Introduction to the theorem of Kerkhoff, Masur and Smillie

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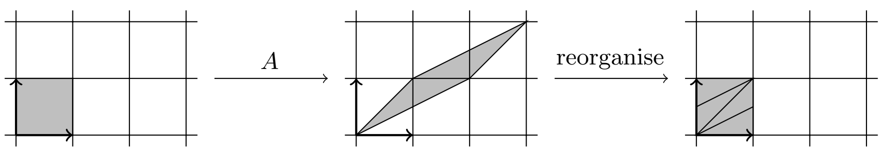
The theorem of Kerkhoff, Masur and Smillie [5] asserts that for any connected translation surface S (in particular for any rational polygonal billiard), and for almost every $\theta \in \mathbb{S}^1$, the flow in the direction θ is uniquely ergodic.

1. Rough idea

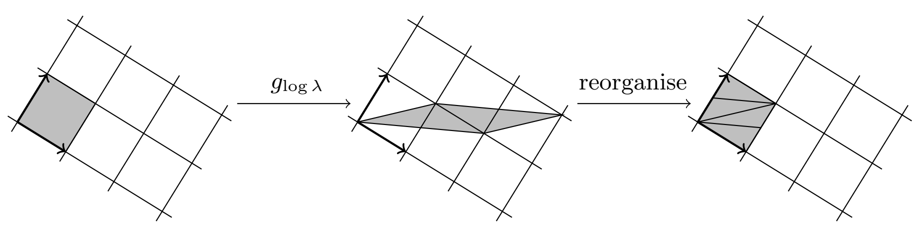
To define the flow on a polygonal billiard, we need the Euclidean notion of angle, whereas for translation surfaces we only need the affine notion of straight line.
In particular, we can apply the matrix $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on a translation surface S without changing the dynamical properties of the flow defined on it.
By contracting the vertical direction, g_t accelerates the time of the vertical flow, so that the asymptotic behaviour of the trajectory $\{g_t S\}$ on the space of translation surfaces will provide some informations about the dynamics of the vertical flow defined on S.

2. Remark

Applying the flow g_t to the standard flat torus $\mathbb{R}^2/\mathbb{Z}^2$ will lead to a degenerate torus (its vertical meridians are shrunk), but it is not always the case, since it is sometimes possible to reorganise the translation surface while applying g_t. For example, let us consider the action of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the standard flat torus, which is well defined since $A \in SL(2, \mathbb{Z})$.



The matrix A is diagonalisable, with two orthogonal eigenlines corresponding to the eigenvalues $\lambda = (3 + \sqrt{5})/2$ and λ^{-1}. If we rotate the torus so that the eigenlines become vertical and horizontal, we obtain a new torus S and the action of A on $\mathbb{R}^2/\mathbb{Z}^2$ corresponds to the action of $g_{\log \lambda}$ on S, hence $g_{\log \lambda} S = S$ and the trajectory $\{g_t S\}$ is periodic.



3. Strategy of the proof

The proof of the theorem is split into three parts:

(1) In order to deal with the asymptotic behaviour of the trajectory $\{g_t S\}$, we will define a convenient topology on the set of translation surfaces and provide a criterion for compactness.

(2) Masur's criterion [6]: if $\{g_t S\}$ does not converge to infinity (that is, if there exists a subsequence $t_n \to \infty$ such that $g_{t_n} S$ stays in a given compact), then the vertical flow is uniquely ergodic.

(3) For any translation surface S and for almost every $\theta \in \mathbb{S}^1$, the flow in the direction θ does not converge to infinity (meaning that the previously discussed degeneration is a rare phenomenon).

We will focus on the first two parts.

