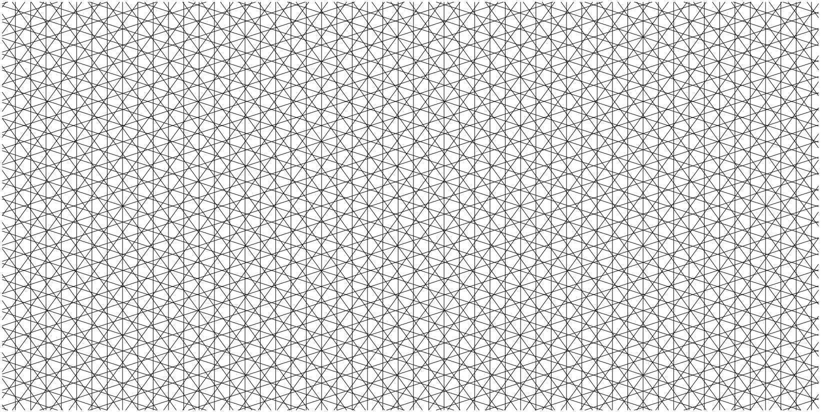
Rhombus Tilings

Thomas Fernique

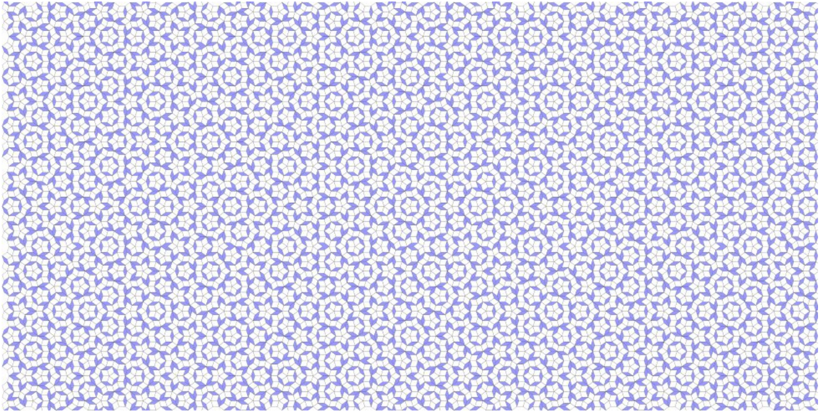
Moscow, Spring 2011

Pentagrids



Penrose tiling $\equiv$ pentagrid with integer-sum shift (de Bruijn).

Pentagrids



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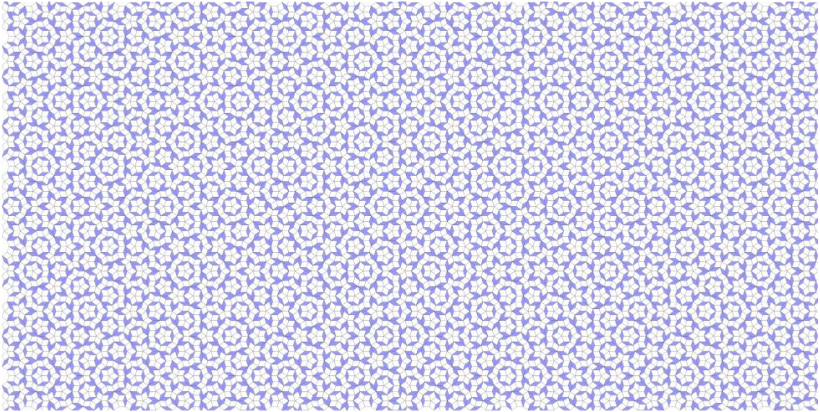
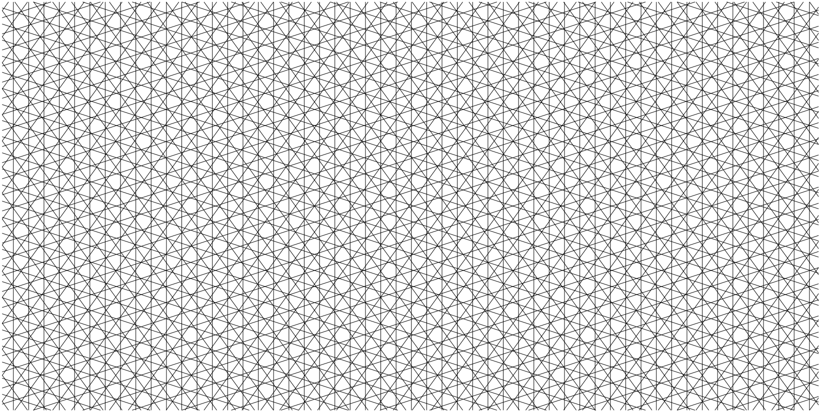


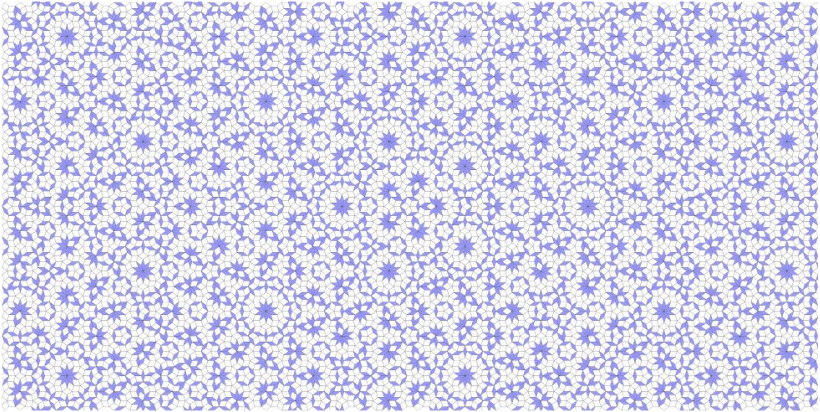
Different integer-sum shifts yield different Penrose tilings.

Playing with the shift



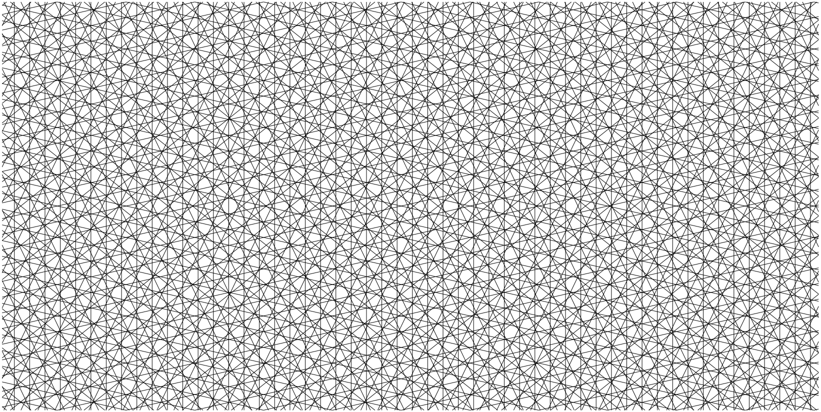
Forgetting the integer-sum condition still yields rhombus tilings.

Playing with the shift



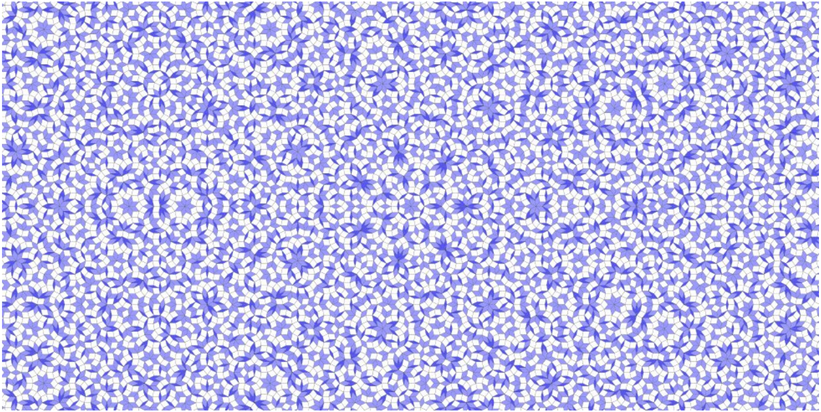
Not Penrose tilings, but so-called *generalized Penrose tilings*.

Playing with the grid number



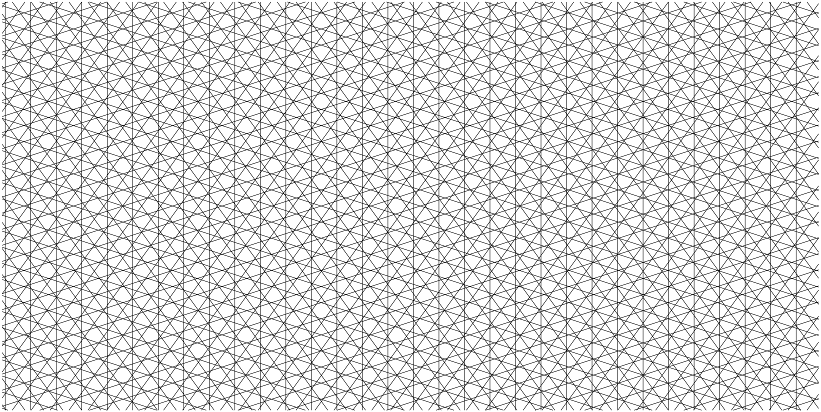
One can actually consider any number $n \geq 2$ of grids (here, $n = 7$).

