficult and intangible definitions, and the core of the thesis refering category theory but, when possible, staying as parenthetic as possible (in order to be skippable). Notice that most of our definitions comes from *the nLab* [nCa].

Sets and classes

In this thesis, we will generally give a set its naive interpretation: a set is composed of potentially infinitely many elements that respect some defined property. However, we know from 1901 that this is not coherent since we do not want to create sets that contains themselves (more exactly we do not want a set of all sets that "do not contain themselves"). That is why we in fact work in the non naive *set theory* which limits the use of sets over sets.

In order to overcome this limitation, we fix an inaccessible cardinal, which is a bigger set than any "set"[1] you can imagine, and we consider that all our *sets*[2] are inside it.

When, at some point, we want to use a collection of objects that is so big that we cannot be sure it is a set anymore, we will call it a *class*. As a result of the "set" theory, a class is also a "set" below another, bigger, inaccessible cardinal.

If, later, we want to apply to a class C a definition that requires a set, then we will restrain the definitions to the *small elements* of C.[3] This means, basically, that we change the working

[1] In order to distinguish our notion of set and the notion of set in the underlying set theory, we will call set the former and "set" the later

[2] See footnote 1

[3] So that we will speak of *small category* or class of *small sets*.

assumption to take a bigger inaccessible cardinal so that what was a class is now a set. Such a dynamical change in our working assumption seems unsafe, but, in actual cases, it always works without any problem.[4]

Categories: basic definitions

A category is basically a mix between a graph and an ordered set. Indeed, it consists of a class of objects related by composable morphisms; in particular, there can be several morphisms.

Definition A.1.0.3. *A category C is the given of:*

- *a class C_0 of* objects *denoted by the meta-variables a , b , c ...,*
- *for each couple of objects $a, b \in C_0$, a set $C_1[a,b]$ of* morphisms *denoted by the meta-variables ϕ , ψ , χ ..., morphisms are generally denoted by arrows so that $\phi \in C_1[a,b]$ is denoted by:*

  $$a \xrightarrow{\phi} b$$

  where a is called the source *of ϕ and b is called the* target *of ϕ,*
- *for each triples of objects $a, b, c \in C_0$, a function*

  $$(\_;\_) : C_1[a,b] \times C_1[b,c] \to C_1[a,c]$$

  called composition and graphically denoted:

  $$a \xrightarrow{\phi;\psi} c \qquad := \qquad a \xrightarrow{\phi} b \xrightarrow{\psi} c$$

- *for each object $a \in C_0$, an identity morphism denoted $id_a \in C_1[a,a]$,*
- *so that the composition is associative in order for the following to never be ambiguous:*

  $$a \xrightarrow{\phi} b \xrightarrow{\psi} c \xrightarrow{\chi} d$$

- *so that id_a is neutral for the composition:*

[4] Basically because the number of upgrades of the working assumptions will not be indexed by the notion of sets itself.

$$a \xrightarrow{\phi} b \xrightarrow{id_b} b \quad = \quad a \xrightarrow{\phi} b \quad = \quad a \xrightarrow{id_a} a \xrightarrow{\phi} b.$$

By abuse of notation, we will generally use C also for C_0 and C_1.

We will call isomorphism *a morphism $\phi : a \to b$ such that there exists $\phi^{-1} : b \to a$ such that $\phi; \phi^{-1} = id_a$ and $\phi^{-1}; \phi = id_b$.*

A subcategory *of a category C is a category $\mathcal{D}$ such that $\mathcal{D}_0 \subseteq C_0$ and $\mathcal{D}_1[a, b] \subseteq C_1[a, b]$ for all $a, b \in \mathcal{D}$. It is called a* full subcategory *if the second inclusion is an equality.*

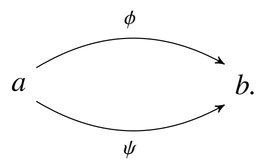
Example A.1.0.4. *The category $\Subset$ has the sets as objects and the inclusion as morphisms. It is an order-degenerated category (or an ordered class) in the sense that the morphisms are entirely described as an order relation on the object.*

The category Set *has the sets as objects and the functions as morphisms. The identity is the identity function and the composition is the composition of functions.*

The category Rel *has the sets as objects and the relations as morphisms. The identity is the relation $id_a := \{(\alpha, \alpha) \mid \alpha \in a\}$ and the composition is the composition of relation:*

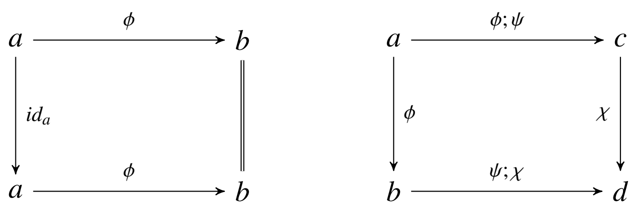
$$\phi; \psi \quad := \quad \{(\alpha, \gamma) \mid \exists \beta,\ (\alpha, \beta) \in \phi,\ (\beta, \gamma) \in \psi\}.$$

Definition A.1.0.5. *A* categorical diagram *or* coherence diagram *is a couple $\phi, \psi : a \to b$ of morphisms with same source and target. We say that such a diagram* commute *if the two morphisms are equal and we denote this equality by a cell:*[5]



When we want to give a name to a commuting diagram, we denote it inside the cell. Moreover, the identity arrow is sometimes denoted by a double line in order to ease the diagram readability.

Example A.1.0.6. *The two following coherence diagrams commute by definition of a category:*



Indeed, the first diagram states that one can precompose and postcompose a morphism ϕ with the identity and still get ψ, and the second states the associativity of the composition.

[5]Remark that often, the morphisms ϕ and ψ are described as compositions of other morphisms.

Definition A.1.0.7. *The* Cartesian product of two categories C *and* $\mathcal{D}$*, is the category* $C \times \mathcal{D}$*:*

- *which objects are the couples over C and* $\mathcal{D}$*:*

$$(C \times \mathcal{D})_0 := C_0 \times \mathcal{D}_0 := \{(a,b) \mid a \in C,\ b \in \mathcal{D})\}$$

- *which morphisms from* (a,b) *to* (a',b') *are the couples of morphisms:*

$$(C \times \mathcal{D})_1[(a,b),(a',b')] := C_1[a,a'] \times \mathcal{D}_1[b,b'].$$

The Cartesian exponent of a category C *by a set* D *is the category* C^D *:*

- *which objects are the fuctions from* D *to* C*:*

$$(C^D)_0 := C_0^D := \{f : D \to C_0\}$$

- *which morphisms from* f *to* g *are the dependant functions:*

$$(C^D)_1[f,g] := \{(\phi_d)_{d \in D} \mid \forall d \in D,\ \phi_d \in C[f(f), g(d)]\}.$$

The Cartesian coproduct of two categories C *and* $\mathcal{D}$*, is the category* $C + \mathcal{D}$*:*

- *which objects are the disjoint unions of the objects of C and* $\mathcal{D}$*:*

$$(C + \mathcal{D})_0 := C_0 \uplus \mathcal{D}_0 := \{(1,a) \mid a \in C\} \cup \{(2,b) \mid b \in \mathcal{D})\}$$

- *which morphisms from* (i,a) *to* (j,b) *are the morphisms from a to b when it makes sens:*

$$(C + \mathcal{D})_1[(1,a),(1,b)] := C_0[a,b] \qquad (C + \mathcal{D})_1[(2,a),(2,b)] := \mathcal{D}_0[a,b]$$
$$(C + \mathcal{D})_1[(1,a),(2,b)] = \emptyset \qquad (C + \mathcal{D})_1[(2,a),(1,b)] := \emptyset$$

The inverse of a category C *is the category* C^{-1} *defined by:*

$$C_0^{-1} := C_0 \qquad C^{-1}[a,b] := C[b,a]$$

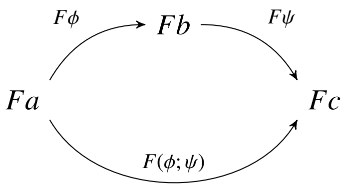
We denote 1 *the neutral element of the Cartesian product which has a single object* $*$ *and a single morphism* id_**.*

A category is an algebraic structure that focus on morphisms, which are basically dynamics, relations or symmetries between structures. One can go further and study the “morphisms” between categories. This is the purpose of functors, that are a sort of morphisms over categories.[6]

[6]We cannot call them “morphisms” in order not to collapse the set hierarchy.

Definition A.1.0.8. *A* functor $F : \mathcal{C} \to \mathcal{D}$ *where* $\mathcal{C}$ *and* $\mathcal{D}$ *are categories, is the given of:*

- *a function on objects* $F : \mathcal{C}_0 \to \mathcal{D}_0$*, with the notation* $Fa := F(a)$*,*
- *for each* $a, b \in \mathcal{C}_0$*, a function on morphism* $F : \mathcal{C}_1[a, b] \to \mathcal{D}_1[Fa, Fb]$*,*
- *such that the composition is preserved:*



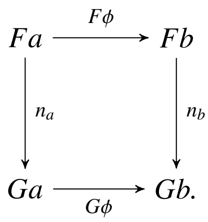
- *and such that the identities are preserved,* i.e. $Fid_a = id_{Fa}$ *for any* $a \in \mathcal{C}$*.*

A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ *is a functor from the opposite category* $F : \mathcal{C}^{op} \to \mathcal{D}$*. A* bi-functor *or simply* functor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{D}$ *is a functor from the product category* $\mathcal{C} \times \mathcal{C}'$ *to* $\mathcal{D}$*. An endofunctor is a functor* $F : \mathcal{C} \to \mathcal{C}$ *with the same source and target.*

If functors are sort of morphisms between categories, *natural transformations* are sort of morphisms between functors. The concept of *natural transformations* carries an idea of duality, stating that morphisms between functors should be morphisms in the target categories that depend on morphisms in the source category.

Definition A.1.0.9. *Given two functors* $F, G : \mathcal{C} \to \mathcal{D}$*, a* natural transformation $n : F \longrightarrow G$ *is the given of:*

- *a morphism* $n_a : Fa \to Ga$ *for each* $a \in \mathcal{C}$*,*
- *such that the following diagram commutes for every* $a, b \in \mathcal{C}$ *and every* $\phi \in \mathcal{C}[a, b]$*:*



Basic categorical constructions

In this section we present some basic constructions over categories; namely:

- initial/final objects (Def. A.1.0.10),

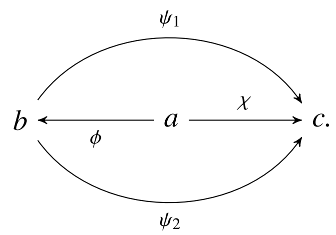
- epimorphisms and monomorphisms (Def. A.1.0.11),
- adjunctions (Def. A.1.0.12),
- symmetric (co-)monoidal (close) categories (Def. A.1.0.13),
- monoids (Def. A.1.0.14),
- (co)monads (Def. A.1.0.15),
- (co)algebras (Def. A.1.0.17),
- Kleisli categories (Def. A.1.0.18)
- and Cartesian (closed) categories (Def. A.1.0.19).

Definition A.1.0.10. *An* initial object *of a category C is an object $i \in C$ such that for any $a \in C$ there is a unique morphism* $\mathrm{init}_a : i \to a$.

A terminal object *is an initial object of the inverse category,* i.e.*, an object $t \in C$ such that for any $a \in C$, there is a unique morphism* $\mathrm{term}_a : a \to t$.

Initials and final objects are unique up-to isomorphisms.

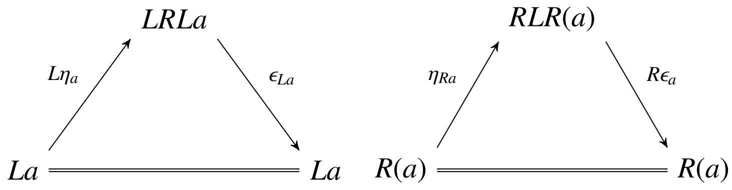
Definition A.1.0.11. *An* epimorphism *is a morphism $\phi : a \to b$ such that for every morphisms $\psi, \psi : b \to c$, if $\phi; \psi_1 = \phi; \psi_2$ then $\psi_1 = \psi_2$. In particular, if $\phi : a \to b$ is* epi *the following is a coherence diagram whenever the two internal cells commute:*



A monomorphism *is an epimorphism in C^{op}.*

Definition A.1.0.12. *Given two categories C and $\mathcal{D}$, an* adjunction *between the functors $L : C \to \mathcal{D}$ and $R : \mathcal{D} \to C$, is the given of:*

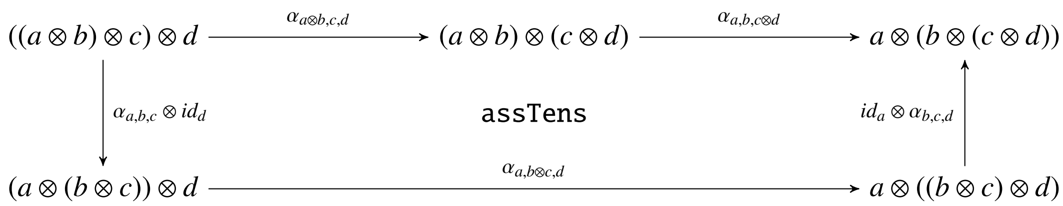
- *a natural transformation $\eta_a : a \longrightarrow RLa$ called* unit*,*
- *a natural transformation $\epsilon_a : LRa \longrightarrow a$ called* counit*,*
- *such that the following diagrams commute*

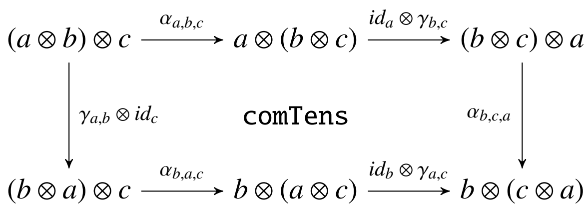


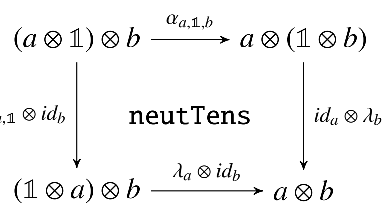
The functor L is called left adjoint *and the functor R is called* right adjoint.