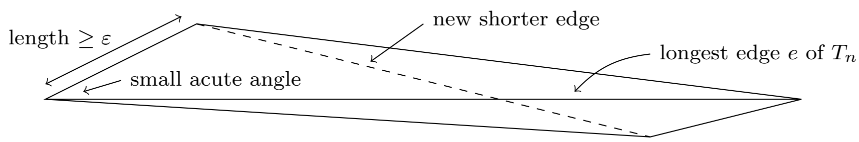
4. Topology on the set of translation surfaces

Any translation surface can be triangulated so that the edges are saddle connections, and any collection of saddle connections having disjoint interiors can be extended to such a triangulation. Thanks to the Euler characteristic, the number of triangles in a triangulation depends only on the number of singularities and on the genus of the surface.

Let S be a translation surface and T be a triangulation of S. We define a small neighbourhood of S by letting the edges of T (viewed as a vectors of $\mathbb{R}^2$) move slightly around their initial position. In particular, two nearby translation surfaces admit triangulations that have the same combinatorics of glueing (and therefore have the same genus). Each saddle connection in S can be written as a sum of edges of T, so that the choice of the triangulation is not relevant.

Let $systole(S)$ denote the length of a shortest saddle connection in S. Let us prove that, given $g \geq 1$ and $\varepsilon > 0$, the set of translation surfaces of genus g (and of constant area 1) satisfying $systole(S) \geq \varepsilon$ is compact.

For this, let S_n be a sequence of translation surfaces of genus g whose systole is larger than ε. To get compactness, we have to ensure that it is possible to find a triangulation of each S_n whose edges have uniformly bounded length. We can achieve this by starting from any triangulation T_n of S_n, and assume that the longest edge e of T_n is very long. This edge bounds two triangles whose edges have length at least ε. Since the area of S_n is 1 and e is the longest edge, the angles that the triangles make with e are very small.



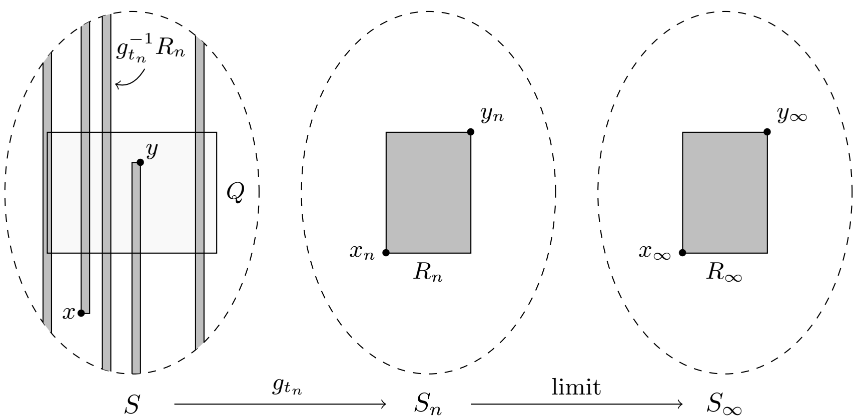
So we can reorganise the triangulation of S to get a better triangulation, by replacing (with a flip) e by a shorter saddle connection whose length if smaller by

a constant of at least $\varepsilon/2$. So after finitely many such reorganisations, we get a triangulation of S_n whose edges have uniformly bounded length (the bound is of the order of $1/\varepsilon$).
Since the number of triangles is bounded, there are only finitely many combinatorics of glueings, so that we can assume that the type of the triangulation is fixed along a subsequence. Up to another subsequence, each edge of each triangle converges, so that we can construct a limit translation surface S_∞.

We also would like to say that two points in two close surfaces are close to each other if they are close in a common triangulation. Some tricky stuff can happen near a surface which admits two symmetric triangulations (different points of the surface will be identified). This problem can be solved by considering marked translation surfaces to break the symmetry, we will not take care about this later.