Playing with the grid number



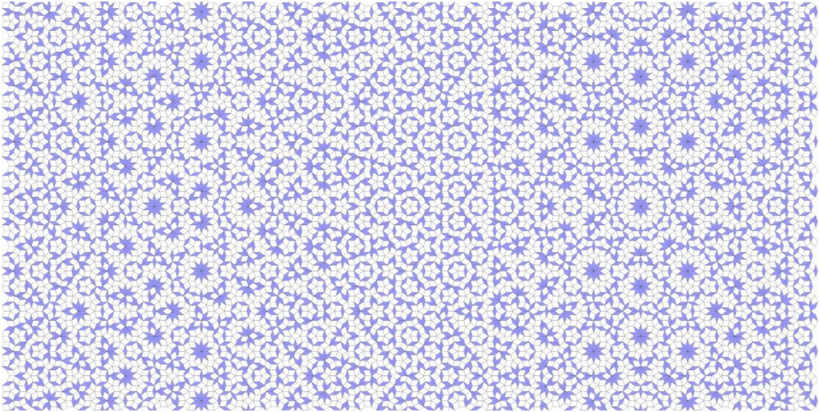
This yields rhombus tilings with arbitrary point-symmetry.

Playing with the grid spacing



Uniform grid spacing (different grids can have a different spacing).

Playing with the grid spacing



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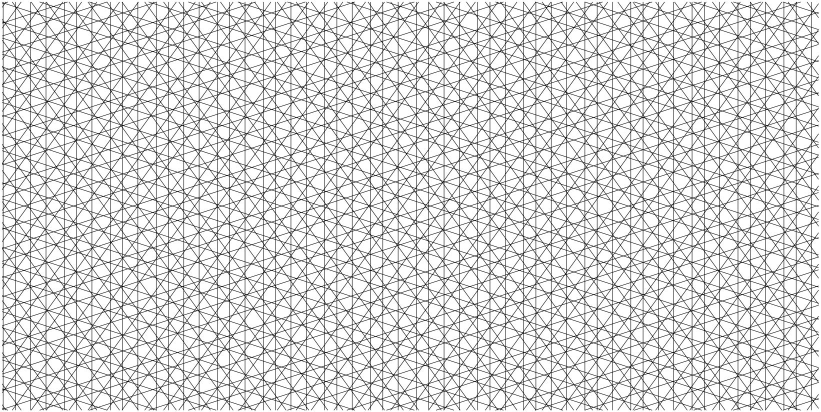
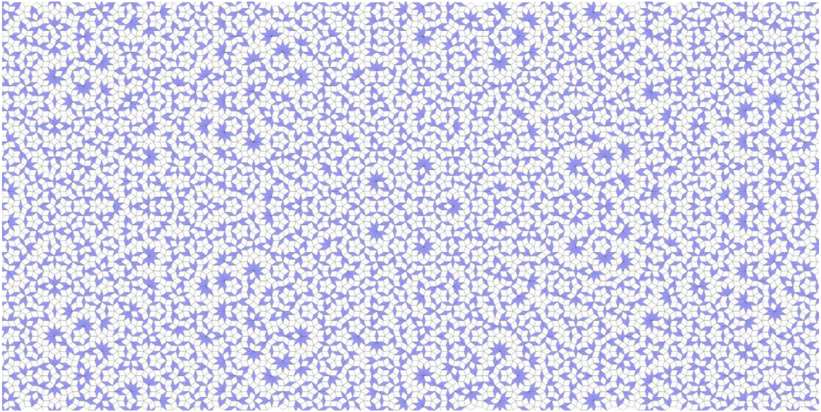


Quasiperiodic grid spacing.

Playing with the grid spacing



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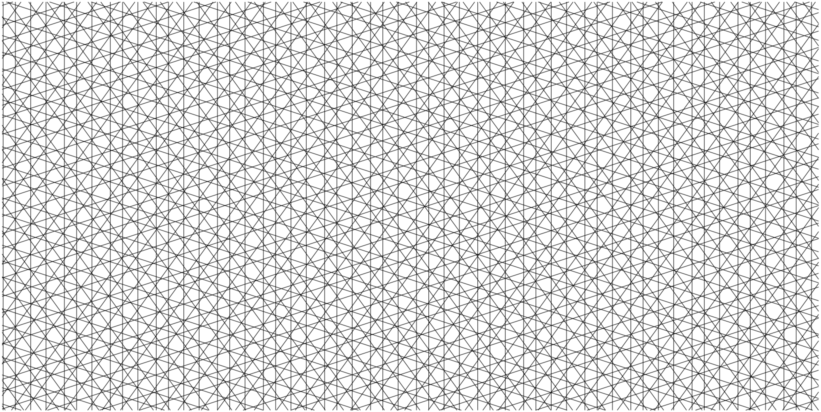
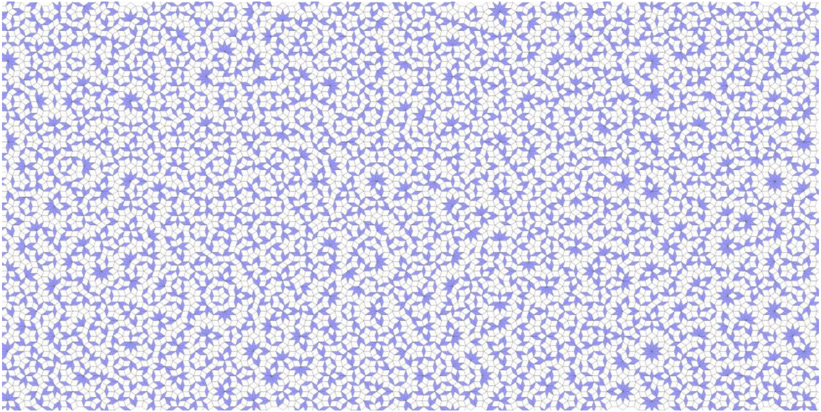


General grid spacing.

Playing with the grid spacing



General grid spacing.

Formally

Definition (Grid)

Let $\vec{g}$ be a unit vector of $\mathbb{R}^2$ and C be a discrete subset of $\mathbb{R}$. The $\vec{g}$*-directed* and C*-spaced grid* is $G := \{\vec{x} \in \mathbb{R}^2 \mid \langle \vec{x} | \vec{g} \rangle \in C\}$.

Let K_G index by integer the strips of G (in the direction of $\vec{g}$).

Definition (Dual of a multigrid $(G_1, \ldots, G_d)$)

To a mesh containing $\vec{x} \in \mathbb{R}^2$ is associated the point $\sum_i K_{G_i}(\vec{x})\vec{g}_i$, and segments connect points associated to edge-adjacent meshes.

This defines tilings of the plane with at most $\binom{d}{2}$ different rhombi.

Formally

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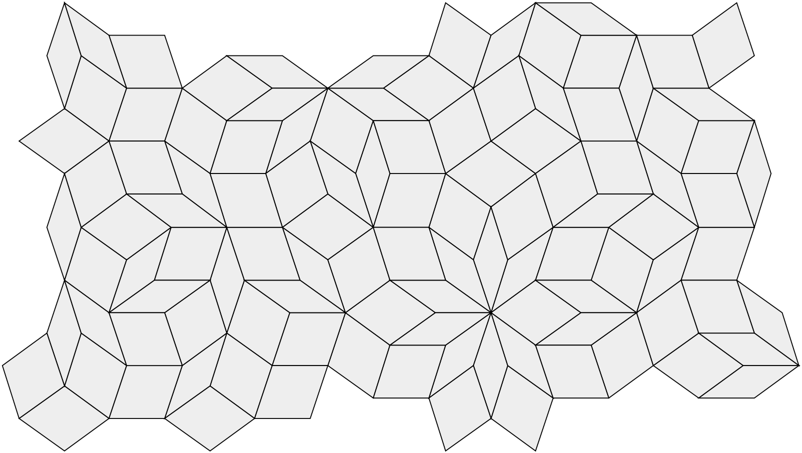
Theorem (de Bruijn, 1986)

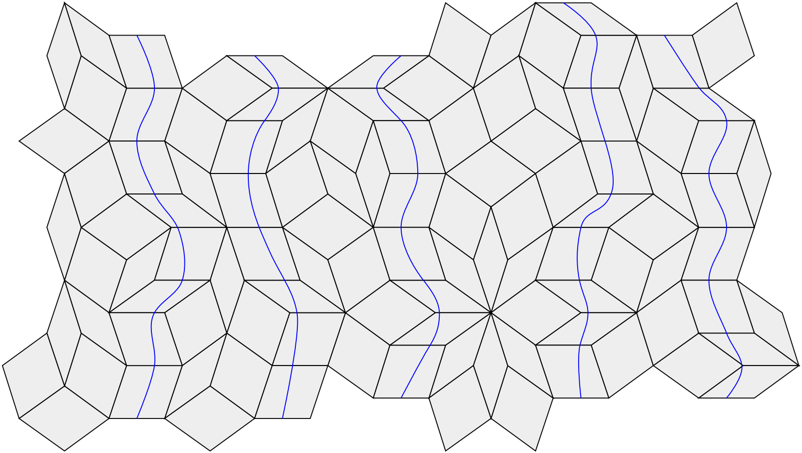
The dualization of a multigrid is a quasiperiodic rhombus tiling if and only if each grid has a quasiperiodic spacing.