Definition A.1.0.13. *A* symmetric monoidal category *(or* SMC*), is a category $\mathcal{L}$ endowed:*

- *with the functors*
  - $\otimes : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$,
  - *and* $\mathbb{1} : 1 \to \mathcal{L}$,
- *and with the natural isomorphisms*
  - $\alpha_{a,b,c} : (a \otimes b) \otimes c \longleftrightarrow a \otimes (b \otimes c)$,
  - $\lambda_a : (\mathbb{1} \otimes a) \longleftrightarrow a$
  - $\gamma_{a,b} : (a \otimes b) \longleftrightarrow (b \otimes a)$,
- *where* $\lambda_{\mathbb{1}} = \gamma_{\mathbb{1},\mathbb{1}}; \lambda_{\mathbb{1}}$ *(*`untTens` *), where* $\gamma^{-1}_{a,b} = \gamma_{b,a}$ *and the following diagrams commute.*







A symmetric monoidal closed category *(or a* SMCC*) is a symmetric monoidal category where* $\_ \otimes a$ *has a right adjoint* $a \multimap \_$ *for every object a.*

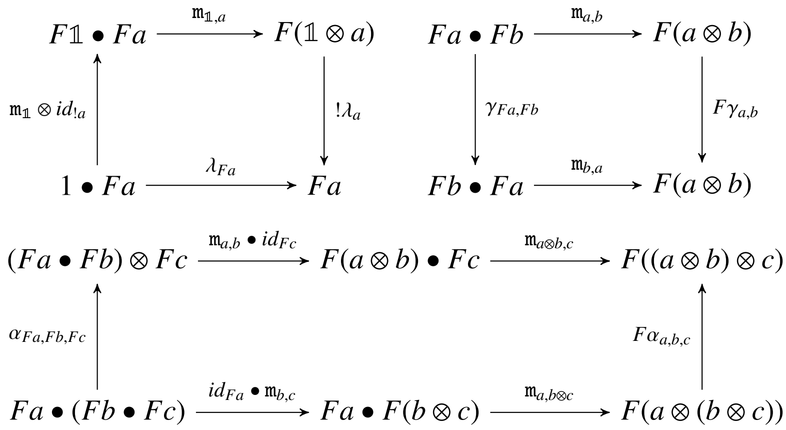
A symmetric comonoidal category *is a category C which inverse* C^{op} *(Def. A.1.0.7) is symmetric monoidal.*

A symmetric monoidal functor *between two monoidal categories* $(C, \otimes, \mathbb{1})$ *and* $(\mathcal{D}, \bullet, 1)$ *is a functor F and two natural transformations*

- $\mathtt{m}_{\mathbb{1}} : 1 \longrightarrow F\mathbb{1}$,

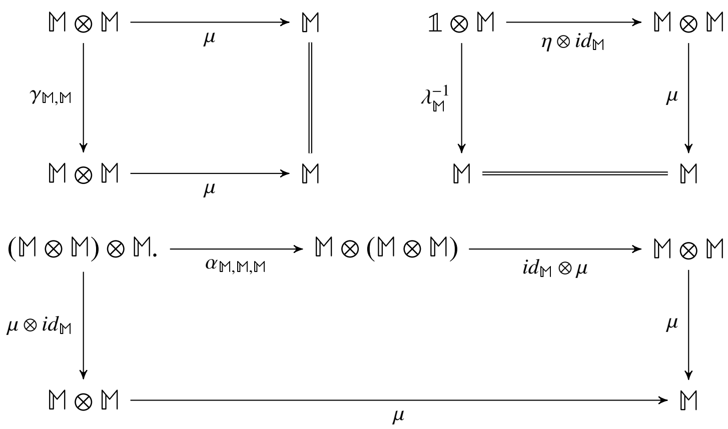
- $\mathfrak{m}_{a,b} : Fa \bullet Fb \longrightarrow F(a \otimes b)$

satisfying the following diagrams



Definition A.1.0.14. *A* commutative monoid *in a symmetric (co-)monoidal category C is an object $\mathbb{M} \in C$ endowed with:*

- *a morphism $\mu : \mathbb{M} \otimes \mathbb{M} \to \mathbb{M}$ called* multiplication*,*
- *a morphism $\eta : \mathbb{1} \to \mathbb{M}$ called* unit*,*
- *such that the following diagrams commute:*

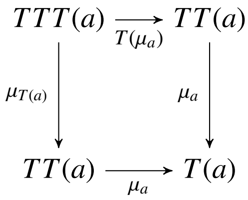


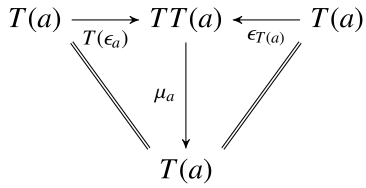
A commutative comonoid *in a symmetric (co-)monoidal category C is a commutative monoid in the inverse category C^{-1} (Def. A.1.0.7).*

Definition A.1.0.15. *A* monad *in a category C is an endofunctor $T : C \to C$ endowed with*

- *a natural transformation $\mu_a : TT(a) \longrightarrow T(a)$ called* multiplication*,*

- *a morphism $\eta_a : a \to T(a)$ called* unit*,*
- *such that the following diagrams commute:*





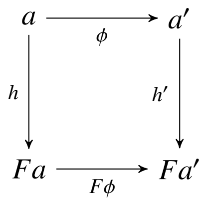
A comonad *is a monad in the inverse category C^{-1} (Def. A.1.0.7).*

Remark A.1.0.16. *A monad (F, μ, η) in C is a monoid of the (non-symmetric) monoidal category of endofunctors over a and natural transformations (with the composition of functors as monoidal product).*

Definition A.1.0.17. *An* algebra on an endofunctor *$F : C \to C$ is an object $a \in C$ endowed with a morphism $h : a \to Fa$. A* coalgebra on an endofunctor *F is an algebra on F in the inverse category.*[7]*A* bialgebra on an endofunctor *F is an algebra (a, h) on F such that h is an isomorphism, so that (a, h^{-1}) is a coalgebra.*

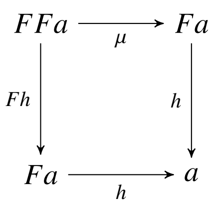
The category of algebras over an endofunctor *$F : C \to C$ is the category:*

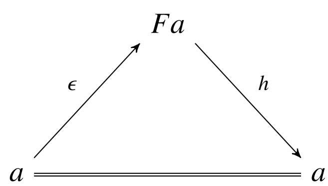
- *which objects are algebras over $F : C \to C$,*
- *which morphisms from (a, h) to (a', h') are the morphisms $\phi : a \to a'$ such that the following diagram commute:*



The dual concept is the category of coalgebras over an endofunctor.

An algebra on a monad *(T, μ, ϵ) is an algebra (a, h) over T (seen as an endofunctor) such that the following diagram commute:*





As usual, a coalgebra on a comonad *is an algebra on the inverse monad in the inverse category.*

The category of algebras over a monad (T, μ, ϵ) *is the full subcategory of the category of algebras over T which objects are algebras over (T, μ, ϵ). The dual concept is the* category of coalgebras over a comonad.

Definition A.1.0.18. *The* Kleisli category C_T *over a monad (T, μ, ϵ) in C consists of the category:*

- *which objects are the objects of C,*
- *which morphisms from a to b are the morphisms from a to Tb in C:*

$$C_T[a, b] \ := \ C[a, Tb],$$

- *which iddentity is the unit:*

$$id^!_a \ := \ \epsilon_a \ \in C[a, Ta],$$

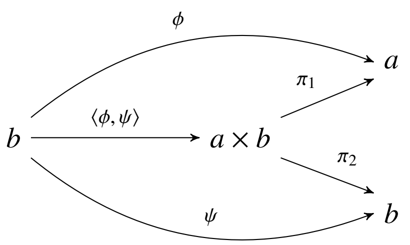
- *which composition is the multiplication (with $\phi \in C[a, Tb]$ and $\psi \in C[b, Tc]$):*

$$\phi \bullet \psi \ := \qquad a \xrightarrow{\ \phi\ } Tb \xrightarrow{\ !\psi\ } TTc \xrightarrow{\ \mu_c\ } Tc$$

Symmetrically, the Kleisli category C_T *over a comonad (T, μ, ϵ) in C is the category with the same objects and with $C[T(a), b]$ as morphisms from a to b.*

Definition A.1.0.19. *A* Cartesian category *consists of:*

- *a category C,*
- *a terminal object $\top$,*
- *a functor $\times : C \times C \to C$ called* Cartesian product*,*
- *two natural transformations $\pi_{1,a,b} : a \times b \longrightarrow a$ and $\pi_{2,a,b} : a \times b \longrightarrow b$,*
- *such that for every $\phi : c \to a$ and $\psi : c \to b$ there is a single morphism $\langle \phi, \psi \rangle : c \to a \times b$ such that:*



[7]Remark that an endofunctor over a category is also an endofunctor over the inverse category.

We will often call category with product *a Cartesian category and* category with coproduct *a category which inverse category is Cartesian.*

A Cartesian closed category *(*CCC *for short) is a Cartesian category which Cartesian product* $(a \times \_)$ *has a right adjoint* $(a \Rightarrow \_)$*. The unit of the adjunctions is called* evaluation *and denoted* $eval : a \times (a \Rightarrow b) \to b$*. For any morphism* $f : (a \times b) \to c$*, we called* Curryfication *of* f*, the morphism* $\Lambda_1 f : b \to (a \Rightarrow c)$ *defined by:*

$$\Lambda_1 f \quad := \qquad b \xrightarrow{\mu} a \Rightarrow (a \times b) \xrightarrow{id \Rightarrow f} a \Rightarrow c$$

where μ *is the counit of the adjunction.*

For any finite set U *with* n *elements, we denote* a^U *the Cartesian product of* n *versions of* a *marked with the elements of* U*. Moreover, given* $V \subseteq U$*, we denote* $\pi^U_V : a^U \to a^V$ *the projections over elements marked in* V*; when the source is clear, we denote it simply* π_V*. Similarly, for* $x \in U$ *anf* $f : a^U \to b$*, we denote* $\Lambda_x f : a^{U-x} \to (a \Rightarrow b)$ *the Currification over the* x*-marked source.*

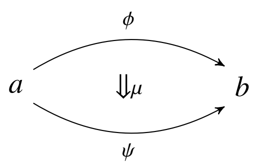
2-categories

In this section we present the concept of 2-categories.[8] If a category is an abstraction that represents dynamic structures and transformations, a 2-category is an abstraction that also represents structures behind transformations over transformations.

Remark A.1.0.20. *The definitions belows are said to be strict. This precision only results from the existence of other, weaker (or lax), definitions that relax some of the constraints.*

Definition A.1.0.21. *[KS74] A (strict)* 2-category C *is given by*

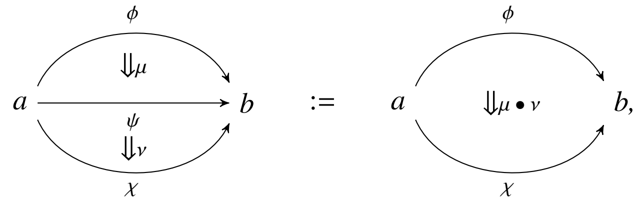
- *a class* C_0 *of objects (or* 0-cells*),*
- *a category (or* 1-category*)* $C_1[a, b]$ *for any* $a, b \in C_0$ *where:*
  - *the objects of* $C_1[a, b]$ *are called* 1-morphisms *(or* 1-cells*) and denoted by simple arrows* $\phi, \psi : a \to b$*,*
  - *the morphisms from* ϕ *to* ψ *are called* 2-morphisms *(or* 2-cells*) and denoted either as 1-dimentional double arrows,* $\mu, \nu : \phi \Rightarrow \psi$*, or depicted as 2-dimentionaly double arrows:*

  - *we moreover denote* $C_2[\phi, \psi] := C_1[a, b][\phi, \psi]$ *the set of 2-morphisms for* ϕ, ψ *objects of* $C_1[a, b]$*,*

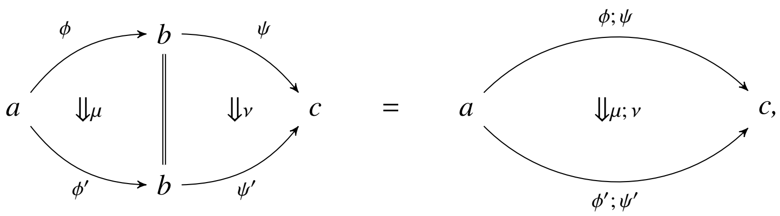
[8]More exactly strict 2-categories which constitute one possible definition over several existing.

– *the composition in* $C_1[a,b]$ *is called* vertical composition *and is denoted by* $\bullet$ *so that we can denote*



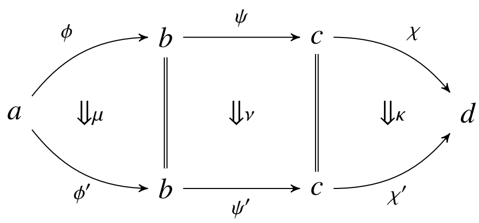
– *and the identities are called* 2-identities *and just denoted* id_ϕ *for* $\phi \in C_1[a,b]$*;*

- *for any* $a,b,c \in C_0$*, an* horizontal composition *functor* $\_;\_ : C_1[a,b] \times C_1[b,c] \to C_1[a,c]$ *so that :*

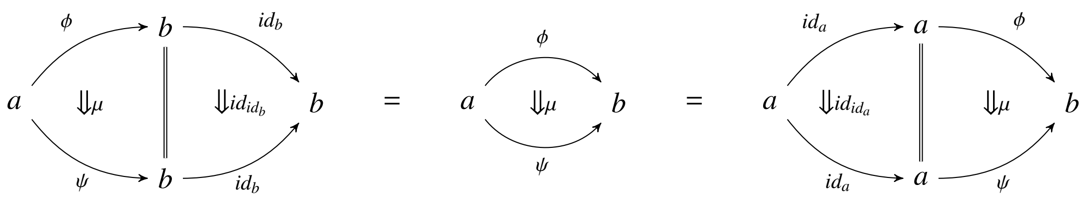


- *for any* $a \in C_0$*, a* 1-identity *object* $id_a \in C_1[a,a]$,

- *so that the horizontal composition is associative, in particular, the following diagram is not ambiguous:*



- *and so that the 2-identities between 1-identities are neutral for the horizontal composition:*



Moreover, we will use the term 3-cell *to refer to identities between 2-cells.*

Remark A.1.0.22. *For any 2-category* C*, the class of 0-cells endowed with the class of 1-cells as* *morphisms forms a category.*[9]

Similarly, the class of 0-cells endowed with the class of 1-cells quotiented by 2-isomorphisms forms a category.[10]

Remark A.1.0.23. *The notion of 2-isomorphisms in 2-categories being similar to the notion of commuting diagrams in 1-categories, we use the same kind of notations.*

Example A.1.0.24. *A traditional example is the example of the 2-category* Cat *:*

- *which objects are small categories,*
- *which 1-morphisms are functors,*
- *which 2-morphisms are natural transformation.*

In Section 3.2, where 2-categories will be at stake, we will in fact use order-degenerated 2-categories. As for the posets being degenerated categories (categories with at most one morphism between each given pair source-target of objects), the interest of order-degenerated 2-categories is that 3-cells always commute and 2-isomorphisms are equalities.

Definition A.1.0.25. *An* order-degenerated 2-category*, or* order-enriched category*, is a 2-category which hom-categories are posets. This means that for any two 1-morphisms ϕ, ψ, there is at most one 2-morphism denoted $(\phi \geq \psi)$ in $C[\phi, \psi]$; moreover $(\phi \geq \psi)$ has no inverse (except for the identity when $\phi = \psi$).*

Example A.1.0.26. *The category* Rel *of sets and relations can be generalized into the order-degenerated category* Rel *:*

- *which 0-cells are the sets,*
- *which 1-cells are the relations,*
- *which 2-cells are the inclusions of relations:*

$$R \Rightarrow R' \quad \textit{iff} \quad R \supseteq R'$$

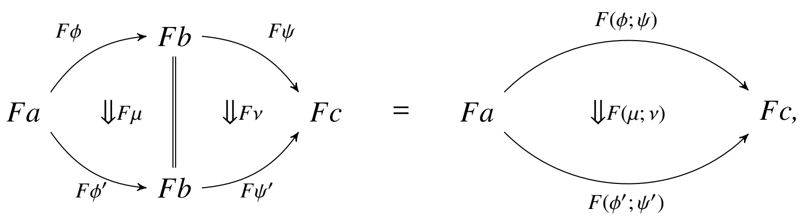
The notions of functors and natural transformations generalize through 2-categorical framework:

Definition A.1.0.27. *A (strict)* 2-functor *$F : C \to \mathcal{D}$ where C and $\mathcal{D}$ are 2-categories, is the given of:*

- *a function on 0-cells $F_0 : C_0 \to \mathcal{D}_0$,*
- *a functor $F : \mathcal{L}_1[a, b] \to \mathcal{L}_1[F_0(a), F_0(b)]$ for each $a, b \in \mathcal{L}_0$,*
- *such that the horizontal composition is preserved:*

[9] Strictly speaking, due to cardinality issues this is only true if $C_1[a, b]$ is a small category for each $a, b \in C_0$.