5. Proof of Masur's criterion

Let S be a translation surface whose trajectory $\{g_t S\}$ does not converge to infinity. Let $\{t_n\}$ be a subsequence such that $S_n := g_{t_n} S$ converges to a translation surface S_∞.
Assume by contradiction that the vertical flow in S is not uniquely ergodic: there exist two distinct ergodic probability measures $\mu \neq \nu$ that are invariant under the vertical flow. Let Q be a horizontal rectangle in S such that $\mu(Q) \neq \nu(Q)$.
Let x be a generic point for μ. We can follow the trajectory $\{x_n\}$ of x on $\{S_n\}$ under g_{t_n}. Passing to a subsequence, we can assume that this trajectory converges to some $x_\infty \in S_\infty$. Do the same for y with ν.
Let us first assume that there exists an open set that does not meet any singularity which contains a rectangle R_∞ in S_∞ such that x_∞ (resp. y_∞) is the lower-left (resp. upper-right) corner of the rectangle. So, for n big enough, we can still embed a rectangle R_n in S_n, whose dimensions (w_n, h_n) are very close to the ones of R_∞, and such that x_n (resp. y_n) is the lower-left (resp. upper-right) corner of it. Let us apply $g_{t_n}^{-1}$ to R_n: we get a very long rectangle in S (of height $e^{t_n} h_n$).

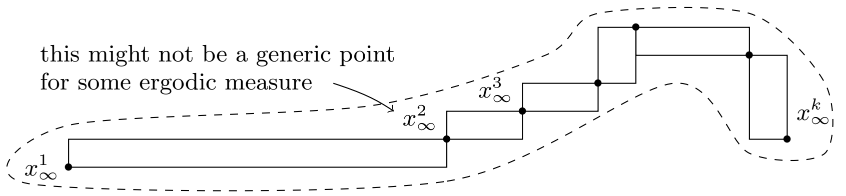


Its left side corresponds to the orbit of x under the vertical flow from time 0 to time $e^{t_n}h_n$, and its right side corresponds to the orbit of y under the vertical flow from time $-e^{t_n}h_n$ to time 0. If ϕ^t denotes the vertical flow on S, Birkhoff's ergodic theorem applied to the characteristic function of Q tells us that

$$\frac{1}{T}\left(\int_{t=0}^{T}\chi_Q(\phi^t(x))dt - \int_{t=-T}^{0}\chi_Q(\phi^t(y))dt\right) \xrightarrow[T\to\infty]{} \mu(Q)-\nu(Q)\neq 0$$

For $T = e^{t_n}h_n$, the parenthesis on the left side is the difference between the length of the intersection of Q with the right side of the rectangle $g_{t_n}^{-1}(R_n)$ and the length of the intersection of Q with the left side of the rectangle $g_{t_n}^{-1}(R_n)$, which is bounded by two times the height of Q (a defect happens when $g_{t_n}^{-1}(R_n)$ is astride a vertical side of Q, which can happens at most twice). So, we get a contradiction when n goes to infinity.

We assumed the possibility to embed a nice rectangle R_∞ in S_∞ with x_∞ and y_∞ as opposite corners. If this is not the case, since x and y are not on the vertical of some singularity, we can ensure (up to shifting some elements of the subsequence $\{t_n\}$) that $\{x_n\}$ and $\{y_n\}$ stay uniformly far from the singularities, so that x_∞ and y_∞ are not singularities of S_∞. Then, since S_∞ is connected, there exists a path in S_∞ between x_∞ and y_∞, which can be surrounded by an open set not meeting any singularity (by compactness). So, there exists a finite sequence $x_\infty = x_\infty^1, x_\infty^2, \dots, x_\infty^k = y_\infty$ such that each rectangle with opposite corners x_∞^i and x_∞^{i+1} lies in the open set.



If, up to taking more subsequences, each x_∞^i is a limit point of the trajectory of some point x^i in S under g_{t_n} that is generic for some invariant ergodic measure μ^i (for ϕ_t), then we can apply the previous reasoning on each rectangle and prove that $\mu = \mu^1 = \mu^2 = \cdots = \mu^k = \nu$, which concludes the proof.
To ensure this, we can notice that each x_∞^i $(1 \leq i \leq k-1)$ can be moved a bit, so, given a small open neighbourhood U_∞^i of x_∞^i in the open set, we have to find a good substitute for x_∞^i in U_∞^i. The open set U_∞^i can be backported to an open set U_n^i in S_n, for n big enough. This set and therefore its preimage $g_{t_n}^{-1}U_n^i$ have uniformly positive Lebesgue measure (in n). Since the Lebesgue measure is an average of ergodic measures, there exists an ergodic measure μ^i (for ϕ_t in S) that gives positive measure to the set of points that belong to infinitely many $g_{t_n}^{-1}U_n^i$, in particular, there exists a generic point x^i for μ^i in S such that the trajectory $\{g_{t_n}x^i\}$ has a limit point x_∞^i in U_∞^i.