Pseudogrids



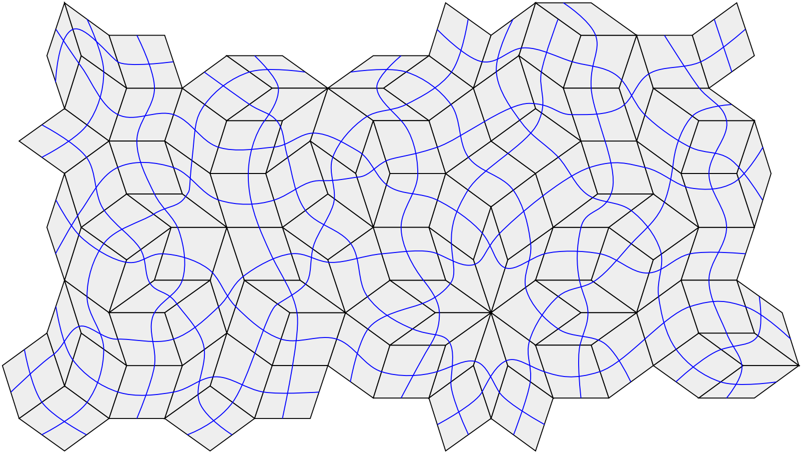
Can any rhombus tiling be obtained as the dual of some multigrid?

Pseudogrids



Given a rhombus tiling, draw *pseudolines* in parallel ribbons of tiles.

Pseudogrids



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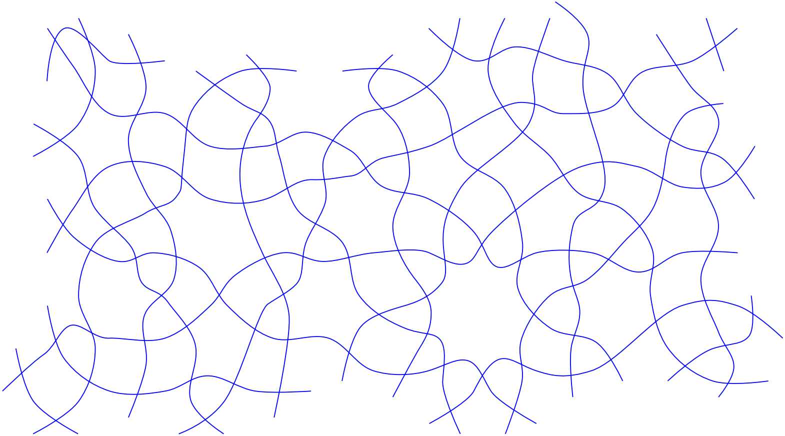
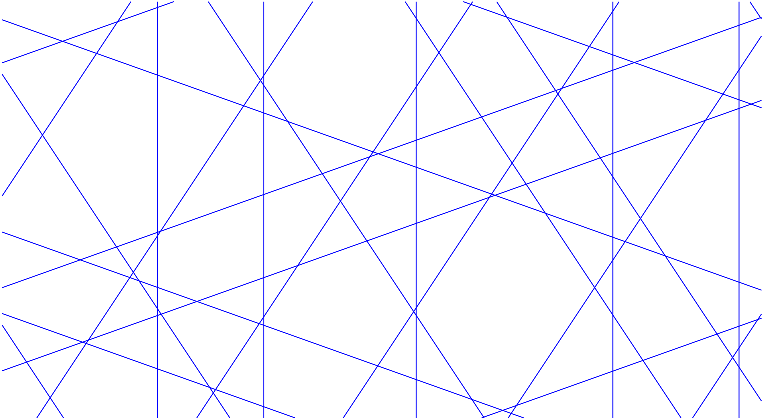


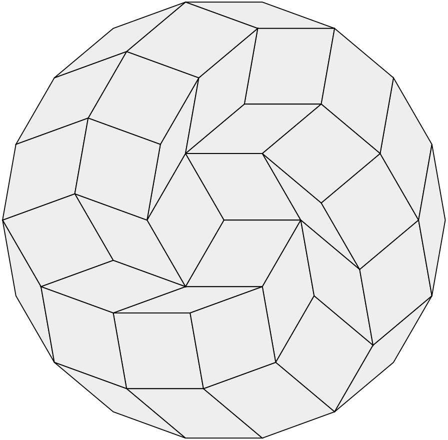
This yields a *pseudogrid*. Is it topologically equivalent to a multigrid?

Pseudogrids



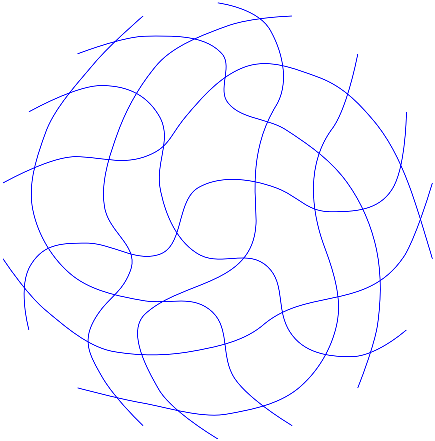
If yes, then the dualization yields back the original rhombus tiling.

Pseudogrids



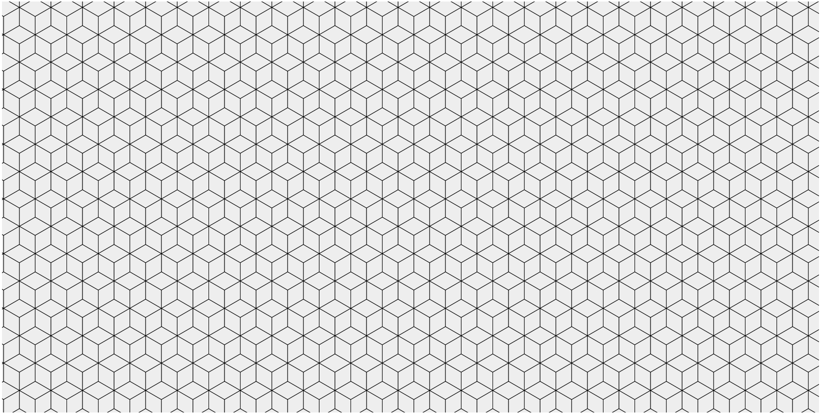
But this does not always holds (Ringel, 1956 – Grünbaum, 1972).

Pseudogrids



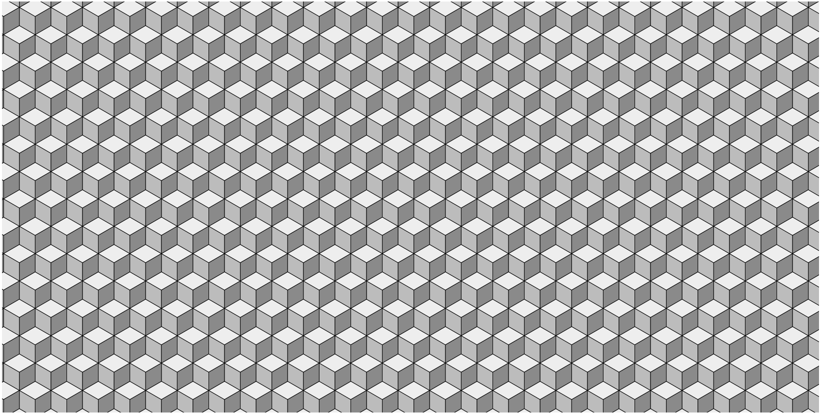
But this does not always holds (Ringel, 1956 – Grünbaum, 1972).

Let there be light!



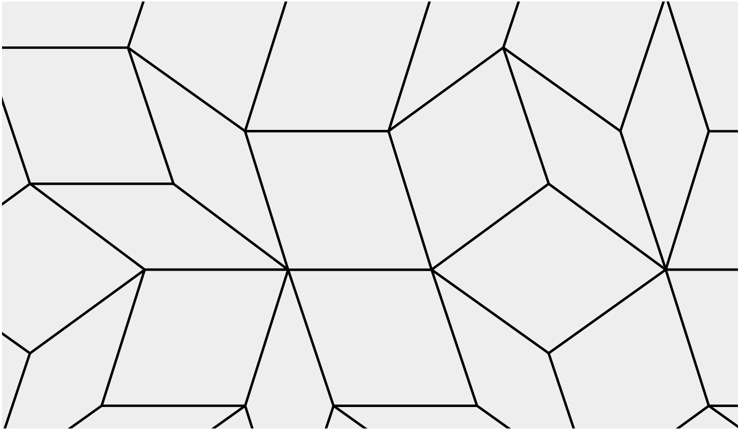
Consider a rhombus tiling defined by three grids (here, 3-fold).

Let there be light!



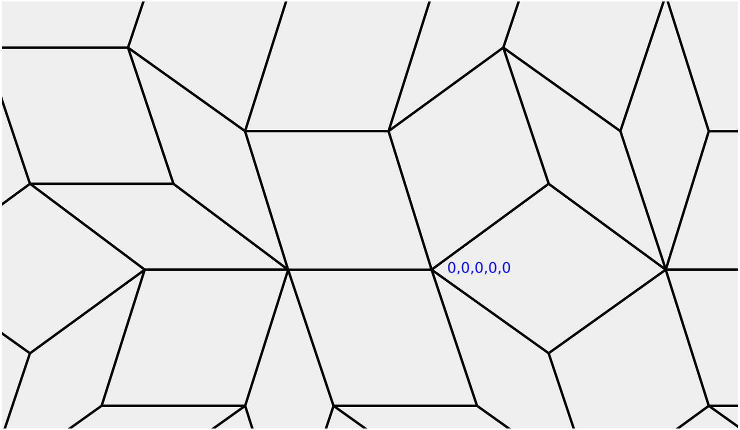
Shadowing $\rightsquigarrow$ kind of digital plane of the Euclidean space!