[10] See Footnote 9

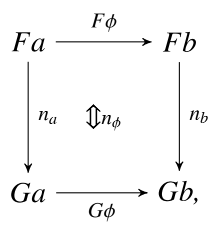


- *and such that the 1-identities are preserved,* i.e. $Fid_a = id_{Fa}$ *for any* $a \in C_0$.

In particular, if C *and* $\mathcal{D}$ *are order-degenerated, the horizontal composition is always preserved.*

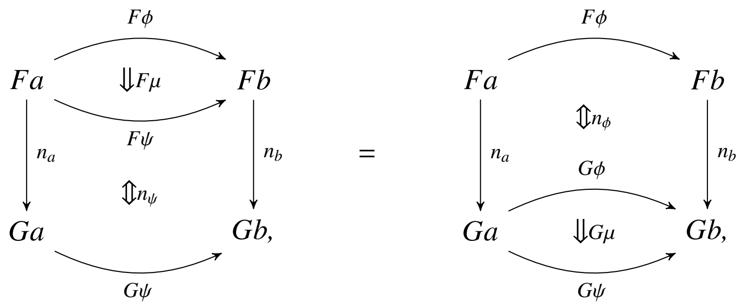
Definition A.1.0.28. *A (strict)* natural 2-transformation $n : FC \to GC$ *is the given of:*

- *a 1-morphism* $n_a : Fa \to Ga$ *for each* $a \in C_0$*,*
- *a 2-isomorphism:*



for any $\phi \in C_1[a, b]$*,*

- *The following diagrams are equal (*i.e.*, the corresponding 3-cell commute):*



- *The transformation respects the horizontal composition, equalizing the following diagrams (*i.e.*, the corresponding 3-cell commute):*

$$
\begin{array}{ccccccccc}
Fa & \xrightarrow{F\phi} & Fb & \xrightarrow{F\psi} & Fc & & Fa & \xrightarrow{F(\phi;\psi)} & Fc \\
\Big\downarrow n_a & \Updownarrow n_\phi & \Big\downarrow n_b & \Updownarrow n_\psi & \Big\downarrow n_c & = & \Big\downarrow n_a & \Updownarrow n_{\phi;\psi} & \Big\downarrow n_c \\
Ga & \xrightarrow[G\phi]{} & Gb, & \xrightarrow[G\psi]{} & Gc & & Ga & \xrightarrow[G(\phi;\psi)]{} & Gc
\end{array}
$$

- *The transformation respects the identity,* i.e. $n_{id_a} = id_{a,a}$ *for any* $a \in C$.

In particular, if C*,* FC *and* GC *are all order-degenerated, then a natural 2-transformation is a natural transformation in the underlying category (where the order has been forgotten).*

A.2. Term rewriting

A.2.1. Grammars and signatures

A recurrent and elementary object in our study is the signature. Signatures are abstract objects that can represent both grammars and formal proof systems.

Definition A.2.1.1. *A* signature Σ *is the given of*

- *a set* Σ_s *of* sorts*, sorts are denoted by* S *,* T *...*
- *and a set* Σ_p *of* function symbols*, which are endowed with a type* $p : S_1 \times \cdot \times S_n \to T$ *with* $S_1, ..., S_n, T \in \Sigma_s$*.*

A signature will be said monosorted *if there is only one sort, and* multisorted *otherwise.*

Function symbols of type $p : S_1 \times \cdot \times S_n \to T$ *are said* n-ary *functions symbols, the sort* T *is the* target *of* p *and* $S_1, \dots, S_n$ *are its* sources. *The* 0*-ary functions symbols are* constant symbols.

Remark A.2.1.2. *In terms of category, a signature* Σ *is an endofunctor*[11] $F_\Sigma : \mathrm{S{\scriptsize ET}}^{\Sigma_s} \to \mathrm{S{\scriptsize ET}}^{\Sigma_s}$*. Indeed, we can set* F_Σ *to be the following polynomial endofunctor:*

$$(F_\Sigma(\vec{X}))_T := \biguplus_{p : S_1 \times \cdots \times S_n \to T} (X_{S_1} \times \cdots \times X_{S_n}).$$

One can generalize a signature at the categorical level to be a polynomial endofunctor. Spelling out, a polynomial endofunctor is a endofunctor $F : C^n \to C^n$*, for* C *a category with products and coproducts, that is a tuple (in* $\mathrm{C{\scriptsize AT}}$*) coproduct of product (in* C*) over projections (in* $\mathrm{C{\scriptsize AT}}$*):*

$$F := \langle \vec{X} \mapsto \coprod_{i \in I_k} \prod_{j \in J_{i,k}} \pi^{\mathrm{C{\scriptsize AT}}}_{u(i,j,k)}(\vec{X}) \mid k \leq n \rangle^{\mathrm{C{\scriptsize AT}}}.$$

In particular, we will often consider for C *the category of small categories.*

[11] Recall from Definition A.1.0.7 that $\mathrm{S{\scriptsize ET}}^{\Sigma_s}$ is the Cartesian exponent of the category SET by the set of sorts

Grammars are used to inductively describe languages, both natural (human) and formal (computer). From a mathematical point of view, a grammar seems to be syntactic sugar coating signatures over universal algebras [MS10]. However, their use is more diverse as grammars can be used to define such algebras, that we call models (Def. A.2.1.13) but also syntactical representations (Def. A.2.1.9).

Definition A.2.1.3. *A* grammar $\mathcal{G}$ *is given by a set* $\mathcal{G}_s$ *of* sorts *and a set* $\mathcal{G}_p$ *of patterns.*

- *A* sort *is basically a* non-terminal symbol.
- *A* pattern p *is given by a* target sort T *and a sequence constituted of* terminal symbols *and non-terminal symbol over some sorts called the* source sorts. *The* type *of a pattern* p *is* $p : S_1 \times \cdots \times S_n \to T$ *where* T *is the target sort and* $S_1, \ldots, S_n \in \mathcal{G}_s$ *are the source sorts with the order and multiplicities of occurrences of their respective non-terminal symbol.*

A deterministic grammar *is a grammar which paterns contains at least one specific terminal symbol*[12] *(that does not appears in others) and which n-ary patterns (for* $n \geq 2$*) are parenthesized.*

For readability, we represent a deterministic grammar using one line per sort. Each sort is then represented by a line of the form

$$(\text{mysort}) \quad \mathbf{K} \quad K := \text{pattern} \quad | \quad \text{pattern} \quad | \quad \cdots$$

with a name (here mysort*), a mathematical symbol (here* $\mathcal{K}$*), a meta-variable (here* K*) and a serial of patterns separated by vertical lines | which target sort is* mysort. *By abuse of notation, we may use two or more non-terminal symbols for a specific sort, those are just synonyms used for readability.*

A (deterministic) grammar will be said monosorted *if there is only one sort, and* multisorted *otherwise.*

By abuse of notation and to be more syntactic, we often use the term "grammar" for "deterministic grammar". There is no ambiguity as we strictly specified that grammars in this thesis are only considered deterministic.

Notice that the first sort of a grammar is of particular importance, so that we will often use the same mathematical symbol to represent both the grammar and this first sort.

Proposition A.2.1.4. *Any deterministic grammar* $\mathcal{G}$ *defines a signature:*

- *which sorts are the sorts of* $\mathcal{G}_s$*,*
- *which function symbols are the patterns* $p \in \mathcal{G}_p$*.*

An other important realization of signatures is the formal inference systems

[12] Notice that the space denoting the application of λ-terms is a terminal symbol.

Definition A.2.1.5. *A formal inference system $\mathcal{S}$ is the given of:*

- *a set $\mathcal{S}_s$ of propositions,*
- *a set $\mathcal{S}_p$ of rules of the form:*

  where $\texttt{Precond}_1$,..., $\texttt{Precond}_n$ are propositions over $\mathcal{S}_s$ called preconditions and where $\texttt{Concl}$ is a propositions over $\mathcal{S}_s$ called conclusion.

Proposition A.2.1.6. *Any formal inference system $\mathcal{S}$ defines a signature:*

- *which sorts are the propositions over $\mathcal{S}_s$,*
- *which function symbols are rules over $\mathcal{S}_p$.*

Remark A.2.1.7. *All these definitions can be extended to patterns of unbounded arity and to deterministic grammars with infinitely many patterns. Such generalizations have no theoretical cost, but make the definitions less readable, thus we will not present them formally. However, we will use them freely along the thesis.*

Representations

When considering a deterministic grammar, we generally intend to use its inductively generated language, *i.e.*, the set of sequences obtained by inductively replacing meta-variables by patterns until no meta-variable remains. However, this presentation lacks of generality because it does not allow to treat infinite objects.

In this section, we are considering the generalization of the inductive language to infinite objects. There is not just one generalization, but a multitude, called *representations* which inductive and coinductive representations are respectively the smallest one and the largest one (for inclusion order).

Definition A.2.1.8. *Let Σ be a signature.*
A (possibly infinite) tree $\mathcal{T}$ is Σ-labeled if:

- *its nodes are labeled by function symbols over Σ_p,*
- *the arity of a node labeled by p the arity of p,*
- *and the i^{th} son of a node labeled by $p : S_1 \times \cdots \times S_n \to T$ is labeled by a function symbol targeting T.*

A Σ-labeled tree $\mathcal{T}$ is said to be of sort T if its root is labeled by a function symbol targeting T.

If $p : S_1 \times \cdots \times S_n \to T$ is a function symbol of Σ_p, and if $\mathcal{T}_1, ..., \mathcal{T}_n$ are Σ-labeled trees of sort $S_1, ..., S_n$, then we denote $p(\mathcal{T}_1, ..., \mathcal{T}_n)$ the Σ-labeled tree which root is labeled by p and which first sons are $\mathcal{T}_1, ..., \mathcal{T}_n$.

Definition A.2.1.9. *A* representation of a signature *is a set $\mathcal{R}$ of Σ-labeled trees such that for any function symbol $p : S_1 \times \cdots \times S_n \to T$ in Σ_p, and any Σ-labeled trees $\mathcal{T}_1, ..., \mathcal{T}_n$ of sort $S_1, ..., S_n$, the following are equivalent:*

- *for a ll $i \leq n$, $\mathcal{T}_i \in \mathcal{R}$*
- *$p(\mathcal{T}_1, ..., \mathcal{T}_n) \in \mathcal{R}$.*

Among representations, there are three particular ones that are often chosen:

- *The* inductive representation*, that is the minimal representation (for the inclusion). It consists of all finite Σ-labeled trees.*
- *The* coinductive representation *that is the maximal representation. It consists of all finite and infinite Σ-labeled trees.*
- *The* recursive representation *that consists on trees that can be recursively[13] describable. This means that we only consider trees for which there is a program, or a machine, that, given any node, can compute what are its sons.*

We will call inductive / coinductive / recursive element *of a sort T an element of its inductive / coinductive / recursive representation.*

Remark A.2.1.10. *In the particular case of a deterministic grammar:*

The inductive representation and the inductively generated language are isomorphic. This is due to the determinism of the grammar (otherwise two trees may corresponds to several words). In the following we will generally use the former, but apply theorems (induction) reserved to the later.

Grammars will always be given with a choice $\mathcal{R}$ of representation (by default the inductive one). By abuse of notation, we denote T the set of trees/words in $\mathcal{R}$ of sort T and its elements are denoted using for meta-variables the non-terminal symbol(s) of T.

Example A.2.1.11. *Numbers in decimal basis can be represented by the following monosorted grammar:*

$$\textit{(numbers)} \quad \overline{\mathbf{10}} \quad n := \quad 0. \mid n0 \mid n1 \mid n2 \mid n3 \mid n4 \mid n5 \mid n6 \mid n7 \mid n8 \mid n9$$

The inductive representation will give the natural numbers in decimal bases. The coinductive representation is composed by possibly infinite streams of numerals that one can see as real numbers between 0 *and* 1 *except that several numbers may have several representatives (for example* $0.5 \simeq 0.5000... \simeq 0.4999...$*) . The recursive representation will give recursive numbers between* 0 *and* 1 *(with possibly several representations of the same number).*

[13] i.e., effectively

Example A.2.1.12. *One can also mix the representations. For example, in the following grammar:*

$$\begin{array}{llll} \textit{(numbers)} & \overline{\mathbf{10}}_n & n := & .d \mid 0n \mid 1n \mid 2n \mid 3n \mid 4n \mid 5n \mid 6n \mid 7n \mid 8n \mid 9n \\ \textit{(decimals)} & \overline{\mathbf{10}}_d & d := & 0d \mid 1d \mid 2d \mid 3d \mid 4d \mid 5d \mid 6d \mid 7d \mid 8d \mid 9d \end{array}$$

We would like to be inductive on numbers and coinductive on decimals in order to form all real numbers.

Definition A.2.1.13. *Notice that a signature Σ defines a set of sorted algebraic operations Σ_p. The* algebraic structures *over this set of operations are composed of:*

- *of a set $[\![T]\!]$ for each sort T,*
- *of a function $[\![p]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![T]\!]$ for each pattern $p : S_1 \times \cdots \times S_n \to T$.*

We call Σ-algebra *of Σ any such algebraic structures.*

Given a representation $\mathcal{R}$ of Σ, a model *of $\mathcal{R}$ is an Σ-algebra $\mathcal{M}$ of Σ together with an interpretation $[\![.]\!] : \mathcal{R} \to \mathcal{M}$ such that*

$$[\![p(s_1, ..., s_n)]\!] = [\![p]\!]([\![s_1]\!], ..., [\![s_n]\!])$$

There is exactly one interpretation of the inductive representation into any model (by application of Proposition A.2.1.19), but there can be several for other representations.

Proposition A.2.1.14. *Any representation of a grammar $\mathcal{G}$ will model $\mathcal{G}$ as well as itself (with the identity as interpretation).*

Remark A.2.1.15. *In terms of category, a representation over Σ is a bialgebra on F_Σ (Rk. A.2.1.2 and Def. A.1.0.17). In particular, inductive and coinductive representations are respectively the initial and final (Def. A.1.0.10) bialgebras on F_Σ (*i.e. *the initial and final objects of the category of bialgebras). In the literature, those are generally described as the initial algebra and the final coalgebra, which is an equivalent definition.*

Remark the absence of categorical definition of the recursive representation. There lies one of the main limits of category theory: representing computability. Some progress have been done to overcome this limit [CH08], but there is not yet any consensus on this point.

Similarly, an algebra over a signature Σ is an algebra on F_Σ. A model over a representation (R, h) (seen as a bialgebra) is an algebra (R, f) together with a morphism of algebra $[\![.]\!] : (R, h) \to (R, f)$.

Definition A.2.1.16. *Let $\mathcal{G}$ be a deterministic grammar, let $S \in \mathcal{G}_s$ be a sort of $\mathcal{G}$, and let $\mathcal{R}$ be a representation of $\mathcal{G}$.*

A S-context over $\mathcal{R}$ is a tree that is an element of $\mathcal{R}$ except that exactly one branch of sort S has been truncated and replaced by a hole $(\!|.|\!)$. A context is a S-context for S the first sort of $\mathcal{G}$.