6. Related results

6.1. **Approximating irrational polygonal billiards by rational ones.** Boshernitzan and Katok [5] proved that the set of n-gons on which the billiard flow is ergodic is a dense G_δ subset of the set of n-gons. The result also holds if we restrict ourselves to a subspace X of n-gons such that for any N, the set of rational tables P with $|G(P)| \geq N$ is dense in X (*e.g.* the set of right-angled triangles). Vorobets [9] gave a quantitative version of this theorem: if P is a polygonal billiard table whose angles $\theta = (\theta_1, \dots, \theta_n)$ are such that there exist infinitely many rationals of the form $P/q = (p_1/q, \dots, p_n/q)$, with $\gcd(p_1, \dots, p_n, q) = 1$ and $||\theta - P/q||_\infty \leq 1/2^{2^{2^q}}$, then the billiard flow is ergodic on P.

6.2. **Hausdorff dimension of the set of non-uniquely ergodic directions.** Masur [6] proved that for any translation surface S, the Hausdorff dimension of the set of non-uniquely ergodic directions is less than or equal to $1/2$. Cheung [2] proved that this bound is sharp: there exists translation surfaces whose set of non-uniquely ergodic directions has Hausdorff dimension equal to $1/2$. Masur and Smillie [7] proved that for any connected component $\mathcal{C}$ of any stratum (in genus at least 2), there exists $\delta > 0$ such that for any generic translation surface S in the component $\mathcal{C}$, the Hausdorff dimension of the set of non-uniquely ergodic directions is δ.

6.3. **Slow divergence still implies unique ergodicity.** Cheung and Eskin [3] proved that there exists $\varepsilon > 0$, depending only on the stratum of the translation surface S, such that the condition $\liminf_{t\to\infty} t^\varepsilon systole(g_t S) > 0$ implies that the vertical flow is uniquely ergodic.

References

[1] Pierre Arnoux. Ergodicité générique des billards polygonaux (d'après Kerckhoff, Masur, Smillie). *Astérisque*, (161-162):Exp. No. 696, 5, 203–221 (1989), 1988. Séminaire Bourbaki, Vol. 1987/88.

[2] Yitwah Cheung. Hausdorff dimension of the set of nonergodic directions. *Ann. of Math. (2)*, 158(2):661–678, 2003. With an appendix by M. Boshernitzan.

[3] Yitwah Cheung and Alex Eskin. Slow Divergence and Unique Ergodicity. Preprint available at `http://arxiv.org/abs/0711.0240`, 2007.

[4] Sébastien Gouëzel and Erwan Lanneau. Un théorème de Kerckhoff, Masur et Smillie : Unique ergodicité sur les surfaces plates. *Séminaires et Congrès (SMF)*, 19, 2008.

[5] Steven Kerckhoff, Howard Masur, and John Smillie. Ergodicity of billiard flows and quadratic differentials. *Ann. of Math. (2)*, 124(2):293–311, 1986.

[6] Howard Masur. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. *Duke Math. J.*, 66(3):387–442, 1992.

[7] Howard Masur and John Smillie. Hausdorff dimension of sets of nonergodic measured foliations. *Ann. of Math. (2)*, 134(3):455–543, 1991.

[8] Howard Masur and Serge Tabachnikov. Rational billiards and flat structures. In *Handbook of dynamical systems, Vol. 1A*, pages 1015–1089. North-Holland, Amsterdam, 2002.

[9] Ya. B. Vorobets. Ergodicity of billiards in polygons: explicit examples. *Uspekhi Mat. Nauk*, 51(4(310)):151–152, 1996.