Lift



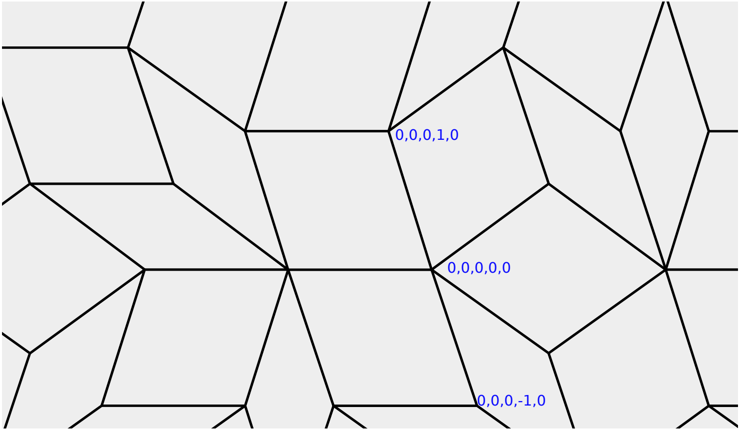
Consider a rhombus tiling where edges can take at most d directions.

Lift



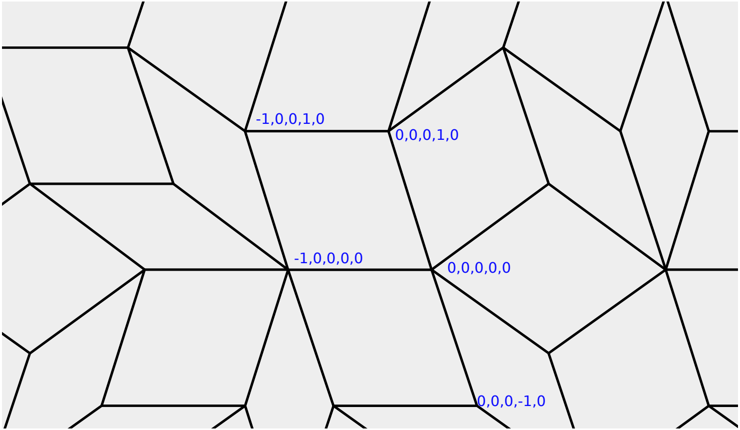
Map an arbitrary vertex onto an arbitrary vector of $\mathbb{Z}^d$.

Lift



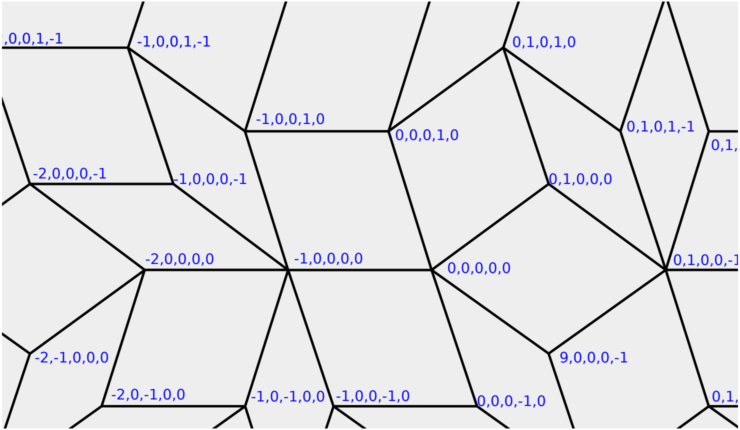
Modify ± 1 the k-th entry when moving along the k-th direction.

Lift



Rhombus vertices are mapped onto vertices of unit d-dim. squares.

Lift



The whole tiling is mapped onto a stepped surface of $\mathbb{R}^2$: its *lift*.

Plane tilings

Definition (Plane tiling)

A rhombus tiling is said to be *plane* if its lift lies inside a "slice" $V + [0,1)^d$, where V is an affine plane of $\mathbb{R}^d$.

The plane $\vec{V}$ is sometimes called *physical* or *real* space, while its orthogonal $\vec{V}^\perp$ is called *reciprocal*, *internal* or *perp-* space.

Parameters of $\vec{V}$ are called *slope* or *phason-strain* of the tiling.

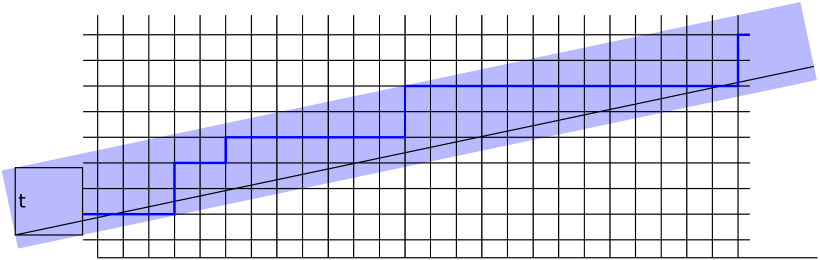
Proposition (Gähler and Rhyner, 1986)

Plane tilings exactly correspond to uniformly spaced multigrids.

Almost plane tilings

Definition (Almost plane tiling)

A rhombus tiling is said to be *almost plane* if its lift lies inside a "slice" $V + [0, t)^d$, where V is an affine plane of $\mathbb{R}^d$ and $t \in \mathbb{R}$.



The smallest possible t is the *thickness* or *fluctuation* of the tiling.

Almost plane tilings

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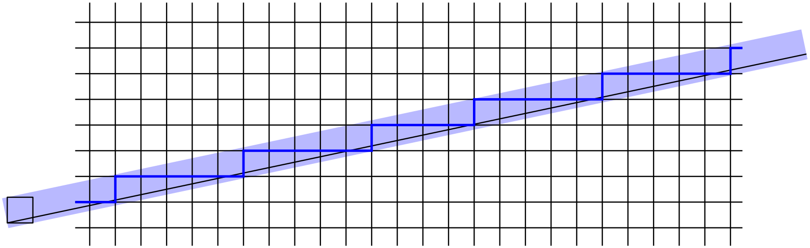
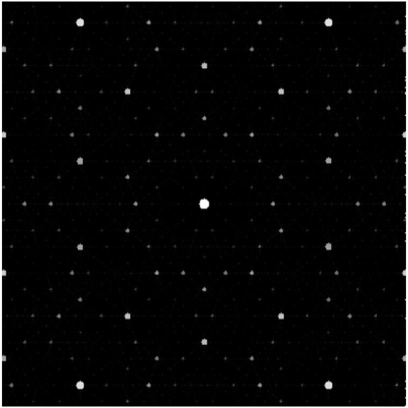


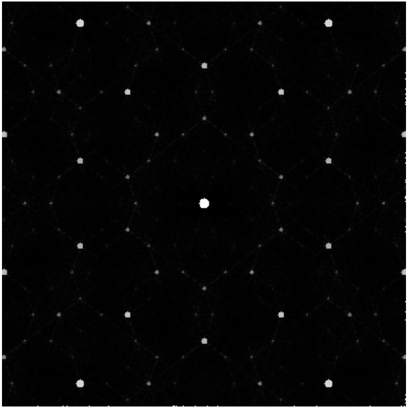
The $t = 1$ case corresponds to plane tilings.

Diffraction



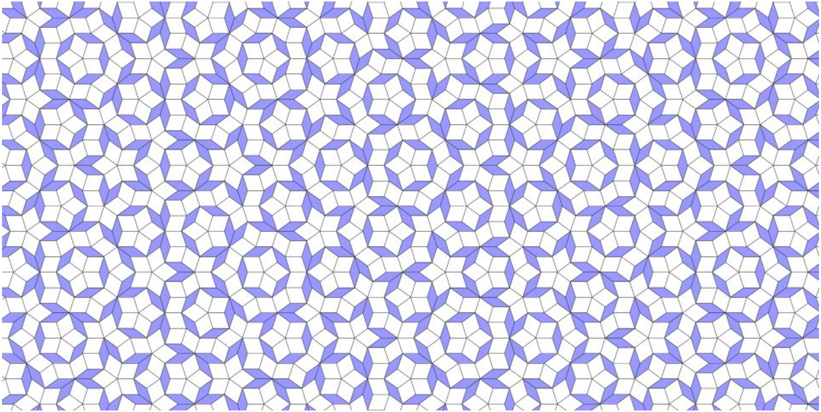
Long-range order of plane tilings yields Bragg peaks.

Diffraction



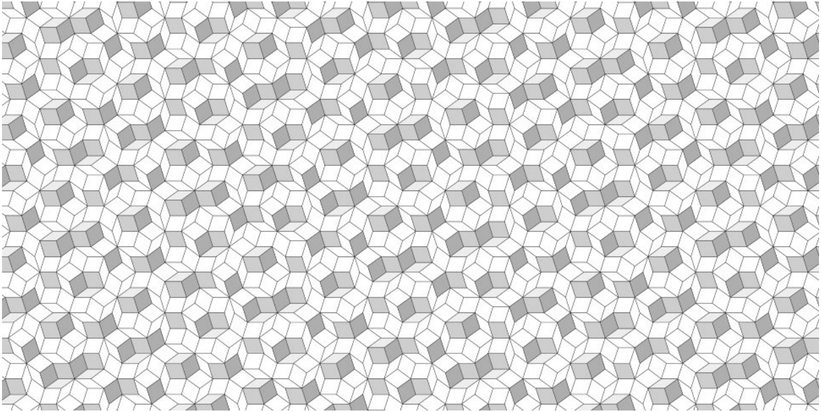
Almost plane tilings still have this long-range order!

