Topological Martin's delirium

Thierry Monteil

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Let A be a nonempty alphabet. We define the following distance on the set of subshifts of $A^{\mathbb{N}}$: $d(X,Y) = 2^{-min\{n \in \mathbb{N} | L_n(X) \neq L_n(Y)\}}$, where $L_n(X)$ denotes the set of factors of length n of some element of the subshift X.

When X is a subshift and σ is a (non-erasing) substitution (*i.e.* a morphism of the free monoid A^* which sends any letter to a nonempty word), we can define $\sigma(X)$ as the smallest subshift that contains $\{\sigma(x) \mid x \in X\}$ (σ can be applied to an infinite word in a natural way). If $L(X)$ denotes the language asociated to X, $L(\sigma(X))$ is the set of factors of the set $\{\sigma(u) \mid u \in L(X)\}$.

Martin is interested to the sets F of subshifts that are :

1. non-empty.
2. stable under the action of any substitution σ defined on A.
3. closed for the topology induced by the previous distance.
4. minimal for those properties.

We will make a lots of easy steps.

Proposition 1 *Any E satisfying 1,2 contains the trivial subshift $\{a^\omega\}$ (where a is any letter in A).*

Proof E is stable under the action of the substitution which sends any letter to the letter a. □

Proposition 2 *Such an F exists and is unique.*

Proof The set of all subshifts satisfies 1,2,3 and the intersection of all E satisfying properties 1,2,3 satisfies 1,2,3,4 (this intersection is non-empty because of the previous step). □

Proposition 3 *F contains all periodic subshifts.*

Proof If X_{u^ω} is the periodic subshift generated by the word u^ω, $X_{u^\omega} = \sigma(\{a^\omega\})$, where σ sends any letter of A to the word u. □

Extending the notion defined for words, we will say that a subshift X is *recurrent* if all of its Rauzy graphs $G_n(X)$ are (strongly) connected.

Proposition 4 *F contains any recurrent subshift.*

Proof Let X be a recurrent sushift, since F is closed it suffice to find a periodic subshift arbitrarily close to X. Let n be an integer, since $G_n(X)$ is strongly connected, there exists a closed path in $G_n(X)$ which meets any vertex of it. Such a path corresponds to a finite word u, hence X and the subshift generated by u^ω are at distance at most 2^{-n}. □

Proposition 5 *The set of recurrent subshifts is closed (in the set of subshifts on $A^{\mathbb{N}}$).*

Proof Let X be a non-recurrent subshift: there exists an integer n such that $G_n(X)$ is not strongly connected. Hence, any recurrent subshift is at distance at least $2^{-(n+1)}$ of X. Hence, the set of non-recurrent subshifts is open. □

Proposition 6 *The set of recurrent subshifts is stable under the action of any substitution.*

Proof Let X be a recurrent subshift, let σ be a non-erasing substitution and let U and V be two elements of $L_n(\sigma(X))$. There exists two elements u and v of $L_n(X)$ (the same n) such that U is a factor of $\sigma(u)$ and V is a factor of $\sigma(v)$. Since $G_n(X)$ is strongly connected, there exists a path $u = u_0 \xrightarrow{w_1} u_1 \xrightarrow{w_2} \dots \xrightarrow{w_m} u_m = v$ in $G_n(X)$, meaning that, for any $i \leq m$, $w_i \in L_{n+1}(X)$ is such that u_{i-1} is a prefix of w_i and u_i is a suffix of w_i. Each $\sigma(w_i)$ is in $L(\sigma(X))$ and has length at least $n+1$, so, the factors of length n of the $\sigma(u_i)$ create a path joining U to V in $G_n(\sigma(X))$.

Note that there can be u' and v' in $L_k(X)$ with $k < n$ such that U is a factor of $\sigma(u')$ and V is a factor of $\sigma(v')$, but we choose u and v in $L_n(X)$ to avoid a problem along the path from u to v. □

Theorem 1 *The set F is the set of recurrent subshifts.*

Proof Put the previous propositions together. □

Note that the poset of sets E satisfying 1,2,3 (ordered by inclusion) is not trivial. Indeed, if we denote by $F(D)$ the smallest set E that contains D and satisfies 1,2,3, we have:

Proposition 7 *The sets $F(\{\{ba^\omega, a^\omega\}\})$ and $F(\{\{b^\omega, a^\omega\}\})$ are not comparable.*

Proposition 8 *The sets $E_n = F(\{X_{(ab)^\omega}, X_{(aabb)^\omega}, \dots, X_{(a^n b^n)^\omega}\})$ form a countable chain.*