Equivalently, a S-context over $\mathcal{R}$ is an element of the inductively generated language over $\mathcal{G}^{(\!|.|\!)_S}$; where $\mathcal{G}^{(\!|.|\!)_S}$ is the deterministic grammar which sorts, denoted $T^{(\!|.|\!)_S} \in \mathcal{G}_s^{(\!|.|\!)_S}$, are the copies of the sorts $T \in \mathcal{G}_s$ of $\mathcal{G}$, and which patterns the following:

- *a constant* $(\!|.|\!) : S^{(\!|.|\!)s}$,
- *a pattern* $p(u_1, ..., u_{i-1}, s, u_{i+1}, ..., u_n) : S_i^{(\!|.|\!)s}$ *for each* $p : S_1 \times \cdots \times S_n \to T$ *in* $\mathcal{G}_p$, *each* $i \leq n$ *and each* $u_j \in S_j$ *for* $j \neq i$.

For any S-context $c \in T^{(\!|.|\!)s}$ *and any* $s \in S$, *we denote* $C(\!|s|\!)$ *the element of T obtained by substituting the sole constant* $(\!|.|\!)$ *in C by s.*

Induction and coinduction

In the following, we will only mainly consider monosorted signatures. However, all this results extend with sorts and higher order.

The inductive and coinductive are particularly important as they offer two fundamental propositions named respectively *induction* and *coinduction*. These theorems allow to prove global properties over the languages looking only at the function symbols, thus requiring a simple case disjunction.

The first (and the simplest) is the induction that reduces a universal property (*i.e.*, a condition in each inductive element) into a requirement on each function symbol.

Proposition A.2.1.17 (Propositional Induction). *Let* Σ *be a signature. Let P be a proposition over the inductive representation of T (the only sort).*
If for every pattern p of signature $T^n \to T$, *the following proposition is true:*

$$P(s_1) \wedge \cdots \wedge (s_n) \quad \Rightarrow \quad P(p(s_1, \ldots, s_n)).$$

Then the initial proposition is universally verified, i.e.*:*

$$\forall t \in T, P(t)$$

This means that to prove any property on an inductive representation, it is sufficient to split the proof and to show that this property is invariant through each of the function symbols.

Remark A.2.1.18. *For multisorted grammars, the proposition P can be different for each sort, except that the definition is identical.*

The induction has a functional counterpart.[14]

Proposition A.2.1.19 (Functional induction). *Let* Σ *be a signature and let X be any set.*
Let $f : (p, y^1, .., y^n) \mapsto x$ *be a function that associate to each function symbol* $p : T^n \to T$ *and each sequence* $y^1, \ldots, y^n \in X$ *an element* $x \in X$.

[14]Notice that functional and propositional inductions can be derived from a same principle in higher order type theory.

This defines a single function f^μ from the inductive elements of T to X respecting:

$$f^\mu(p(s_1, \ldots, s_n)) = f(p, f^\mu(s_1), \ldots, f^\mu(s_n)).$$

Remark A.2.1.20. *For multisorted signatures, the definition is identical except that the set X and the function f can depend on the sort.*

Remark A.2.1.21. *In terms of category, this means that for any algebra (X, f) on $F_{\mathcal{G}}$, there is a single algebra morphism from the initial bialgebra to (X, f). This is immediate once we remark that the initial bialgebra is also the initial algebra.*

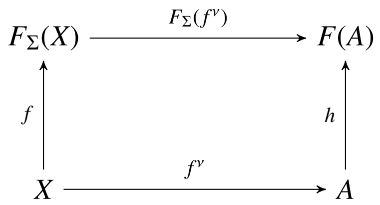
If the induction propagate an information from the sources to the target, the coinduction goes reversely.

Proposition A.2.1.22 (Functional coinduction). *Let Σ a signature and X be a set.*
Let $f : x \mapsto (p_x, y_x^1, \cdots, y_x^n)$ be a function that associate to each $x \in X$ a function symbol $p_x : T^n \to T$ and a sequence $y_x^1, \ldots, y_x^n \in X$.
This defines a single function f^ν from X to the coinductive elements of T respecting:

$$f^\nu(x) = p_x(f^\nu(y_x^1), \ldots, f^\nu(y_x^n)).$$

Remark A.2.1.23. *In terms of category:*

Requiring a function $f : X \to F_\Sigma(X)$ makes (X, f) a coalgebra on F_Σ. Recalling that the coinductive algebra (A, h) is the final coalgebra, it is immediate that there exists a single function $f^\nu : X \to A$ such that the following commutes:



Remark that the commutation of this diagram exactly states that:

$$f^\nu(x) = p_x(f^\nu(y_x^1), \ldots, f^\nu(y_x^n)).$$

The propositional coinduction, however, does not exist. Nonetheless, the coinduction can be used to derive coinductive proofs, *e.g.*, proofs over the coinductive representation of a formal inference system (Def. A.2.1.5).

Relations

Formally, a *relation* between a set X and a set Y is a set of couples over those $\_ \rhd \_ \subseteq (X \times Y)$, so that we denote $x \rhd y$ whenever $(x, y) \in \_ \rhd \_$. We denote $\rhd: X \to Y$ a relation from the set X called the *source* and the set Y called the *target*.

The *inverse relation* will be denoted $\_\triangleright^{-1}\_ := \{(y,x) \mid x \triangleright y\} : Y \to X$. The *image* of a set $x \in X$ by a relation $\triangleright$ is the set of all y that are in relation with $x \in X$: $(X \triangleright \_) := \{y \in Y \mid \exists x \in X,\ x \triangleright y\}$, the *pre-image* of a set $y \in Y$ by a relation $\triangleright$ is similarly defined by $(\_ \triangleright Y) := \{x \in X \mid \exists y \in Y,\ x \triangleright y\}$. In particular, the *domain* and *codomain* of $\triangleright\colon X \to Y$ are respectively the image of X and the pre-image of Y:

$$\mathrm{dom}(\triangleright) := \{x \in X \mid \exists y \in Y,\ x \triangleright y\} \qquad \text{and} \qquad \mathrm{cod}(\triangleright) := \{y \in Y \mid \exists x \in X,\ x \triangleright y\}.$$

We call *reflexive closure* of a relation $\triangleright\colon X \to X$ the smallest reflexive relation $\triangleright^?\colon X \to X$ that contains $\triangleright$:

$$x \triangleright^? y \quad \text{iff} \quad (x = y) \text{ or } x \triangleright y.$$

We call *transitive closure* of a relation $\triangleright\colon X \to X$ the smallest transitive relation $\triangleright^+\colon X \to X$ that contains $\triangleright$:

$$x \triangleright^+ y \quad \text{iff} \quad \exists z_1, \dots, z_n,\ x \triangleright z_1 \triangleright \cdots \triangleright z_n \triangleright y.$$

We call *reflexive transitive closure* of a relation $\triangleright\colon X \to X$ the smallest preorder $\triangleright^*\colon X \to X$ that contains $\triangleright$:

$$x \triangleright^* y \quad \text{iff} \quad \exists z_1, \dots, z_n,\ x = z_1 \triangleright z_2 \triangleright \cdots \triangleright z_{n-1} \triangleright z_n = y.$$

We call *composition* of two relations $\triangleright\colon X \to Y$ and $\blacktriangleright\colon Y \to Z$ the relation $\triangleright\blacktriangleright\colon X \to Z$ obtained by composing links:

$$x \triangleright\blacktriangleright z \quad \text{iff} \quad \exists y \in Y,\ x \triangleright y \blacktriangleright z.$$

We call *n^{th} composition* of a relation $\triangleright\colon X \to X$ the relation $\triangleright^n\colon X \to X$ obtained by composing it n times:

$$x \triangleright^n z \quad \text{iff} \quad \exists y_0, .., y_n \in X,\ x = y_0 \triangleright y_1 \triangleright \cdots \triangleright y_{n-1} \triangleright y_n = z.$$

We call *weak n^{th} composition* of a relation $\triangleright\colon X \to X$ the relation $\triangleright^{\leq n}\colon X \to X$ obtained by composing it up to n times:

$$x \triangleright^{\leq n} z \quad \text{iff} \quad \exists m \leq n, x \triangleright^m z.$$

Definition A.2.1.24. *Let Σ be a signature and X, Y be two models/representations of Σ.*
A relation $\triangleright$ between X and Y is sorted *if whenever $x \triangleright y$ the sorts of x and the sort of y are the same. In other words, a sorted relation is the disjoint unions of relations over each sorts.*

Definition A.2.1.25. *Let $\mathcal{R}$ a representation over a signature Σ.*
A (sorted) relation $\triangleright \in \mathcal{R} \times \mathcal{R}$ is contextually closed *if $\triangleright$ distributes locally with every function symbol. This means that for any function symbol $p : S_1 \times \cdots S_n \to T$, and for any $s_i, s_i' \in S_i$ (for some $i \leq n$) such that $s_i \triangleright s_i'$, we have*

$$p(s_1, \dots, s_i, \dots, s_n) \triangleright p(s_1, \dots, s_i', \dots, s_n).$$

When a contextually closed relation is an equivalence relation, we call it a congruence. *When it is just a pre-order, we call it an* inequational congruence.
The contextual closure *of a sorted relation $\triangleright$ is the smallest contextually closed relation that contains $\triangleright$. Equivalently, the contextual closure of $\triangleright$ is the relation inductively generated*

by the following rules (for each pattern $p : S_1 \times \cdots \times S_n \to T$):

$$\frac{s \rhd s'}{s \blacktriangleright s'} \qquad \frac{i \le n \qquad s_1 \blacktriangleright s'_1 \qquad s_n \blacktriangleright s'_n}{p(s_1, \dots, s_n) \blacktriangleright p(s'_1, \dots, s'_n).}$$

The contextual coclosure *of a sorted relation $\triangleright$ is the largest contextually closed relation that is contained in $\triangleright$.*

Remark A.2.1.26. *In the particular case of a representation $\mathcal{R}$ over a deterministic grammars $\mathcal{G}$, the relation $\rhd \in \mathcal{R} \times \mathcal{R}$ is* contextually closed *if for every sort S and T and for any S-context $C \in T^{(\!|.|\!)S}$ over $\mathcal{R}$:*

$$s \rhd s' \quad \Rightarrow \quad C(\!|s|\!) \rhd C(\!|s'|\!).$$

In particular, the contextual closure corresponds to the following closure:

$$t \blacktriangleright t' \quad \mathit{iff} \quad \left(\exists S, \exists C \in T^{(\!|.|\!)S}, \exists s, s' \in S,\ t = C(\!|s|\!),\ t' = C(\!|s'|\!),\ \mathit{and}\ s \rhd s'\right)$$

and the contextual coclosure corresponds to the following closure:

$$s \blacktriangleright s' \quad \mathit{iff} \quad \forall T, \forall C \in T^{(\!|.|\!)S}, C(\!|s|\!) \rhd C(\!|s'|\!),$$

Definition A.2.1.27. *Let $\mathcal{G}$ be a grammar and $\mathcal{R}$ and $\mathcal{M}$ be, respectively, a representation and a model of $\mathcal{G}$.*

An interpretation *of $\mathcal{R}$ into $\mathcal{M}$ is function $[\![.]\!]$ from $\mathcal{R}$ to $\mathcal{M}$,* i.e. *such that:*

$$[\![p(s_1, \dots, s_n)]\!] = [\![p]\!]([\![s_1]\!], \dots, [\![s_n]\!]).$$

Remark A.2.1.28. *In terms of category, this means that the interpretation is a morphism of algebra. The uniqueness of the interpretation of the inductive representation is immediate since the inductive representation is the initial algebra.*

Calculi and rewriting systems

A calculus is basically a relation (called rewriting) over a deterministic grammar, generally denoted by an arrow $\to$ and a set of normal forms denoted nf. When grammars represent programming languages, rewriting relations represent their dynamism and the normal forms the possible results. We assume here that all internal states of your computer can be seen as a specific program in your grammar. Then, one step of reduction $M \to N$ means that a computer executing the program M will reach a state represented by N.[15]

Similarly to grammars that were abstracted by signatures, calculi are abstracted by *abstract term rewriting systems*.

[15] Of course, these assumptions are generally unrealistic, but the gap between the theory and practice here can be overcome in different aspects.

Definition A.2.1.29. *An* abstract rewriting system *(for short* ARS*) is a set C of terms together with a relations $\to$ over $C \times C$ called* small steps reduction*. If $M \to N$ we say that M* reduces *in one step to N. We call* transitive reduction *the reflexive transitive closure $\to^*$ of $\to$, and if $M \to^* N$, we say that M* reduces *to N.*

Remark A.2.1.30. *In some cases, it is more natural to consider other transitive reductions. For example, if C is a co-inductive representation, it seems more natural to define a transitive reduction $\to^\omega$ by the coinductive proofs over:*[16]

However, such a generalization require to redo numerous usual proofs over ARS.

Remark that the reduction of a calculus/ARS is generally *not deterministic* in the sense that one can have a *split* $N_1 \leftarrow M \to N_2$. However, a weaker notion of determinism is often required: *the confluence*. The confluence basically state that reductions can be performed in any order (with a potential cost).

Definition A.2.1.31. *A small steps reduction $\to$ is* confluent *if every split ${N_1}^* \leftarrow M \to^* N_2$ can be closed in a finite number of steps:*

$$\begin{array}{ccc} M & \to^* & N_2 \\ \downarrow_* & \rightsquigarrow & \downarrow_* \\ N_1 & \to^* & M', \end{array}$$

where $\to^*$ is the reflexive transitive closure of $\to$.

An important notion related to calculi is the notion of *termination*. Indeed, it may be possible to form an infinite chain $M_1 \to M_2 \to \cdots$. Such a chain represents a never ending computation which is generally an ill behavior. That is why such a situation deserves a special treatment.

Definition A.2.1.32. *Let $(C, \to)$ an ARS. A term $M \in C$ is a* normal forms *if there is no N such that $M \to N$, the set of normal forms is denoted* nf*. We denote by $M \Downarrow N$ the* big step reduction *of M into N if N is a normal form and $M \to^* N$. We say that M* converges *(or is* weakly normalizing*) if there is N such that $M \Downarrow N$; convergence is denoted $M\Downarrow$. Reciprocally, the* divergence *$\Uparrow M$ of M is the absence of a normal form N such that $M \Downarrow N$*

To keep track of the converging times, we denote $M\Downarrow_{\mathrm{n}}$ whenever there is a normal form N such that $M \to^{\leq n} N$.

Remark A.2.1.33. *In certain cases, the definition of normal forms can be larger. For example, in order to represent may-non-determinism, we can consider that a term is in normal form if one of its component is.*

[16]Here we assume that $\to$ is contextually closed.

Proposition A.2.1.34. *The convergence is an inductive property. Indeed, a proof of convergence of M is an inductive proof over the following rules:*

$$\frac{M \in \mathrm{nf}}{M\Downarrow} \qquad \frac{M \to N \qquad N\Downarrow}{M\Downarrow}$$

Similarly, the divergence of M is a coinductive proof over the following rules:

$$\frac{M \notin \mathrm{nf} \qquad \forall N \in (M \to \_), \qquad N\Uparrow}{M\Uparrow}$$

where $(M \to \_)$ *is the set of terms* N *such that* $M \to N$*. To prove that a set* X *of terms diverges, it suffices to show that for any term* $M \in X$*,* M *is not a normal form and can only reduce inside* X*.*

For calculi with a reduction that is not contextually close, we are generally tempted to compare the reduction to its contextual closure. This is because it is generally much easier to work with the contextual closure and then to insure that the result extends to the original relation.

Definition A.2.1.35. *Let* $(C, \to_\alpha)$ *and* $(C, \to_\beta)$ *be two ARS over the same set* C*. We say that* $\to_\alpha$ *and* $\to_\alpha$ decomposes $\to_{\alpha\beta} := \to_\alpha \cup \to_\beta$ *if:*

$$\to_{\alpha\beta}^{*} \quad = \quad \to_{\alpha}^{*}\to_{\beta}^{*} .$$

This property is the basis for a well known property called *standardization*.