Shadows



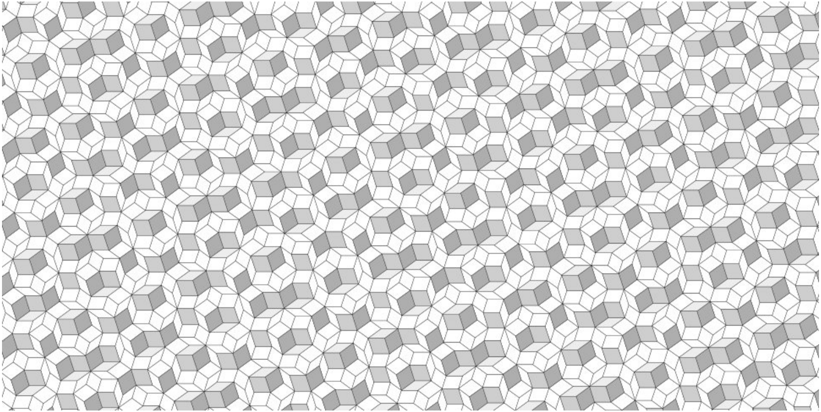
Rhombus tilings are projection of d-dim. unit squares (remind lift).

Shadows



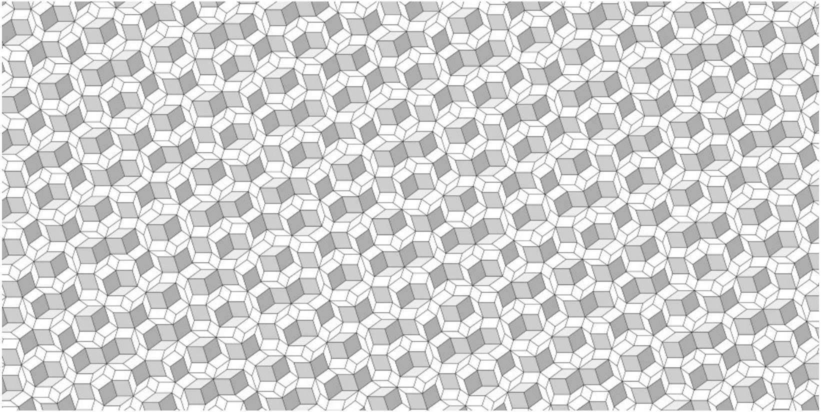
Select three edge directions and emphasize rhombi defined by them.

Shadows



Rotate in $\mathbb{R}^d$ until all the remaining edges orthogonally project.

Shadows



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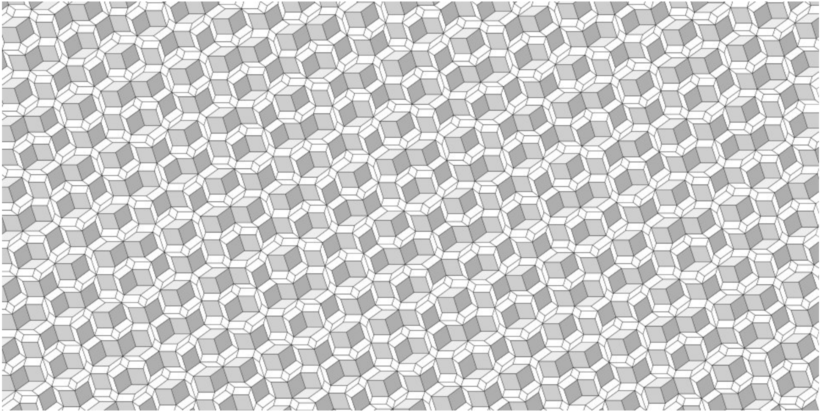

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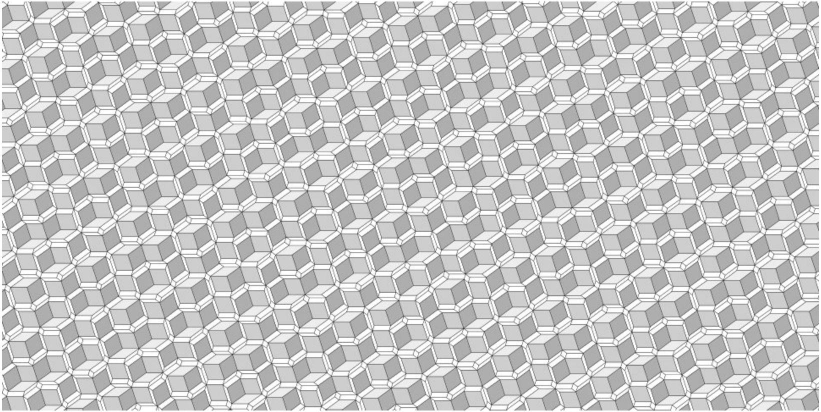

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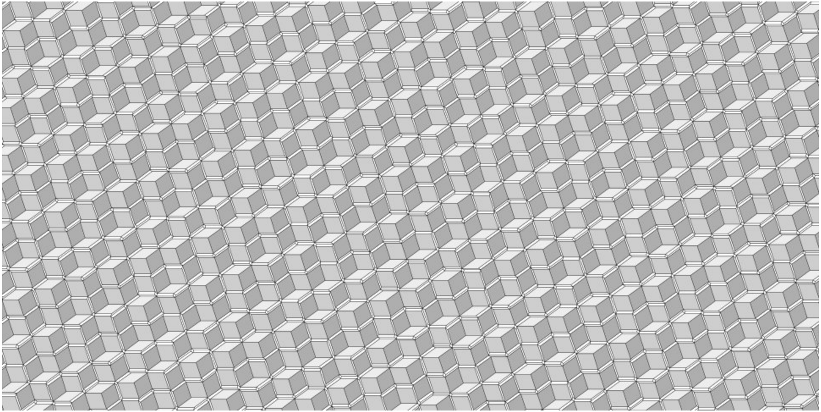

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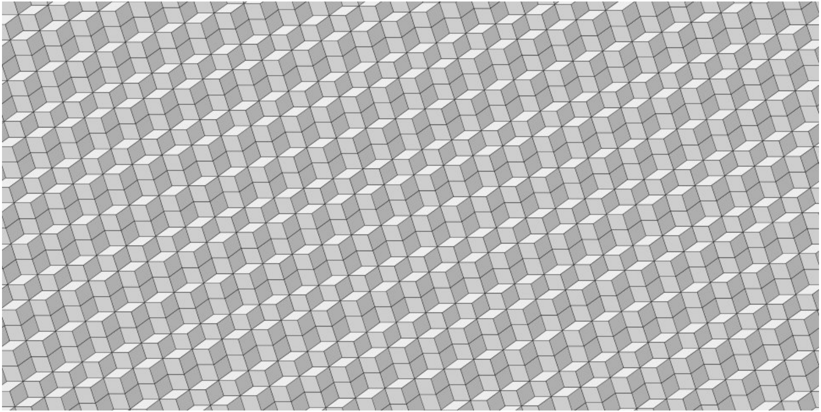


This yields a rhombus tiling, called a *shadow*, whose lift is in $\mathbb{R}^3$.

Local rules

Terminology:

- set T of tiles $\rightsquigarrow$ set X_T of tilings;
- *r-pattern* of a tiling: tiles lying inside a ball of radius $r > 0$;
- *r-atlas* of $X \subset X_T$: r-patterns of tilings in X (up to isometry).

Definition (Local rules)

$X \subset X_T$ admits *local rules* if it is characterized by a r-atlas, $r > 0$.

Dynamical systems terminology:

- X_T: *fullshift* over T;
- $X \subset X_T$ translation-invariant and closed: *shift*;
- X admits local rules $\equiv$ X is a shift of *finite type*.

Decorated local rules

Terminology:

- Decorated tiling: tiles can be colored, labelled, notched *etc.*;
- locally derivable from $\equiv$ image under a local map of.

Definition (Decorated local rules)

$X \subset X_T$ admits *decorated local rules* if it is locally derivable from a set of decorated tilings which admits local rules.

Dynamical systems terminology:

- locally derivable from $\equiv$ *topological factor* of;
- X admits decorated local rules $\equiv$ X is a *sofic* shift.

One-dimensional examples

Consider the fullshift $\{a, b\}^{\mathbb{Z}}$.

Local rules that admit these subshifts?

1. the sequences with no more than 10 consecutive b;
2. the sequences with at most one b-run;
3. the centro-symmetric sequences;
4. the non-periodic sequences.

Strong and weak local rules

Distinction introduced by Levitov for rhombus tilings:

Definition (Strong and weak local rules)

Local rules which define a set of rhombus tilings are said to be

- *strong* if the tilings are all parallel plane tilings;
- *weak* if the tilings are parallel almost plane tilings.

Remind: bounded fluctuations do not destroy long-range order!

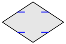
One-dimensional examples

Fullshift over $\{a, b\} \equiv$ one-dimensional rhombus tilings.

Type of these local rules (and subshifts they define)?

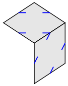
1. $\{aba,\ \ bab\}$
2. $\{aa,\ \ ab,\ \ ba\}$
3. $\{aabb, abba, bbaa, baab\}$
4. $\{a_i a_{i+1},\ \ a_i b_i,\ \ b_i a_{i-p}\}_{1 \leq i \leq q}$

Two-dimensional examples



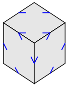
Consider this decorated rhombus.