An other property that one would like is the invariance of the convergence.[17] It states that a forbidden reduction never extends the converging time of a converging term.

Definition A.2.1.36. *Let* $(C, \to)$ *be an ARS and* $\rightsquigarrow$ *a relation on* C*. We say that* $\Downarrow$ *is* invariant *through* $\rightsquigarrow$ *if for any* $M \rightsquigarrow N$ *and any* $n \in \mathbb{N}$*:*

$$M\Downarrow_n \quad \Rightarrow \quad N\Downarrow_n$$

The main interest of this invariance is the following property that allows one to treat divergence up-to any contextual reduction.

Proposition A.2.1.37. *If* $\Downarrow^{\mathrm{h}}$ *is invariant through* $\rightsquigarrow$ *then a proof of divergence corresponds to a coinductive derivation in the following proof system:*

$$\frac{M \notin \mathrm{nf} \qquad M \rightsquigarrow^{*} N \qquad \forall L \in (N \to \_), \quad L\Uparrow}{M\Uparrow}$$

[17] To my knowledge, this property is an original one.

where $(N \to \_)$ is the set of terms L such that $N \to L$. To prove that a set X of terms diverges, it suffices to show that for any term $M \in X$, M is not a normal form and $M \rightsquigarrow^* N$ that can only reduce inside X.

Proof. Any proof over the system of Proposition A.2.1.34 is convertible in a proof over this system (with $N = M$), thus if M do not converges then one can form a coinductive proof of $M\Uparrow^{\mathrm{h}}$ in this system.

Conversely, if M converges, then there is n such that $M\Downarrow_n$. By induction, we show that there is no proof of $M\Uparrow$ in this system:

- if $n = 0$ then $M \in \mathrm{nf}$ and the rule cannot apply,
- otherwise, for any N such that $M \rightsquigarrow^* N$, we have $N\Downarrow_n$ by invariance of the convergence, thus there is L such that $N \to L$ and $L\Downarrow_{n-1}$.

□

A.3. Linear logic

A.3.1. The logic

In this thesis, we will only use the *intuitionistic linear logic* (*ILL*). This is the asymmetric logical fragment of the full *linear logic* that treats negations indirectly via the linear arrow. This fragment is logically as powerful as the whole logic, but lacks of symmetry and fail at modeling notions where duality is at stakes. Our choice for such a restriction is the simplicity (sequent calculus is halved so that cut elimination is greatly simplified) and a focus on the main points for the thesis.

Intuitionistic linear logic can be decomposed in three fragments, the *multiplicative fragment*, the *exponential fragment* and the *additive fragment*.

- The multiplicative fragment can be thought as "a logic of continuous transformation": one can stick points/formulas together, translate an existing point/formula, transport a translation or even a translation between translations.
- The exponential fragment is "a logic of resource management and irreversibility": one can erase or duplicate formulas under the exponential modality !, so that this modality intend to break/direct the causality symmetry.
- The additive fragment is "a logic of choice and superposition".

It is the interaction of these three fragments that gives the linear logic its full potential. Nonetheless, we will mainly focus on the multiplicative exponential fragment along Chapter 3. The additive fragment is definitely as important as the other two, but the multiplicative exponential fragment is already sufficiently entertaining for our task.

$$\frac{}{A \vdash A}\,\texttt{Ax} \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,\texttt{Cut}$$

$$\frac{\Gamma \vdash C}{\Gamma, \mathbb{1} \vdash C}\,\mathbb{1}_{\mathrm{L}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes_{\mathrm{L}} \qquad \frac{}{\Gamma, \Delta \vdash \mathbb{1}}\,\mathbb{1}_{\mathrm{R}} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,\otimes_{\mathrm{R}}$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,\multimap_{\mathrm{L}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap_{\mathrm{R}}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\,\texttt{Weak} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,\texttt{Der} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\,\texttt{Contr} \qquad \frac{!A_1, \cdots, !A_n \vdash B}{!A_1, \cdots !A_n \vdash !B}\,\texttt{Prom}$$

$$\frac{}{\Gamma \vdash \top}\,\top_{\mathrm{L}} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C}\,\&_{\mathrm{L1}} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\,\&_{\mathrm{L2}} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&_{\mathrm{R}}$$

$$\frac{}{\Gamma, \mathbb{0} \vdash a}\,\mathbb{0}_{\mathrm{L}} \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,\oplus_{\mathrm{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,\oplus_{\mathrm{R1}} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,\oplus_{\mathrm{R2}}$$

Figure A.1.: The sequent calculus of ILL. In a sequent $\Gamma \vdash A$, Γ is supposed to be a multiset of formulas (no implicit contraction rule is admitted).

Definition A.3.1.1. *The* intuitionistic linear logic, ILL *the logic given by:*

- *the* formulas *inductively defined by the grammar:*

$$\textit{(formulas)} \quad \mathrm{LL} \quad A, B, C := \mathbb{1} \mid A \otimes B \mid A \multimap B \mid !A \mid A \,\&\, B \mid A \oplus B \mid \mathbb{0} \mid \top$$

- *the* proof system *inductively given by the sequent calculus of in Figure A.1.*
- *the* cut-elimination procedure *defined in detail by Bierman [Bie94].*

The fragment with formulas generated by the tensorial unity $\mathbb{1}$*, the* tensorial product $\otimes$ *and the* linear arrow $\multimap$ *is called the* multiplicative fragment*. The fragment with formulas generated by the* exponential modality ! *is called the* exponential fragment*. The fragment with formulas generated by the* Cartesian unit $\mathbb{0}$*, the* Cartesian co-unit $\top$*, the* Cartesian product $\oplus$ *and the* Cartesian co-product & *is called the* additive fragment*.*

The interaction between multiplicative and additive fragments seems very sparse since no sequent nor cut-elimination rule does mix them. In fact, an interaction activelly applies at the level of provability and expressivity; this interaction is visible, in particular, in the following fundamental equivalence (provable in ILL):

$$!(a \,\&\, b) \quad \simeq \quad !a \otimes !b. \tag{A.1}$$

The interaction between multiplicative and exponential fragments, however, is rich at every level. This fact is hidden behind the use of the comma in the context that is, at some point, another notation for the tensorial product $\otimes$. That is the reason why Section 3, which main point is the study of a quantitative refinement of the exponential modality (Sec. 1.3), will generally focus the multiplicative exponential fragment.

A.3.2. Linear categories

A model for ILL

The notion of *linear category* has been introduced by Bierman [Bie94] and revisited several times later on [BBdPH93]. It gives sufficient (but non necessary) conditions for a category to be a model of intuitionistic linear logic.There are other axiomatisations such that the new-Seely, the linear-non-linear and the Lafont categories, but those are equivalent to the linear axiomatisation if not weaker.

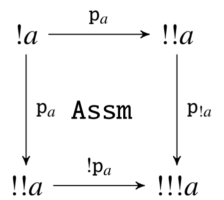
In this thesis, we will be particularly interested in the multiplicative exponential fragment so that we will not call *linear category* the full axiomatisation of Bierman but only the multiplicative exponential fragment.

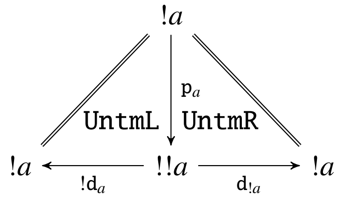
A *linear category* $\mathcal{L}$ consists of:

- a symmetric monoidal closed category (Def. A.1.0.13) for the multiplicative structure,
- and a functor $! : \mathcal{L} \to \mathcal{L}$ which has:
  - a comonad structure (Def. A.1.0.15): *i.e.*, the natural transformations

$$\mathtt{d}_a : !a \longrightarrow a,$$
$$\mathtt{p}_a : !a \longrightarrow !!a)$$

satisfying the following diagrams

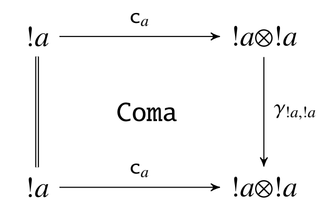


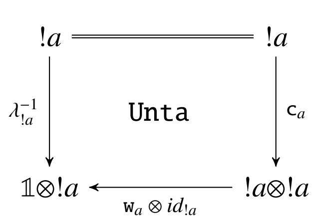


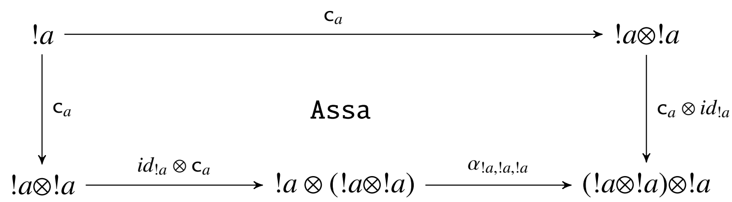
  - a commutative comonoidal structure on each $!a$ (Def. A.1.0.14): *i.e.*, the natural transformations

$$\mathtt{w}_a : !a \longrightarrow \mathbb{1},$$
$$\mathtt{c}_a : !a \longrightarrow !a \otimes !a$$

satisfying the following diagrams



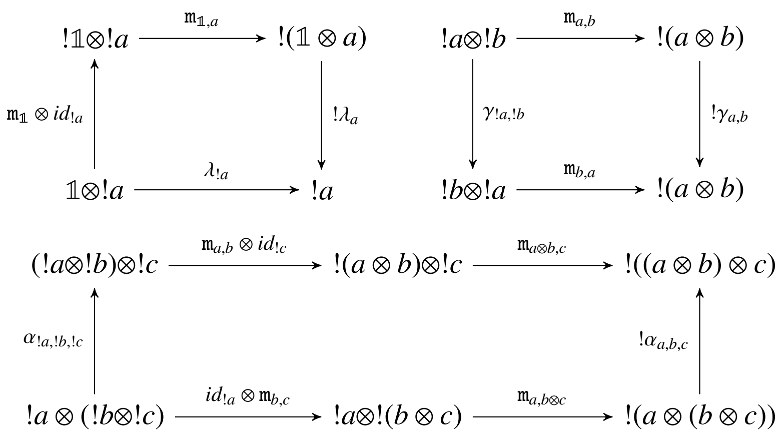




– and a structure of symmetric monoidal functor, *i.e.*, the natural transformations

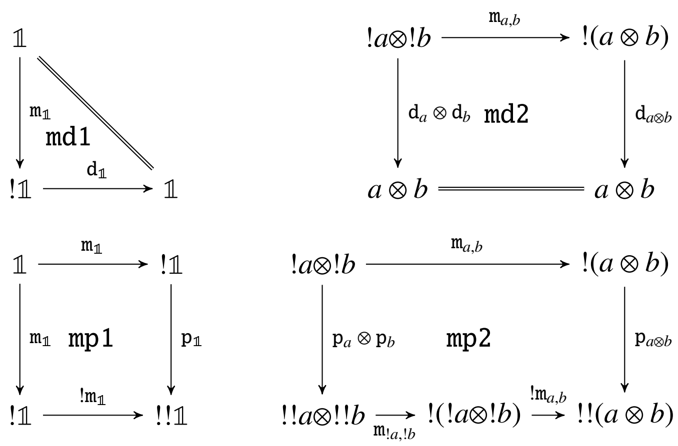
$$\mathtt{m}_{\mathbb{1}} : \mathbb{1} \longrightarrow !\mathbb{1}, ,$$
$$\mathtt{m}_{a,b} : !a \otimes !b \longrightarrow !(a \otimes b)$$

satisfying the following diagrams

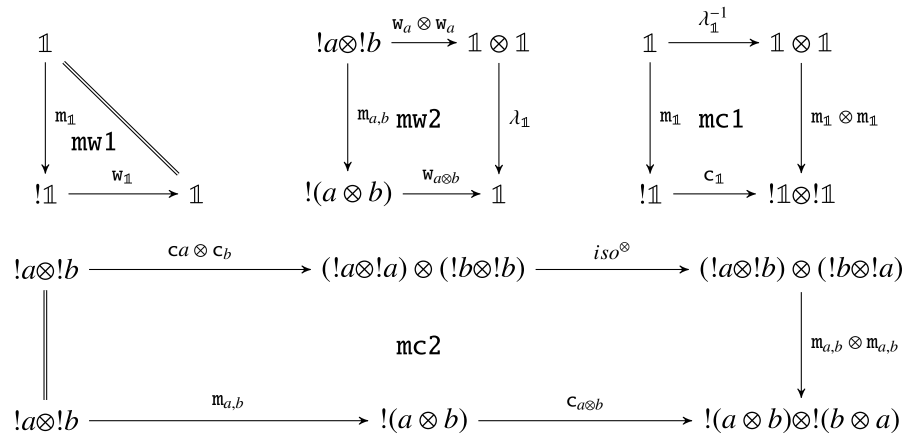


and where this structures interact nicely:

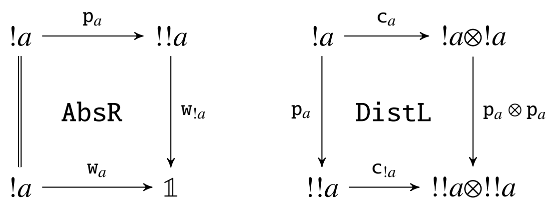
– the natural transformations d and p of the commonad are monoidal natural transformation, spelled out this means that the following diagram should commute:



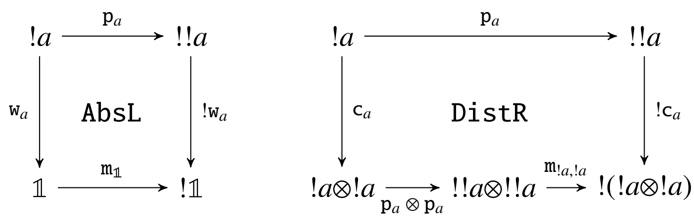
– the natural transformations w and c are monoidal natural transformation, spelled out this means that the following diagram should commute:



– every free coalgebra should be a comonoid morphism; in particular the natural transformation p, this means that the following diagrams should commute:



– the natural transformations w and c are coalgebra morphisms between coalgebras $(\mathbb{1}, \mathtt{m}_{\mathbb{1}})$, $(!a, \mathtt{p})$ and $(!a \times !a, (\mathtt{p} \otimes \mathtt{p}); \mathtt{m})$, spelled out this means that the following diagram should commute:



Proposition A.3.2.1 ([Bie94]). *A linear category with a Cartesian product and a Cartesian co-product is a model of* ILL.

A.3.3. The linear categories REL and COH

In this section, we will investigate two different categories that can be turned into linear categories: the relational category REL and the category COH of coherent spaces.

The linear category $\textsc{Rel}^{\mathbb{N}}$

Recall that he category $\textsc{Rel}$ (Ex. A.1.0.4) has sets as objects and relations as morphisms , i.e. $\textsc{Rel}(a, b) := \mathcal{P}(a \times b)$. Composition and identities are given by:

$$\phi; \psi := \left\{(\alpha, \beta) \;\middle|\; \exists\gamma, (\alpha, \gamma) \in \phi, (\gamma, \beta) \in \psi\right\}, \qquad id_a := \left\{(\alpha, \alpha) \;\middle|\; \alpha \in a\right\}.$$

In particular, remark that $\textsc{Rel}^{op} = \textsc{Rel}$.