Two-dimensional examples



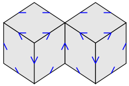
Two rhombi match if they form an arrow on their common edge.

Two-dimensional examples



This allows only one plane tiling $\rightsquigarrow$ strong (decorated) rules.

Two-dimensional examples



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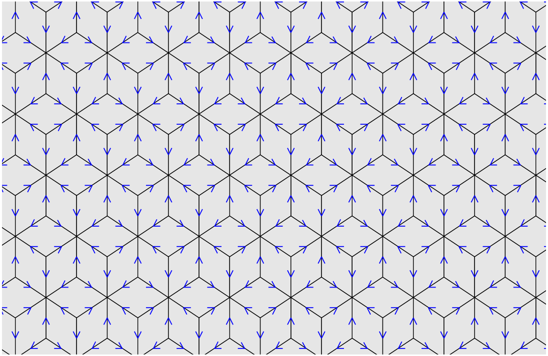

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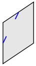
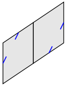


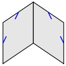
Consider now this decorated rhombus.

Two-dimensional examples



Matching are free on empty edges, as before on arrowed ones.

Two-dimensional examples



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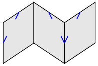

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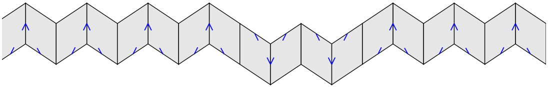
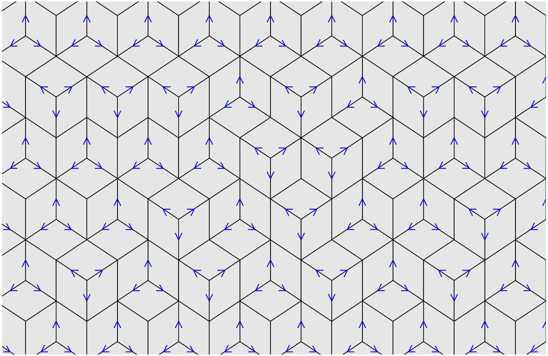


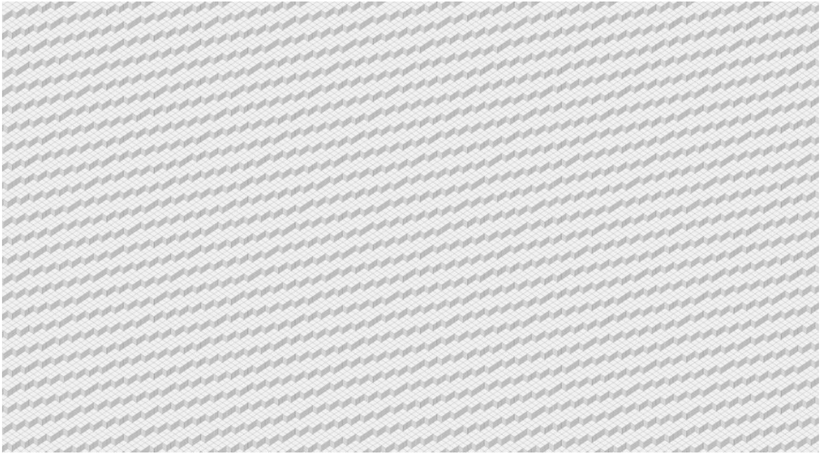
This allows only small fluctuations on tile ribbons.

Two-dimensional examples



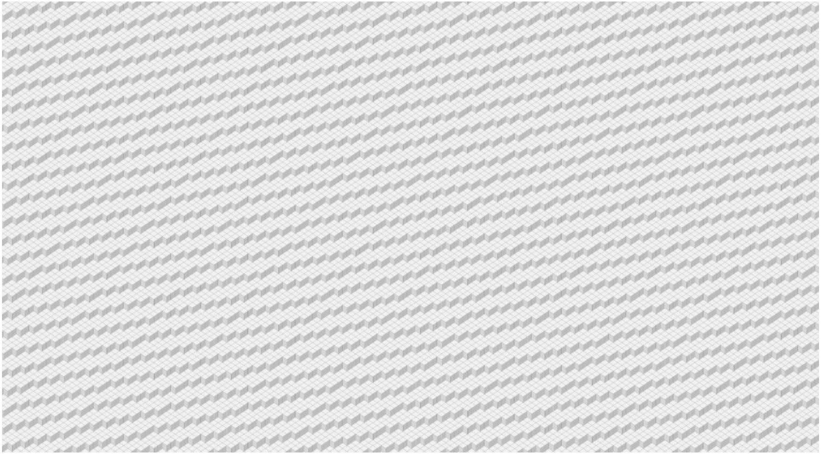
The same thus holds for the whole tiling $\rightsquigarrow$ weak decorated rules.

Shifting the cut of a fully periodic shadow



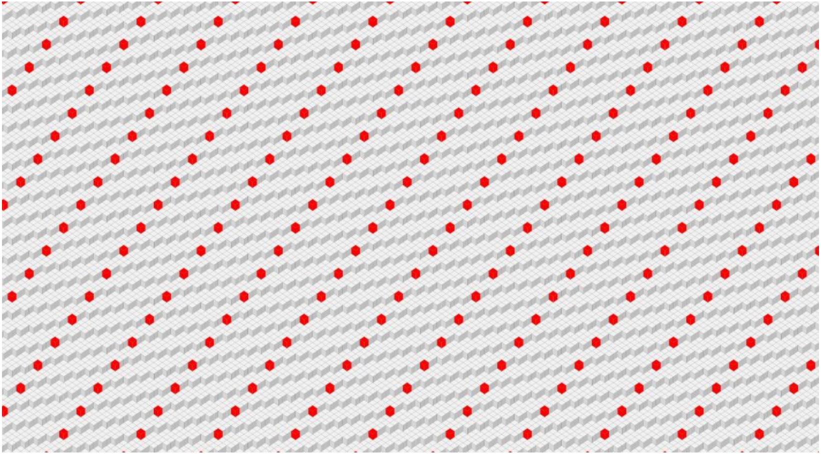
Consider a plane tiling obtained by a rational cut in $\mathbb{R}^3$.

Shifting the cut of a fully periodic shadow



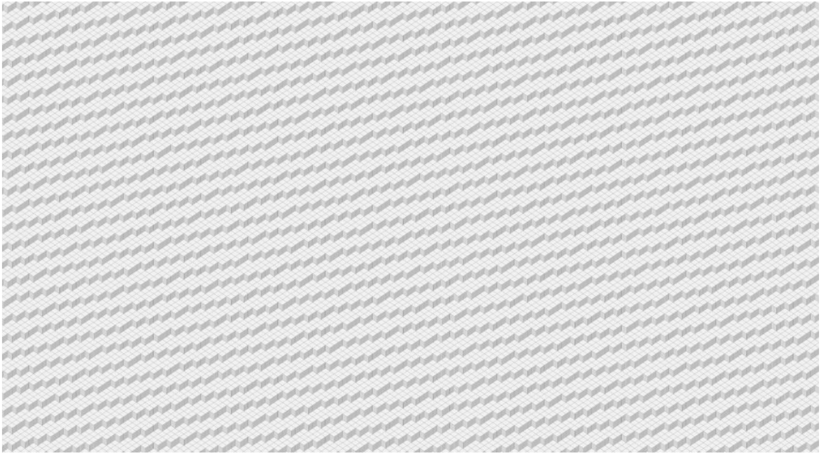
Shifting (in $\mathbb{R}^3$) the cut just shifts (in $\mathbb{R}^2$) the tiling.

Shifting the cut of a fully periodic shadow



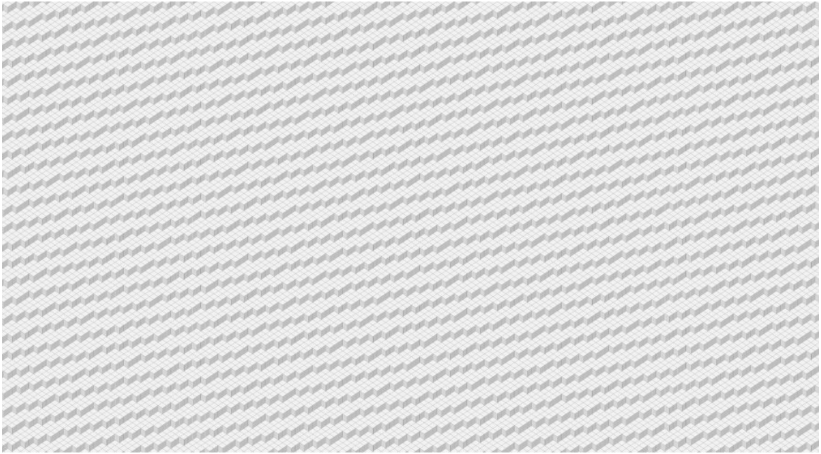
This corresponds to local rearrangements (*flip*) on a 2-dim. lattice.

Shifting the cut of a non-periodic shadow



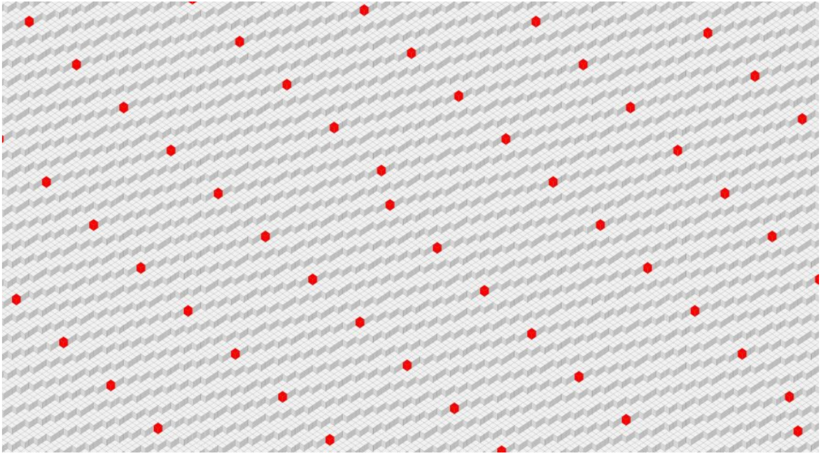
Consider a plane tiling obtained by an irrational cut in $\mathbb{R}^3$.

Shifting the cut of a non-periodic shadow



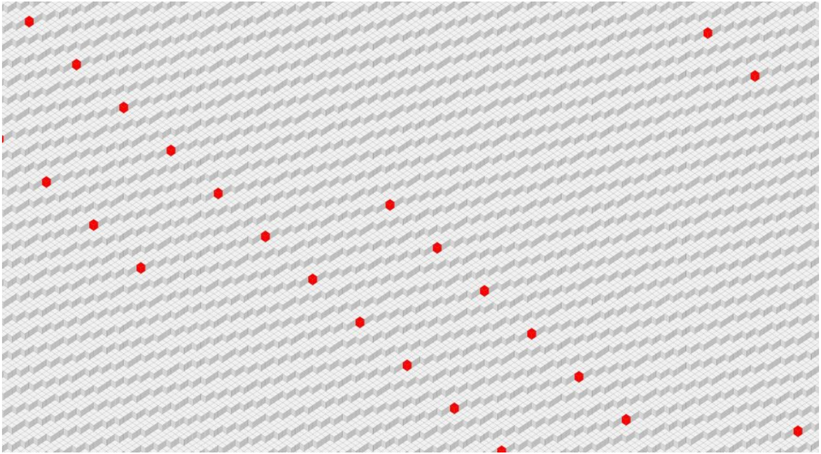
Shifting the cut modifies the tiling but not the finite patterns.

Shifting the cut of a non-periodic shadow



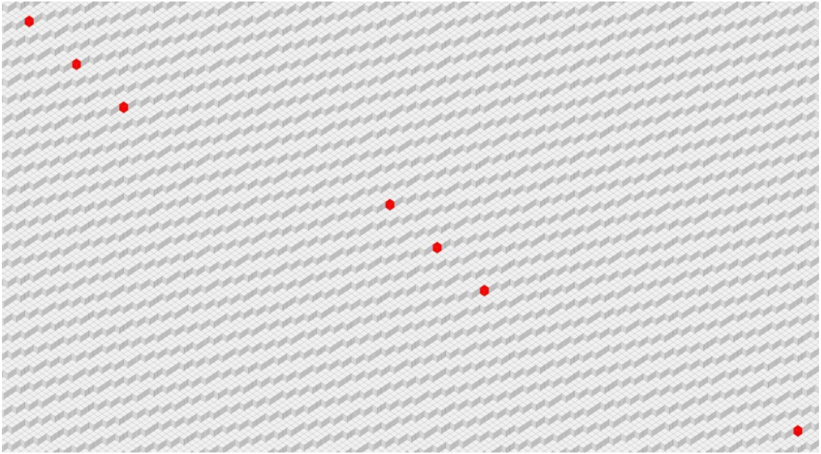
Modifications are quasiperiodically spaced flips.

Shifting the cut of a non-periodic shadow



The smaller is the shift, the sparser are these flips.

Shifting the cut of a non-periodic shadow



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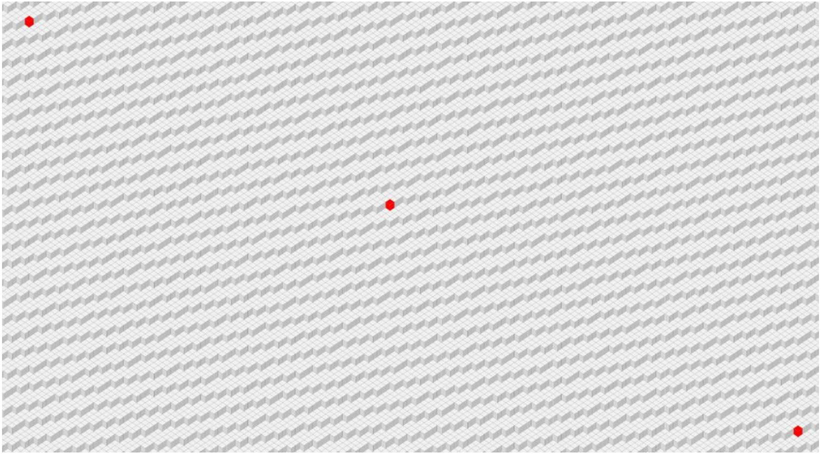
Shifting the cut of a non-periodic shadow



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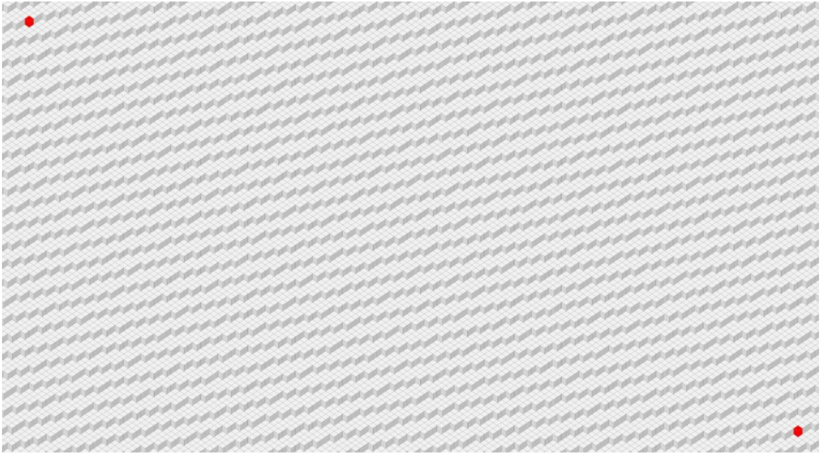
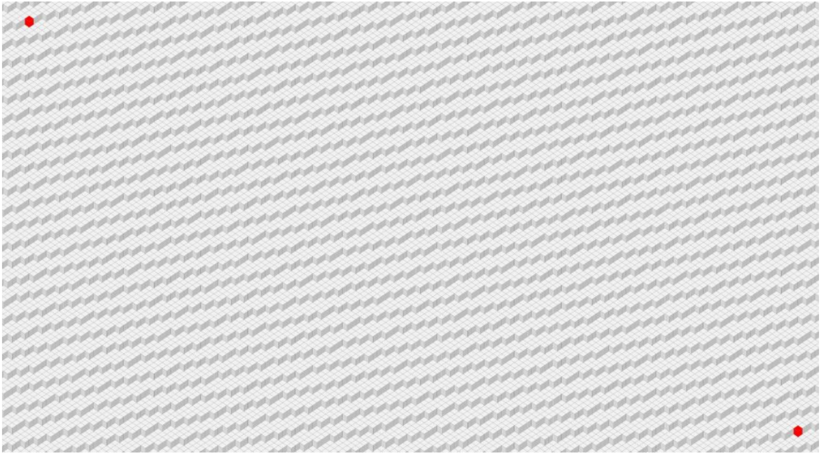
Shifting the cut of a non-periodic shadow



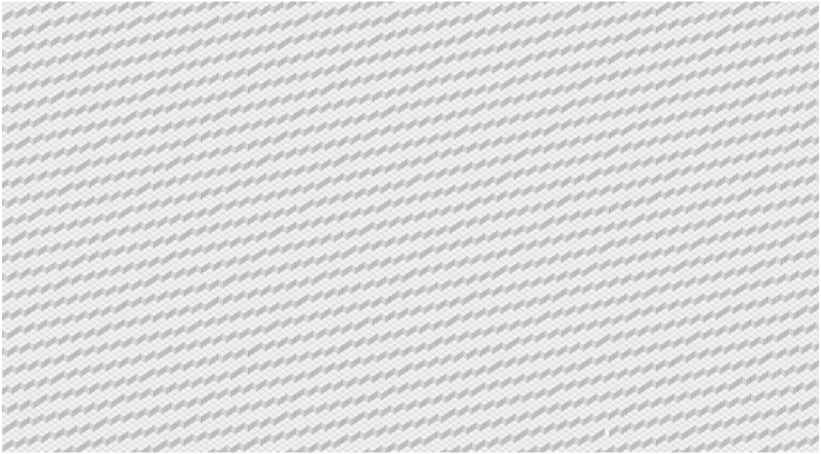
Removing a single flip increases the thickness $\rightsquigarrow$ non-plane tiling.