Proposition A.3.3.1. *The category* $\textsc{Rel}$ *is a symmetric monoidal closed category when endowed with:*

- *the tensor product:*

$$a \otimes b := a \times b, \qquad \phi \otimes \psi := \left\{((\alpha, \alpha'), (\beta, \beta')) \;\middle|\; (\alpha, \beta) \in \phi,\ (\alpha', \beta') \in \psi\right\}.$$

  which neutral object is the singleton $\mathbb{1} := \{*\}$.

- *the natural bijections:*

$$\begin{aligned}
\alpha_{a,b,c} &:= \left\{((\alpha, (\beta, \gamma)), ((\alpha, \beta), \gamma)) \;\middle|\; \alpha \in a, \beta \in b, \gamma \in c\right\} \in \textsc{Rel}(a \otimes (b \otimes c), (a \otimes b) \otimes c)\\
\gamma_{a,b} &:= \left\{((\alpha, \beta), (\beta, \alpha)) \;\middle|\; \alpha \in a, \beta \in b\right\} \in \textsc{Rel}(a \otimes b, b \otimes a)\\
\lambda_a &:= \left\{(\alpha, (*, \alpha)) \;\middle|\; \alpha \in a\right\}
\end{aligned}$$

- *the linear arrow which is equal to the tensor product:*

$$a \multimap b := a \otimes b, \qquad \phi \multimap \psi := \phi \otimes \psi$$

  with the evaluation morphism

$$eval_{a,b} := \left\{(((\alpha, \beta), \alpha), \beta) \;\middle|\; \alpha \in a, \beta \in b\right\} \in \textsc{Rel}((a \multimap b) \otimes a, b).$$

Proposition A.3.3.2. *The category* $\textsc{Rel}$ *is a Cartesian category with infinite product:*

- *with the functor*

$$\mathop{\&}_{i \in I} a_i := \bigcup_{i \in I} (\{i\} \times a_i) \qquad \mathop{\&}_{i \in I} \phi_i := \left\{((i, \alpha), (i, \beta)) \;\middle|\; i \in I, (\alpha, \beta) \in \phi_i\right\}$$

- *with the empty set as terminal object* $\top := \emptyset$*,*

- *with the projection given by:*

$$\pi_{(a_i)_{i\in I},j} := \left\{((j,\alpha),\alpha) \;\middle|\; \alpha \in a_j\right\} \in \mathrm{REL}\Big(\operatorname*{\&}_{i\in I} a_i, a_j\Big)$$

- *and for any family $(\phi_i)_{i\in I}$ such that $\phi_i \in \mathrm{REL}(b,a_i)$, the pairing:*

$$\langle \phi_i \mid i \in I\rangle := \left\{\beta,(i,\alpha) \;\middle|\; i\in I, (\beta,\alpha)\in\phi_i\right\} \in \mathrm{REL}\Big(b, \operatorname*{\&}_{i} a_i\Big),$$

The product is also an (infinite) Cartesian co-product since $\mathrm{REL}^{op} = \mathrm{REL}$ *so that*

$$\bigoplus_{i\in I} a_i := \operatorname*{\&}_{i\in I} a_i.$$

Definition A.3.3.3. *We call* $\mathrm{REL}^{\mathbb{N}}$ *the linear category (Sec. A.3.2) given by the exponential:*

- *which functor is the finite multiset functor*

$$!a := \mathbb{N}_f\langle a\rangle, \qquad !\phi := \left\{([\alpha_1,...,\alpha_n],[\beta_1,...,\beta_n]) \;\middle|\; n\in\mathbb{N},\ \forall i\leq n, (\alpha_i,\beta_i)\in\phi\right\}$$

- *with the natural transformation:*

$$\begin{aligned}
\mathtt{d}_a &:= \left\{([\alpha],\alpha) \;\middle|\; \alpha\in a\right\} & \mathtt{p}_a &:= \left\{(\Sigma_{i\leq n}u_i,[u_1,...,u_n]) \;\middle|\; n\in\mathbb{N},\forall i, u_i\in !a\right\}\\
\mathtt{w}_a &:= \left\{([],*)\right\} & \mathtt{c}_a &:= \left\{(u+v,(u,v)) \;\middle|\; u,v\in !a\right\}\\
\mathtt{m}_\perp &:= \left\{(*,u) \;\middle|\; u\in !\mathbb{1}\right\} & \mathtt{m}_{a,b} &:= \Big\{(([\alpha_i|i\leq n],[\beta_i|i\leq n]),[(\alpha_i,\beta_i)|i\leq n])\\
&&&\qquad \Big|\; n\in\mathbb{N},\ \forall i,\ \alpha_i\in a,\ \beta_i\in b\Big\}
\end{aligned}$$

The Kleisli category $\mathrm{REL}^{\mathbb{N}}_{!}$

As we have seen in Theorem 1.2.3.2, any linear category with products and coproducts can be turned into a CCC by taking the Kleisli category.

Definition A.3.3.4. *We define the Cartesian closed category* $\mathrm{REL}^{\mathbb{N}}_{!}$ *[Hut94, Win99, Ehr12]:*

- *the* objects *are the sets.*
- *the* morphisms *from a to b are relations between* $\mathbb{N}_f\langle a\rangle$ *and b.*

The Cartesian product *is the disjoint sum of sets. The* terminal object $\top$ *is the empty set. The* exponential object $a \Rightarrow b$ *is* $\mathbb{N}_f\langle a\rangle \times b$.

Coherent spaces

An other important example of linear category is the first historical model of linear logic: the coherent spaces [Gir88]. In fact those where even prior to linear logic that was deduced from their internal structure.

Definition A.3.3.5. *A* coherent space *a consists of*

- *a set* $|a|$ *called its* web*,*
- *a symmetric reflexive relation* $\frown\!\!\!\!\smile_a \subseteq |a| \times |a|$ *called its coherence.*

Given a coherent space a, we denote $\frown_a$ *the* strict coherence *defined by* $\alpha \frown_a \beta$ *if* $\alpha \frown\!\!\!\!\smile_a \beta$ *and* $\alpha \neq \beta$*. Similarly, we denote* $\asymp_a$ *the* incoherence *which is the negation of the strict coherence.*

We call cliques *of a any subset* $\phi \subseteq a$ *such that* $\alpha \frown\!\!\!\!\smile_a \beta$ *for every* $\alpha, \beta \in \phi$*. Similarly, we call* multicliques *any multiset* ϕ *over a which support is a clique. The sets of cliques and multicliques over a are denoted* $C(a)$ *and* $C_m(a)$ *.*

Given two coherent spaces, we define the linear arrow $a \multimap b$ *to be the coherent space:*

- *which web is the Cartesian product in* Set $|a \multimap b| := |a| \times |b|$,
- *which coherence is defined by* $(\alpha,\beta) \frown\!\!\!\!\smile_{a\multimap b} (\alpha',\beta')$ *iff :*
  - $\alpha \frown\!\!\!\!\smile_a \alpha'$ *implies* $\beta \frown\!\!\!\!\smile_\alpha \beta'$*,*
  - $\alpha \frown_a \alpha'$ *implies* $\beta \frown_\alpha \beta'$*.*

Definition A.3.3.6. *The category* Coh *of* coherent spaces *is the SMCC with products and coproducts:*

- *which objects are the coherent spaces (Def. A.3.3.5),*
- *which morphisms from a to b are the cliques of their linear arrow*

$$\textsc{Coh}(a,b) := C(a \multimap b),$$

- *which iddentities and compositions are the relational iddentities and compositions:*

$$id_a := \{(\alpha,\alpha) \mid \alpha \in a\} \qquad \phi;\psi := \{(\alpha,\gamma) \mid \exists\beta,\ (\alpha,\beta) \in \phi,\ (\beta,\gamma) \in \psi\}$$

- *which monoidal product and unit are defined by*

$$a \otimes b := (|a| \times |b|, \frown\!\!\!\!\smile_a \times \frown\!\!\!\!\smile_b) \qquad \mathbb{1} := (\{*\}, \{(*,*)\}),$$

- *which linear arrow is the linear arrow over coherent spaces,*
- *which Cartesian product and terminal object are defined by*

$$a \,\&\, b := (|a| \uplus |b|, \frown_a \uplus \frown_b) \qquad\qquad top := (\emptyset, \emptyset),$$

- *which Cartesian coproduct is defined by* $|a \oplus b| := |a| \uplus |b|$ *and* $\asymp_{a\oplus b} := \asymp_a \uplus \asymp_b$ *with the empty space as initial object* $\bot := (\emptyset, \emptyset)$,
- *and with the following natural isomorphisms*

$$\begin{aligned}
\alpha_{a,b,c} &:= \{(((\alpha,\beta),\gamma)\ ,\ (\alpha,(\beta,\gamma))) \mid \alpha \in |a|, \beta \in |b|, \gamma \in |c|\}\\
\lambda_a &:= \{((*,\alpha)\ ,\ \alpha) \mid \alpha \in |a|\}\\
\gamma_{a,b} &:= \{((\alpha,\beta)\ ,\ (\beta,\alpha)) \mid \alpha \in a, \beta \in |b|\}\\
\pi_{1,a,b} &:= \{((\alpha,\beta),\alpha) \mid \alpha \in |a|,\ \beta \in |b|\}\\
\iota_{1,a,b} &:= \{(\alpha,(\alpha,\beta)) \mid \alpha \in |a|,\ \beta \in |b|\}
\end{aligned}$$

as well as the morphisms (for $\phi : c \to a$, $\psi : c \to b$, $\phi' : a \to c$ *and* $\psi' : b \to c$*):*

$$\begin{aligned}
\langle \phi, \psi \rangle &:= \{(\gamma,(\alpha,\beta)) \mid (\gamma,\alpha) \in \phi,\ (\gamma,\beta) \in \psi\}\\
[\phi', \psi'] &:= \{((\alpha,\beta),\gamma) \mid (\alpha,\gamma) \in \phi',\ (\beta,\gamma) \in \psi'\}
\end{aligned}$$

There are two different exponentials appearing in the literature that make Coh a linear category. Those are the clique and the multiclique comonads.

Definition A.3.3.7. *We call* $\text{Coh}^{\mathbb{B}}$ *the linear category (Sec. A.3.2) given by the exponential:*

- *which functor is the finite clique functor*

$$\begin{aligned}
!a &:= (C(a), \{(\phi,\psi) \mid \forall \alpha \in \phi, \forall \beta \in \psi, \alpha \frown \beta\}),\\
!\phi &:= \left\{(\{\alpha_1,...,\alpha_n\},\{\beta_1,...,\beta_n\}) \in C(a) \times C(b) \;\middle|\; n \in \mathbb{N},\ \forall i \leq n, (\alpha_i,\beta_i) \in \phi\right\},
\end{aligned}$$

- *with the natural transformation:*

$$\begin{aligned}
\mathtt{d}_a &:= \left\{(\{\alpha\},\alpha) \;\middle|\; \alpha \in a\right\} & \mathtt{p}_a &:= \left\{(\bigcup_{i\leq n} u_i, \{u_1,...,u_n\}) \;\middle|\; n \in \mathbb{N}, \forall i, u_i \in !a,\ \forall i,j, u_i \frown_{!a} u_j\right\}\\
\mathtt{w}_a &:= \left\{(\emptyset, *)\right\} & \mathtt{c}_a &:= \left\{(u \cup v, (u,v)) \;\middle|\; u, v \in !a,\ u \frown_{!a} v\right\}\\
\mathtt{m}_\bot &:= \left\{(*, u) \;\middle|\; u \in !\mathbb{1}\right\} & \mathtt{m}_{a,b} &:= \left\{((u,v), u \times v) \;\middle|\; u \in !a,\ v \in !b\right\}
\end{aligned}$$

We call $\text{Coh}^{\mathbb{N}}$ *the linear category (Sec. A.3.2) given by the exponential:*

- *which functor is the finite multiclique functor*

$$!a := (C_m(a), \{(\phi,\psi) \mid \forall\alpha\in\phi, \forall\beta\in\psi, \alpha \frown \beta\}),$$
$$!\phi := \left\{([\alpha_1,...,\alpha_n],[\beta_1,...,\beta_n]) \;\middle|\; n\in\mathbb{N},\ \forall i\leq n, (\alpha_i,\beta_i)\in\phi\right\},$$

- *with the natural transformation:*

$$\mathtt{d}_a := \left\{([\alpha],\alpha) \;\middle|\; \alpha\in a\right\} \qquad \mathtt{p}_a := \left\{(\Sigma_{i\leq n}u_i, [u_1,...,u_n]) \;\middle|\; n\in\mathbb{N}, \forall i, u_i \in !a\right\}$$
$$\mathtt{w}_a := \left\{([],*)\right\} \qquad \mathtt{c}_a := \left\{(u+v,(u,v)) \;\middle|\; u,v\in !a\right\}$$
$$\mathtt{m}_\perp := \left\{(*,u) \;\middle|\; u\in !\mathbb{1}\right\} \qquad \mathtt{m}_{a,b} := \left\{(([\alpha_i|i\leq n],[\beta_i|i\leq n]),[(\alpha_i,\beta_i)|i\leq n]) \;\middle|\; n\in\mathbb{N},\ \forall i,\ \alpha_i\in a,\ \beta_i\in b\right\}$$

A.3.4. The linear category ScottL

Order relations

A *partially ordered sets* (or *poset*) is a couple $D = (|D|, \leq_D)$ where $|D|$ is a set and $\leq_D$ is an order, *i.e.* a reflexive, transitive and symmetric relation (remark that an order do not need to be total).

Given two posets $D = (|D|, \leq_D)$ and $E = (|E|, \leq_E)$, we will denote:

- $D^{op} = (|D|, \geq_D)$ the *reverse-ordered set*.
- $D\times E = (|D|\times|E|, \leq_{D\times E})$ the *Cartesian product* endowed with the pointwise order:

$$(\delta,\epsilon) \leq_{D\times E} (\delta',\epsilon') \quad \text{iff} \quad \delta\leq_D\delta' \quad \text{and} \quad \epsilon\leq_E\epsilon'.$$

- $\mathcal{A}_f(D) = (\mathcal{A}_f(|D|), \leq_{\mathcal{A}_f(D)})$ the set of *finite antichains* of D (*i.e.*, finite subsets whose elements are pairwise incomparable) endowed with the order :

$$a \leq_{\mathcal{A}_f(D)} b \quad \text{iff} \quad \forall\alpha\in a, \exists\beta\in b,\ \alpha\leq_D\beta$$

In the following we will use D for $|D|$ when there is no ambiguity. Initial Greek letters α,β,γ... will vary on elements of ordered sets. Capital initial Latin letters A, B, C... will vary over subsets of ordered sets. And finally, initial Latin letters a, b, c... will denote finite antichains.

An *order isomorphism* between D and E is a bijection $\phi : |D| \to |E|$ such that ϕ and ϕ^{-1} are monotone.

Given a subset $A \subseteq |D|$, we will denote $\downarrow A = \{\alpha \mid \exists\beta\in A, \alpha\leq\beta\}$. We denote by $\mathcal{I}(D)$ the set of *initial segments of* D, that is $\mathcal{I}(D) = \{\downarrow A \mid A\subseteq|D|\}$.

A *complete lattice* is a poset D where *sup*s are defined for any subsets; *i.e.* when for every $S\subseteq D$, there is $\bigvee S$ such that

$$\forall\alpha\in S,\quad \alpha\leq_D \bigvee S \qquad \text{and} \qquad \forall\beta,\quad (\forall\alpha\in s,\ \alpha\leq_D\beta) \Rightarrow \bigvee S \leq\beta$$

Remarks that a complete lattice automatically also has every *inf*s.

A complete lattice D is *completely distributive* if sups and infs distribute one with each other. It is *prime algebraic* if there is a subset P called *prime elements* that are not sups (if $\alpha = \bigvee S$ and $\alpha \in P$ then $\alpha \in S$) and such that any $\alpha \in D$ is the sup of the primes that it majors $\alpha = \bigvee(P \cap \downarrow\alpha)$.