Shifting the cut of a non-periodic shadow



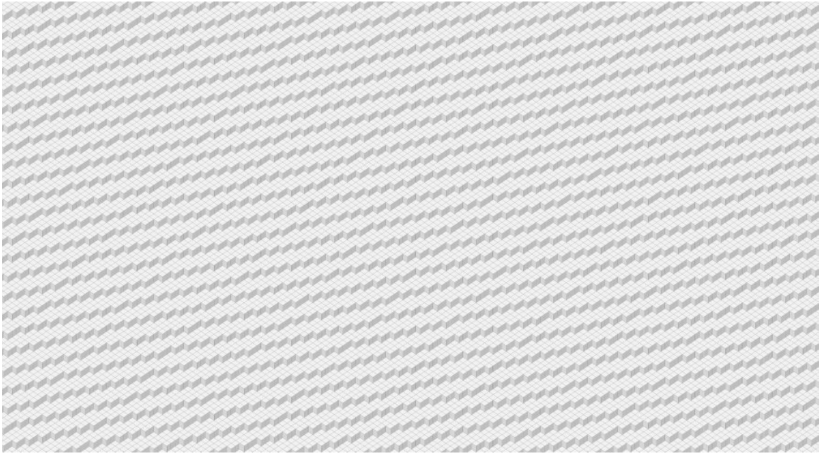
To forbid this, strong rules should be larger than the flip-spacing. . .

Shifting the cut of a semi-periodic shadow



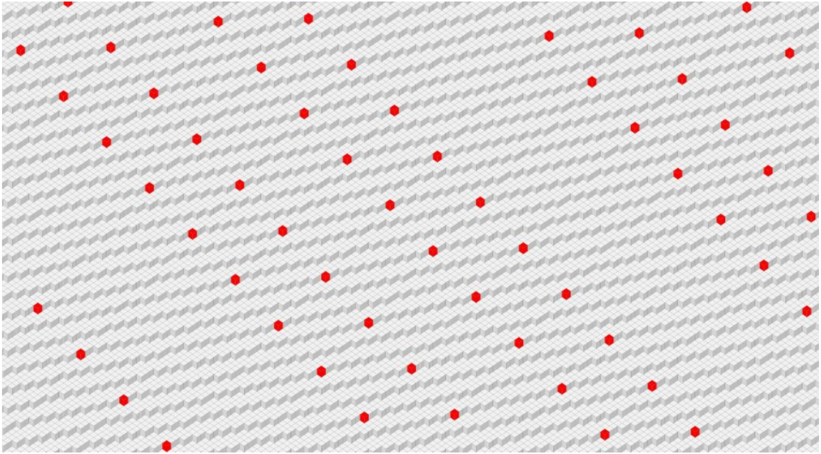
Consider now the intermediary case.

Shifting the cut of a semi-periodic shadow



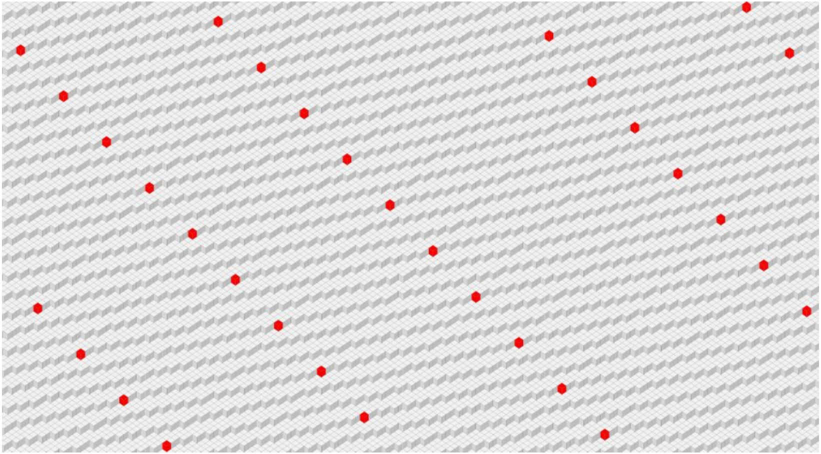
Shifting the cut modifies the tiling but not the finite patterns.

Shifting the cut of a semi-periodic shadow



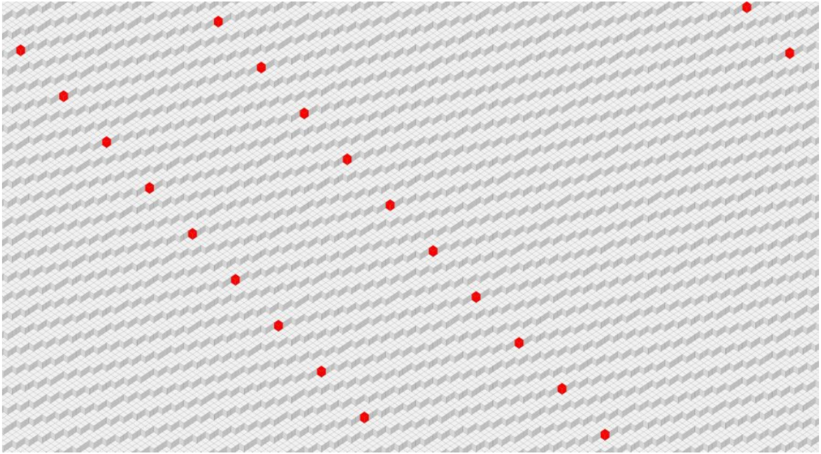
Modifications are quasiperiodically spaced periodic lines of flips.

Shifting the cut of a semi-periodic shadow



The smaller is the shift, the sparser are these lines of flips.

Shifting the cut of a semi-periodic shadow



For similar reasons, this is incompatible with strong rules.

Necessary condition for strong rules

Theorem (Levitov, 1988)

If a rhombus tiling has strong rules, then its shadows are periodic.

Proof:
Assume that there are non-periodic shadows and strong rules.

1. by a sufficiently small shift on the cut (in $\mathbb{R}^n$):
   - fully periodic shadows are unchanged (for a suitable shift);
   - flips in non-periodic shadows are at dist. $\geq R$ from each other;
   - flip lines in semi-periodic shadows are sufficiently spaced to be at dist. $\geq R$, in the tiling, of a flip of non-periodic shadows.
2. show that there is k indep. from R s.t. each diameter R ball in the tiling contains at most k flips of non-periodic shadows;
3. deduce that strong rules should have diameter $\frac{R}{2k}$, for any R.

Periodic shadows yield $\{3,4,5,6,8,10,12\}$-fold tilings

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1,\ldots,u_n)+\mathbb{R}(v_1,\ldots,v_n)$,

$$u_k=\cos\left(\frac{2k\pi}{n}\right)\quad\text{and}\quad v_k=\sin\left(\frac{2k\pi}{n}\right).$$

Periodicity of shadows yields $\cos(2\pi/n)\in\mathbb{Q}(\sqrt{D})$. Possible cases:

$$\begin{array}{lll}
\cos(2\pi/n)\in\mathbb{Q} & \text{if} & n=3,4,6\\
\cos(2\pi/n)\in\mathbb{Q}(\sqrt{2}) & \text{if} & n=8\\
\cos(2\pi/n)\in\mathbb{Q}(\sqrt{3}) & \text{if} & n=12\\
\cos(2\pi/n)\in\mathbb{Q}(\sqrt{5}) & \text{if} & n=5,10
\end{array}$$

Periodic shadows yield $\{3, 4, 5, 6, 8, 10, 12\}$-fold tilings

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1, \ldots, u_n) + \mathbb{R}(v_1, \ldots, v_n)$,

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These symmetries are exactly those yet experimentally observed!

Sufficient condition for weak rules

The *ijk*-shadow of a plane tiling of slope $\mathbb{R}\vec{u}+\mathbb{R}\vec{v}$ is periodic iff:

$$\exists \vec{p}_{ijk} \in \mathbb{Z}^3 \backslash \{\vec{0}\}, \quad \det(\vec{u}_{ijk}, \vec{v}_{ijk}, \vec{p}_{ijk}) = (\vec{u}_{ijk} \wedge \vec{v}_{ijk}).\vec{p}_{ijk} = 0.$$

This can be seen as an equation for three entries of $\vec{u}$ and $\vec{v}$.

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Theorem (Levitov-Socolar mix)

If periodic shadows of a plane tiling yield equations characterizing its slope, then this tiling does admit weak rules.

Proof:

- the periodicity of a shadow can be enforced by local rules;
- the hypothesis ensure that this characterizes the tiling slope;
- no control on the intertwining of shadows $\rightsquigarrow$ only weak rules.

Further results

Tiling	undecorated rules	decorated rules
$5, 10$-fold	strong	strong[1]
8-fold	none[2]	strong[3]
12-fold	none[3]	strong[4]
$(4 \nmid n)$-fold	weak[5]	strong?
quadratic slope in $\mathbb{R}^4$	a.e. weak[6]	strong[7]
non-algebraic slope	none[8]	?

(1): Penrose, 1974
(2): Burkov, 1988
(3): Le, 1992
(4): Socolar, 1989
(5): Socolar, 1990
(6): Levitov, 1988
(7): Le *et al.*, 1992
(8): Le, 1997

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(8): Le, 1997

Conjecture

A plane tiling admits decorated rules iff its slope is computable.

Some references for this lecture:

- Nicolaas Govert de Bruijn, *Dualization of multigrids*, J. Phys. France **47** (1986).
- Leonid Levitov, *Local rules for quasicrystals*, Comm. Math. Phys. **119** (1988).
- Joshua Socolar, *Weak matching rules for quasicrystals*, Comm. Math. Phys. **129** (1990).
- Thang Tu Quoc Le, *Local rules for quasiperiodic tilings*, in: The Mathematics of long-range aperiodic order, 1995.

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`