Example A.3.4.1. *The set $\mathcal{I}(D)$ is a prime algebraic complete lattice with respect to the set-theoretical inclusion. The sups are given by the unions and the prime elements are the downward closure of the singletons. The compact elements are the downward closure of finite antichains.*

A function $f : D \to E$ between complete lattices is *linear* if it preserves finite sups:

$$\forall S \subseteq_f D, \quad f(\bigvee S) = \bigvee f(S)$$

A function $f : D \to E$ between complete lattices is *Scott-continuous* if it preserves all sups:

$$\forall S \subseteq D, \quad f(\bigvee S) = \bigvee f(S)$$

The domain of a function f is denoted by $\mathrm{dom}(f)$.

The *linear graph* of a linear function $f : \mathcal{I}(D) \to \mathcal{I}(E)$ is injectivelly defined by

$$\mathrm{graph}_l(f) := \{(\alpha,\beta) \in D{\times}E \mid \beta \leq f(\alpha)\} \tag{A.2}$$

The *Scott graph* of a Scott-continuous function $f : \mathcal{I}(D) \to \mathcal{I}(E)$ is injectivelly defined by

$$\mathrm{graph}_s(f) := \{(a,\beta) \in \mathcal{A}_f(D)^{op}{\times}E \mid \beta \in f(\downarrow a)\} \tag{A.3}$$

Proposition A.3.4.2. *Elements of $\mathcal{I}(\mathcal{A}_f(D)^{op}{\times}E)$ are in one-to-one correspondence with the linear functions from $\mathcal{I}(D)$ to $\mathcal{I}(E)$.*

Elements of $\mathcal{I}(\mathcal{A}_f(D)^{op}{\times}E)$ are in one-to-one correspondence with the Scott-continuous functions from $\mathcal{I}(D)$ to $\mathcal{I}(E)$.

Proof. A linear (resp. Scott-continuous) function is entirely defined by its linear (resp. Scott) graph:

$$\begin{array}{ll} \forall I \in \mathcal{I}(D{\times}E) & fun_l(I) := (A \mapsto \{\beta \mid \exists \alpha \in A,\ (\alpha,\beta) \in I\}) \\ \forall I \in \mathcal{I}(\mathcal{A}_f(D){\times}E) & fun_s(I) := (A \mapsto \{\beta \mid \exists a \in \mathcal{A}_f(A),\ (a,\beta) \in I\}) \end{array}$$

□

The category ScottL

Definition A.3.4.3. *We define the category* ScottL *[Ehr12]:*

- *An* object *is a partially ordered set.*
- *A* morphism *from D to E is a linear function between the complete lattices $\mathcal{I}(D)$ and $\mathcal{I}(E)$ of initial segments over D and E.*

Equivalently, a morphism from D to E is an initial segment of $D^{op} \to E$ (see Prop. A.3.4.2).

Proposition A.3.4.4. *The category* ScottL *is a symmetric monoidal closed category when endowed with:*

- *the product of posets as tensor product:*

$$D \otimes E := D \times E, \qquad f \otimes g := (I \mapsto f(\pi_1(I)) \times g(\pi_2(I))).$$

  which the neutral object is the singleton $\mathbb{1} := \{*\}$.

- *the left-contravariant product of posets linear arrow:*

$$D \multimap E := D^{op} \times E, \qquad f \multimap g := f \otimes g$$

  with the evaluation morphism

$$eval_{D,E} := (I \mapsto \{\beta \mid \exists \alpha, ((\alpha, \beta), \alpha) \in I\}).$$

Proposition A.3.4.5. *The category* ScottL *is a Cartesian category with infinite products*

$$\operatorname*{\&}_{i \in I} D_i := \bigcup_{i \in I} (\{i\} \times D_i) \qquad \operatorname*{\&}_{i \in I} f_i := \prod_{i \in I} f_i.$$

Remark that $\mathcal{I}(\&_i D_i) = \prod_i \mathcal{I}(D_i)$ with the projection and pairing given by the projection and pairing in Set.

The product is also an (infinite) Cartesian co-product since $\text{ScottL}^{op} = \text{ScottL}$ *so that*

$$\bigoplus_{i \in I} D_i := \operatorname*{\&}_{i \in I} D_i$$

with the co-projections and co-pairing given by:

$$\iota(A) := A \times \{\emptyset\}^{I-i} \qquad [f_i \mid i \in I](i, A) := f_i(A)$$

Proposition A.3.4.6. *The category* ScottL *is linear (Sec. A.3.2) when endowed with:*

- *the antichain functor as exponential functor:*

$$!D := \mathcal{A}_f(D); \qquad !f := A \mapsto \{b \mid \exists a \in A, f(\downarrow a) \supseteq b\}$$

remark that $f^{-1}(a)$ is an antichain by monotonicity of f,

- *with the natural transformation:*

$$\begin{aligned} \mathtt{d}_D(A) &:= \bigcup A & \mathtt{p}_D(A) &:= \downarrow\{\{a\} \mid a \in A\} \\ \mathtt{w}_D(A) &:= \{* \mid A \neq \emptyset\} & \mathtt{c}_D(A) &:= \downarrow\{(a,a) \mid a \in A\} \\ \mathtt{m}_{\mathbb{1}}(A) &:= \{\{*\}, \emptyset \mid A = \{*\}\} & \mathtt{m}_{D,E}(C) &:= \downarrow\{a \times b \mid (a,b) \in C\}. \end{aligned}$$

A.4. Semi-rings

A.4.1. Definitions and examples

Definition A.4.1.1. *A* monoid *is given by* $(\mathbb{M}, \cdot, 1)$ *where* $\mathbb{M}$ *is a set, the product* $\cdot$ *is an associative binary operation with a neutral element* $1 \in \mathcal{S}$*.*

A semiring *is given by* $(\mathcal{S}, \cdot, 1, +, 0)$ *where* $\mathcal{S}$ *is a set, the product* $\cdot$ *is an associative binary operation with a neutral element* $1 \in \mathcal{S}$ *and the sum* $+$ *is an associative commutative binary operation distributing over* $\cdot$ *with a neutral element* $0 \in \mathcal{S}$ *(that is absorbing for* $\cdot$*).*

An ordered semiring $(\mathcal{S}, \cdot, 1, +, 0, \leq)$ *is a semiring* $(\mathcal{S}, \cdot, 1, +, 0)$ *with a partial order* $\leq$ *such that sum and product are increasing monotone.*

A right-semiring $(\mathcal{S}, \cdot, 1, +, 0, \leq)$ *is similar to an ordered semiring, except that the* right-distribution *and* left-absorption *axioms (*i.e.*,* $(I + J)\cdot K = (I\cdot K) + (J\cdot K)$ *and* $0\cdot I = 0$*) are not required.*

A lax-semiring $(\mathcal{S}, \cdot, 1, +, 0, \leq)$ *is an intermediate stage where the right-distribution and the left-absorption axioms are only required in their oriented form:* $(I+J)\cdot K \leq (I\cdot K)+(J\cdot K)$ *and* $0\cdot I \leq 0$.

We use the meta-variables $\mathcal{S}$ and $\mathcal{R}$ for (ordered) (lax-)(left-)semiring; their elements are denoted by capital Latin letters I, J... in the first case and by lowercase Latin letters p, q... for the second. Monoids are denoted by the meta-variable $\mathbb{M}$, and its elements are written with lowercase Latin letters g, h...

Remark A.4.1.2. *Notice that because of the monotonicity of the multiplication,* $0_\mathcal{S} \leq 1_\mathcal{S}$ *(resp.* $1_\mathcal{S} \leq 0_\mathcal{S}$*) implies that* $0_\mathcal{S}$ *is the bottom (resp. top) element of* $\mathcal{S}$*. However, we will often consider examples of ordered semirings where the two neutral elements are incomparable. In [BGMZ14] the authors impose* $0_\mathcal{S}$ *to be the bottom element, but this condition is not necessary.*

Example A.4.1.3.
- *The* trivial semiring *is the one point semiring* $(\{*\}, \cdot, *, +, *)$ *with* $*\cdot* = *+* = *$*. It is generally not very interesting but gives easy examples.*

- *The two elements lax-semirings are:*

- *the* Boolean semiring $\mathbb{B} = (\{tt, ff\}, \wedge, tt, \vee, ff, \{ff \leq tt\})$,
- *the* reversed Boolean semiring $\mathbb{B}^{op} = (\{tt, ff\}, \wedge, tt, \vee, ff, \{tt \leq ff\})$,
- *the* discrete Boolean semiring $\mathbb{B}_d = (\{tt, ff\}, \wedge, tt, \vee, ff, \{\})$,
- *the* two member field $\mathbb{Z}_2 = (\{0,1\}, \cdot_{\mathbb{Z}_2}, 1, +_{\mathbb{Z}_2}, ff, \{\})$ *with* $1 +_{\mathbb{Z}_2} 1 = 2$.

- *The* natural numbers semiring $\mathbb{N} := (\mathbb{N}, \cdot_{\mathbb{N}}, 1, +_{\mathbb{N}}, 0, \leq_{\mathbb{N}})$ *forms a semiring. When endowed with the discrete order (the identity), we get the* discrete natural numbers semiring $\mathbb{N}_d$.

- *Any semiring $\mathcal{S}$ semiring can be turned into a lax-semirings*

$$\mathcal{S}_\perp := (\mathcal{S} \uplus \{\perp\}, +_{\mathcal{S}_\perp}, 0_{\mathcal{S}}, \cdot_{\mathcal{S}\cup\perp}, 1_{\mathcal{S}}, \leq_{\mathcal{S}} \cup \{\perp \leq \alpha \mid \alpha \in \mathcal{S}_\perp\})$$

by adding a bottom element $\perp$ that is absorbing for the sum, absorbing for the right multiplication and absorbing for the left multiplication except that $0_{\mathcal{S}} *_{\mathcal{S}_\perp} \perp = 0_{\mathcal{S}}$. Such an extention is called the bottomed version *of $\mathcal{S}$ and denoted $\mathcal{S}_\perp$ except for:*

  - *the bottomed discrete Boolean lax-semiring that is called* flat Boolean lax-semiring *and denoted $\mathbb{B}_f$,*
  - *The bottomed discete natural numbers* lax-semiring *that is called* flat natural numbers lax-semiring *and denoted* $\mathbb{N}_f$.

- *We call* diamond *the lax-semiring $\Diamond$ with four elements $|\Diamond| := \{0, 1, \perp, \top\}$ defined by (for any $I \in \Diamond$):*

$$1 + 1 = 1 + \top = 1 \qquad I + \perp = \perp \qquad I \cdot \perp = \perp \qquad \top \cdot \top = \top$$

and by $\perp \cdot I = \perp$ for $I \neq 0$.

- *The* completed natural numbers semiring $\bar{\mathbb{N}} := (\mathbb{N} \cup \{\omega\}, \cdot_{\bar{\mathbb{N}}}, 1, +_{\bar{\mathbb{N}}}, 0)$*, where $\omega + n = \omega \cdot (n+1) = \omega$, forms a semiring. Endowed with the usual order over $\mathbb{N}$, it is an ordered semiring.*

- *The* tropical semiring $\mathtt{Trop} = (\mathbb{N} \cup \{-\infty\}, +_{\mathbb{N}}, 0, \min, -\infty, \leq)$ *forms an ordered semiring. There, the role of the multiplication is played by the usual addition and the role of the addition is played by the* min *operator; the order is the usual order over $\mathbb{N}$ with $-\infty$ that is a bottom.*

- *The* multisets of natural numbers $(\mathbb{N}_f\langle\mathbb{N}_*\rangle, \times_{\mathbb{N}_f\langle\mathbb{N}_*\rangle}, [1], +_{\mathbb{N}_f\langle\mathbb{N}_*\rangle}, [])$ *form a semiring (Prop. A.4.2.2). The sum is the sum of multisets while the product is the Dirichlet convolution of the internal product:*

$$(I +_{\mathbb{N}_f\langle\mathbb{N}_*\rangle} J)(n) := I(n) +_{\mathbb{N}} J(n),$$
$$(I \cdot_{\mathbb{N}_f\langle\mathbb{N}_*\rangle} J)(n) := \sum_{\substack{m_1, m_2 \in \mathbb{N} \\ m_1 \cdot m_2 = n}} I(m_1) \cdot I(m_2).$$

- *The* powersets of natural numbers $(\mathcal{P}(\mathbb{N}), \otimes, \{0\}, \oplus, \{1\})$ *form a lax-semiring (Prop. A.4.2.6). The sum is the Dirichlet convolution and product is the dependent sum:*

$$I \oplus J := \{m+n \mid m \in I, n \in J\},$$
$$I \otimes J := \{\sum_{i=1}^{m} n_i \mid m \in J, \forall i, n_i \in I\}.$$

The resulting lax-semiring is not a semiring. Indeed, it does not have the right-distribution:

$$(\{1\} \oplus \{1\}) \odot \{1, 2\} = \{2, 4\}, \qquad (\{1, 2\} \odot \{1\}) \oplus (\{1, 2\} \odot \{1\}) = \{2, 3, 4\}.$$

The following is a generalization of the multiset to any free $\mathcal{S}$ semi-module over a set D:

Definition A.4.1.4. *Let $\mathcal{S}$ be a semiring and D a set.*
We denote $\mathcal{S}_f\langle D\rangle$ the set which elements are the functions $f : D \mapsto \mathcal{S}$ with finite support (where $\operatorname{supp}(f) = \{d \in D \mid f(d) \neq 0_{\mathcal{S}}\}$*)*
The free $\mathcal{S}$ semi-module *over D is the set $\mathcal{S}_f\langle D\rangle$ endowed with the sum over functions and the product $(d\cdot f)(d) := d\cdot f(d)$.*
We denote $[\,]$ the constant function with value $0_{\mathcal{S}}$, $[d]$ is the function with value $1_{\mathcal{S}}$ on d and $0_{\mathcal{S}}$ everywhere else. More generally, we denote inductively $[p_0\cdot d_0, ...p_n\cdot d_n] := p_0\cdot[d_0] + [p_1\cdot d_1...p_n\cdot d_n]$.

Remark that any element $f \in \mathcal{S}_f\langle D\rangle$ has a canonical notation $[d\cdot f(d) \mid d \in \operatorname{supp}(()f)]$.

A.4.2. A few propositions

Definition A.4.2.1. *Let $\mathbb{M}$ be a monoid and $\mathcal{S}$ be a semiring.*
*The $\mathcal{S}$-*linear semiring over *$\mathbb{M}$ is the semi-module $\mathcal{S}_f\langle\mathbb{M}\rangle$ endowed with the operators:*

$$0_{\mathcal{S}_f\langle\mathbb{M}\rangle} := [\,], \qquad (I +_{\mathcal{S}_f\langle\mathbb{M}\rangle} J)(g) := I(g) +_{\mathcal{S}} J(g),$$
$$1_{\mathcal{S}_f\langle\mathbb{M}\rangle} := [1_{\mathbb{M}}], \qquad (I \cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle} J)(g) := \sum_{\substack{g',g''\in\mathbb{M}\\ g'\cdot_{\mathbb{M}} g''=g}} I(g')\cdot_{\mathcal{S}} J(g''),$$

Proposition A.4.2.2. *Given a monoid $\mathbb{M}$ and a semiring $\mathcal{S}$, the $\mathcal{S}$-linear semiring over $\mathbb{M}$ is a semiring.*

Proof.
- $+_{\mathcal{S}_f\langle\mathbb{M}\rangle}$ is associative:

$$\begin{aligned}(I +_{\mathcal{S}_f\langle\mathbb{M}\rangle} (J +_{\mathcal{S}_f\langle\mathbb{M}\rangle} \kappa))(g) &= I(g) +_{\mathcal{S}} (J(g) +_{\mathcal{S}} \kappa(g)) && \text{(by def.)}\\ &= (I(g) +_{\mathcal{S}} J(g)) +_{\mathcal{S}} \kappa(g) && \text{(ass. of } +_{\mathcal{S}})\\ &= ((I +_{\mathcal{S}_f\langle\mathbb{M}\rangle} J) +_{\mathcal{S}_f\langle\mathbb{M}\rangle} \kappa)(g) && \text{(by def.)}\end{aligned}$$

- $\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}$ is associative:

$$
\begin{aligned}
(I\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}(J\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}\kappa))(g) &= \sum_{\substack{g_1,g_2\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}g_2=g}} I(g_1)\cdot_{\mathcal{S}}(\sum_{\substack{g_3,g_4\in\mathbb{M}\\ g_3\cdot_{\mathbb{M}}g_4=g_2}} J(g_3)\cdot_{\mathcal{S}}\kappa(g_4)) && \text{(by def.)}\\
&= \sum_{\substack{g_1,g_2\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}g_2=g}} \sum_{\substack{g_3,g_4\in\mathbb{M}\\ g_3\cdot_{\mathbb{M}}g_4=g_2}} I(g_1)\cdot_{\mathcal{S}}(J(g_3)\cdot_{\mathcal{S}}\kappa(g_4)) && \text{(dist in } \mathcal{S})\\
&= \sum_{\substack{g_1,g_3,g_4\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}(g_3\cdot_{\mathbb{M}}g_4)=g}} I(g_1)\cdot_{\mathcal{S}}(J(g_3)\cdot_{\mathcal{S}}\kappa(g_4)) && \text{(ass. of } +_{\mathcal{S}})\\
&= \sum_{\substack{g_1,g_3,g_4\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}(g_3\cdot_{\mathbb{M}}g_4)=g}} (I(g_1)\cdot_{\mathcal{S}}J(g_3))\cdot_{\mathcal{S}}\kappa(g_4) && \text{(ass. of } \cdot_{\mathcal{S}})\\
&= \sum_{\substack{g_1,g_3,g_4\in\mathbb{M}\\ (g_1\cdot_{\mathbb{M}}g_3)\cdot_{\mathbb{M}}g_4=g}} (I(g_1)\cdot_{\mathcal{S}}J(g_3))\cdot_{\mathcal{S}}\kappa(g_4) && \text{(ass. of } \cdot_{\mathbb{M}})\\
&= \sum_{\substack{g_5,g_4\in\mathbb{M}\\ g_5\cdot_{\mathbb{M}}g_4=g}} \sum_{\substack{g_1,g_3\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}g_3=g_5}} (I(g_1)\cdot_{\mathcal{S}}J(g_3))\cdot_{\mathcal{S}}\kappa(g_4) && \text{(ass. of } +_{\mathcal{S}})\\
&= \sum_{\substack{g_5,g_4\in\mathbb{M}\\ g_5\cdot_{\mathbb{M}}g_4=g}} (\sum_{\substack{g_1,g_3\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}g_3=g_5}} I(g_1)\cdot_{\mathcal{S}}J(g_3))\cdot_{\mathcal{S}}\kappa(g_4) && \text{(dist in } \mathcal{S})\\
&= ((I\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}J)\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}\kappa)(g) && \text{(by def.)}
\end{aligned}
$$

- $0_{\mathcal{S}_f\langle\mathbb{M}\rangle}$ is the unity of $+_{\mathcal{S}_f\langle\mathbb{M}\rangle}$:

$$
\begin{aligned}
(0_{\mathcal{S}_f\langle\mathbb{M}\rangle} +_{\mathcal{S}_f\langle\mathbb{M}\rangle} I)(g) &= 0_{\mathcal{S}_f\langle\mathbb{M}\rangle}(g) +_{\mathcal{S}} I(g) && \text{(by def.)}\\
&= 0_{\mathcal{S}} +_{\mathcal{S}} I(g) && \text{(by def.)}\\
&= I(g) && \text{(unity in } \mathcal{S})
\end{aligned}
$$

- $1_{\mathcal{S}_f\langle\mathbb{M}\rangle}$ is the left unity of $\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}$:

$$
\begin{aligned}
(1_{\mathcal{S}_f\langle\mathbb{M}\rangle}\cdot_{\mathcal{S}_f\langle\mathbb{M}\rangle}I)(g) &= \sum_{\substack{g_1,g_2\in\mathbb{M}\\ g_1\cdot_{\mathbb{M}}g_2=g}} (1_{\mathcal{S}_f\langle\mathbb{M}\rangle}(g_1)\cdot_{\mathcal{S}}I(g_2)) && \text{(by def.)}\\
&= \sum_{\substack{g_2\in\mathbb{M}\\ 1_{\mathbb{M}}\cdot_{\mathbb{M}}g_2=g}} (1_{\mathcal{S}}\cdot_{\mathcal{S}}I(g_2)) && \text{(by def. et abs. } 0_{\mathcal{S}})\\
&= \sum_{\substack{g_2\in\mathbb{M}\\ 1_{\mathbb{M}}\cdot_{\mathbb{M}}g_2=g}} I(g_2) && \text{(unity in } \mathcal{S})\\
&= I(g) && \text{(left unity in } \mathbb{M})
\end{aligned}
$$

- $1_{\mathcal{S}_f\langle \mathbb{M}\rangle}$ is the right unity of $\cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle}$:

$$
\begin{aligned}
(I \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} 1_{\mathcal{S}_f\langle \mathbb{M}\rangle})(g) &= \sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} (I_{\mathcal{S}_f\langle \mathbb{M}\rangle}(g_1) \cdot_{\mathcal{S}} 1(g_2)) && \text{(by def.)} \\
&= \sum_{\substack{g_1 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} 1_{\mathbb{M}} = g}} (I(g_1) \cdot_{\mathcal{S}} 1_{\mathcal{S}}) && \text{(by def. et abs. } 0_{\mathcal{S}}) \\
&= \sum_{\substack{g_1 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} 1_{\mathbb{M}} = g}} I(g_1) && \text{(unity in } \mathcal{S}) \\
&= I(g) && \text{(right unity in } \mathbb{M})
\end{aligned}
$$

- $\cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle}$ left distribute over $+_{\mathcal{S}_f\langle \mathbb{M}\rangle}$:

$$
\begin{aligned}
(I \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} (J +_{\mathcal{S}_f\langle \mathbb{M}\rangle} \kappa))(g) &= \sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} I(g_1) \cdot_{\mathcal{S}} (J(g_2) +_{\mathcal{S}} \kappa(g_2)) && \text{(by def.)} \\
&= \sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} (I(g_1) \cdot_{\mathcal{S}} J(g_2)) +_{\mathcal{S}} (I(g_1) \cdot_{\mathcal{S}} \kappa(g_2)) && \text{(dist. in } \mathcal{S}) \\
&= (\sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} I(g_1) \cdot_{\mathcal{S}} J(g_2)) +_{\mathcal{S}} (\sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} I(g_1) \cdot_{\mathcal{S}} \kappa(g_2)) && \text{(comm., ass. of } +_{\mathcal{S}}) \\
&= ((I \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} J) +_{\mathcal{S}_f\langle \mathbb{M}\rangle} (I \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} \kappa))(g) && \text{(by def.)}
\end{aligned}
$$

- $\cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle}$ right distribute over $+_{\mathcal{S}_f\langle \mathbb{M}\rangle}$:

$$
\begin{aligned}
((J +_{\mathcal{S}_f\langle \mathbb{M}\rangle} \kappa) \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} I)(g) &= \sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} (J(g_1) +_{\mathcal{S}} \kappa(g_1)) \cdot_{\mathcal{S}} I(g_2) && \text{(by def.)} \\
&= \sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} (J(g_1) \cdot_{\mathcal{S}} I(g_2)) +_{\mathcal{S}} (\kappa(g_1) \cdot_{\mathcal{S}} I(g_2)) && \text{(dist. in } \mathcal{S}) \\
&= (\sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} J(g_1) \cdot_{\mathcal{S}} I(g_2)) +_{\mathcal{S}} (\sum_{\substack{g_1, g_2 \in \mathbb{M} \\ g_1 \cdot_{\mathbb{M}} g_2 = g}} \kappa(g_1) \cdot_{\mathcal{S}} I(g_2)) && \text{(comm., ass. of } +_{\mathcal{S}}) \\
&= ((J \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} I) +_{\mathcal{S}_f\langle \mathbb{M}\rangle} (\kappa \cdot_{\mathcal{S}_f\langle \mathbb{M}\rangle} I))(g) && \text{(by def.)}
\end{aligned}
$$

- $+_{\mathcal{S}_f\langle \mathbb{M}\rangle}$ is commutative:

$$
\begin{aligned}
(I +_{\mathcal{S}_f\langle \mathbb{M}\rangle} J)(g) &= I(g) +_{\mathcal{S}} J(g) && \text{(by def.)} \\
&= J(g) +_{\mathcal{S}} I(g) && \text{(comm. of } +_{\mathcal{S}}) \\
&= (J +_{\mathcal{S}_f\langle \mathbb{M}\rangle} I)(g) && \text{(by def.)}
\end{aligned}
$$

□

Example A.4.2.3. *The* polynomial semiring $(\mathbb{N}[X_i]_{i\in\mathbb{N}}, \times, 1, +, 0)$ *can be recovered as* $\mathbb{N}_f\langle \mathbb{N}_+\rangle$ *associating* $f \in \mathbb{N}_f\langle \mathbb{N}_+\rangle$ *with* $\Sigma_{n\in \mathrm{supp}(()f)} f(n)X^n$.

Remark A.4.2.4. *Example A.4.2.3 is quite universal since a the linear S-semiring over $\mathbb{M}$ can be seen as a generalization of polynomials $I := \sum_{g\in\mathbb{M}} I(g)\cdot X^g$ where X^g is a formal exponent. Then, the sums and products are the same as the sums and products of polynomials.*

Definition A.4.2.5. *The powersets lax-semiring of natural numbers is the structure $(\mathcal{P}(\mathbb{N}), \otimes, \{0\}, \oplus, \{1\})$ where the sum is the Dirichlet convolution and product is the dependent sum:*

$$I \oplus J := \{m{+}n \mid m \in I, n \in J\},$$
$$I \otimes J := \{\sum_{i=1}^{m} n_i \mid m \in J, \forall i, n_i \in I\}.$$

The resulting lax-semiring is not a semiring, indeed, it does not have the right-distribution:

$$(\{1\} \oplus \{1\}) \odot \{1,2\} = \{2,4\}, \qquad (\{1,2\} \odot \{1\}) \oplus (\{1,2\} \odot \{1\}) = \{2,3,4\}.$$

Proposition A.4.2.6. *The powerset lax-semiring of natural numbers is a lax-semiring.*

Proof.

- $\oplus$ is associative and commutative: immediate by associativity and commutativity of $\mathbb{N}$.
- $\{0_{\mathbb{N}}\}$ is neutral for $\oplus$: immediate by neutrality of $0_{\mathbb{N}}$ for $+_{\mathbb{N}}$.
- $\{1_{\mathbb{N}}\}$ is left-neutral for $\odot$:

$$\begin{aligned}\{1_{\mathbb{N}}\} \odot J &= \Big\{\sum_{i=1}^{k} n_i \mid k \in J, \forall i \leq k, n_i \in \{1_{\mathbb{N}}\}\Big\}\\ &= \Big\{\sum_{i=1}^{k} 1 \mid k \in J\Big\}\\ &= J.\end{aligned}$$

- $\{1_{\mathbb{N}}\}$ is right-neutral for $\odot$:

$$\begin{aligned}I \odot \{1_{\mathbb{N}}\} &= \Big\{\sum_{i=1}^{k} n_i \mid k \in \{1_{\mathbb{N}}\}, \forall i \leq k, n_i \in I\Big\}\\ &= \Big\{n_1 \mid n_1 \in I\Big\}\\ &= I.\end{aligned}$$

- $\odot$ left-distributes over $\oplus$:

$$\begin{aligned}
I \odot (J \oplus K) &= \Big\{\sum_{i=1}^{k} n_i \mid k \in (J \oplus K), \forall i \leq k, n_i \in I\Big\} \\
&= \Big\{\sum_{i=1}^{k+k'} n_i \mid k \in J,\ k' \in K,\ \forall i \leq k, n_i \in I\Big\} \\
&= \Big\{\sum_{i=1}^{k} n_i + \sum_{j=1}^{k'} m_j \mid k \in J,\ k' \in K,\ \forall i \leq k, n_i \in I, \forall j \leq k', m_i \in I\Big\} \\
&= (I \odot J) \oplus (I \odot K)
\end{aligned}$$

- $\odot$ is associative:

$$\begin{aligned}
(I \odot J) \odot K &= \Big\{\sum_{i=1}^{k} n_i \mid k \in K, \forall i \leq k, n_i \in (I \odot J)\Big\} \\
&= \Big\{\sum_{i=1}^{k} \sum_{j=1}^{k_i} n_{i,j} \mid k \in K,\ \forall i \leq k, k_i J,\ \forall j \leq k, n_{i,j} \in I\Big\} \\
&= \Big\{\sum_{j=1}^{k'} n_j \mid k' = \sum_{i=1}^{k} k', k \in K,\ \forall i \leq k, k_i J,\ \forall j \leq k', n_j \in I\Big\} \\
&= \Big\{\sum_{j=1}^{k'} n_j \mid k' \in J \odot K,\ \forall j \leq k', n_j \in I\Big\} \\
&= I \odot (J \odot K)
\end{aligned}$$

- $\{0_{\mathbb{N}}\}$ is right absorbing for $\oplus$:

$$\begin{aligned}
I \odot \{0_{\mathbb{N}}\} &:= \Big\{\sum_{i=1}^{k} n_i \mid k \in \{0_{\mathbb{N}}\},\ \forall i \leq k, n_i \in I\Big\} \\
&:= \{0_{\mathbb{N}}\}
\end{aligned}$$

- $\odot$ right-distribute over $\oplus$ in the lax way:

$$\begin{aligned}
(I \oplus J) \odot K &:= \Big\{\sum_{i=1}^{k} n_i \mid k \in K,\ \forall i \leq k, n_i \in (I \oplus J)\Big\} \\
&:= \Big\{\sum_{i=1}^{k} (m_i + n_i) \mid k \in K,\ \forall i \leq k, m_i \in I, n_i \in J\Big\} \\
&:= \Big\{\sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i \mid k \in K,\ \forall i \leq k, m_i \in I, n_i \in J\Big\} \\
&\subseteq \Big\{\sum_{i=1}^{k} m_i + \sum_{j=1}^{k'} n_j) \mid k, k' K,\ \forall i \leq k, m_i \in I, \forall j \leq k' n_i \in J\Big\} \\
&= (I \odot K) \oplus (J \odot K)\}
\end{aligned}$$

- $\{0_{\mathbb{N}}\}$ is left absorbing for $\oplus$ in the lax way:

$$\begin{aligned}
\{0_{\mathbb{N}}\}\odot &:= \Big\{\sum_{i=1}^{k} n_i \mid k \in K,\ \forall i \leq k, n_i \in \{0_{\mathbb{N}}\}\Big\} \\
&:= \Big\{\sum_{i=1}^{k} 0_{\mathbb{N}} \mid k \in K\Big\} \\
&\subseteq \{0_{\mathbb{N}}\}
\end{aligned}$$

□