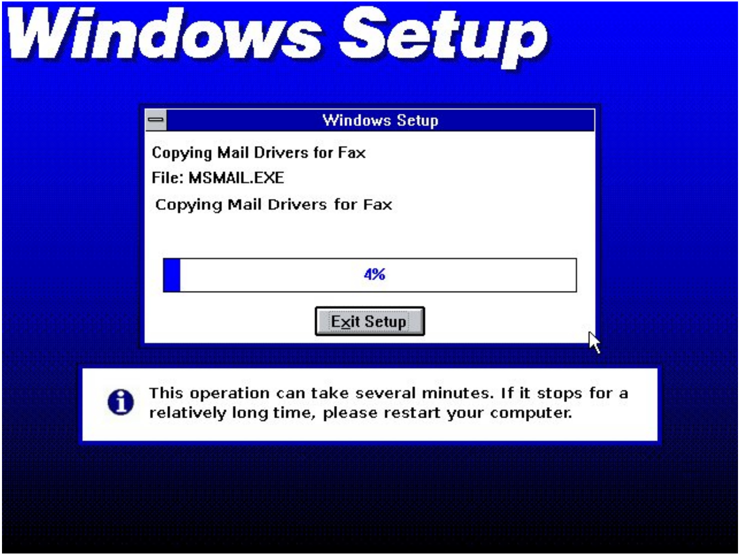
Robinson Tilings

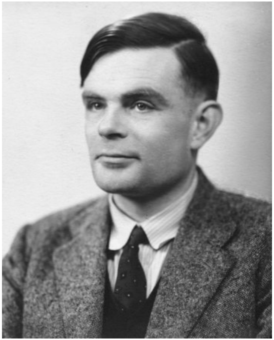
Thomas Fernique

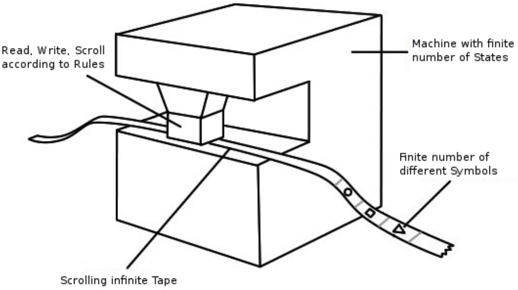
Moscow, Spring 2011

The Halting problem for dummies (Windows 3.11, 1993)



Turing machines (1937)





Rule: read s in state $q \rightsquigarrow$ write s', scroll tape, goto state q'.

Turing machine: finite set of rules $(q, s, s', \leftrightharpoons, q')$.

Input: finite symbol sequence written on the tape.

Example

What does compute this Turing machine?

	on 0			on 1		
state	write	scroll	goto	write	scroll	goto
q	1	$\longrightarrow$	$\square$	0	$\longrightarrow$	q

Examples

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	on 0			on 1		
state	write	scroll	goto	write	scroll	goto
q	1	$\rightarrow$	$\square$	0	$\rightarrow$	q

How many steps of computation on a 0-filled tape for this one?

	on 0			on 1		
state	write	scroll	goto	write	scroll	goto
A	1	$\rightarrow$	B	1	$\rightarrow$	$\square$
B	1	$\leftarrow$	B	0	$\rightarrow$	C
C	1	$\leftarrow$	C	1	$\leftarrow$	A

Decidability

Decision problem: does the input satisfies a given property?

Example (parity): does the input encodes an even number?

Decidable problem: there exists a Turing machine which decides (writes yes/no on the tape and halts) the problem for any input.

Example: parity problem is decidable.

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Example: parity problem is decidable.

Example: decision problems in P and NP are equally decidable.

The Halting problem

Halting problem: does Turing machine M halts on input w?

Theorem (Turing, 1937)

The halting problem is undecidable.

Proof:

- assume M_H decides the halting problem for any input $M; w$;
- let $D(M)$: if $M_H(M; M) = \text{yes}$ then loops, otherwise halts;
- $D(D)$ halts $\Leftrightarrow$ $M_H(D; D) = \text{no}$ $\Leftrightarrow$ $D(D)$ does not halt.

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Halting problem bis: does Turing machine M halts on empty input?

Busy beavers

Challenge: Among fixed size Turing machines (the beavers), find the one with the longest output on an empty input (the busiest).

Try to beat this one:

	on 0			on 1			on 2		
state	write	scroll	goto	write	scroll	goto	write	scroll	goto
A	1	$\rightarrow$	B	2	$\leftarrow$	A	1	$\leftarrow$	C
B	0	$\leftarrow$	A	2	$\rightarrow$	B	1	$\leftarrow$	B
C	1	$\rightarrow$	$\square$	1	$\rightarrow$	A	1	$\rightarrow$	C

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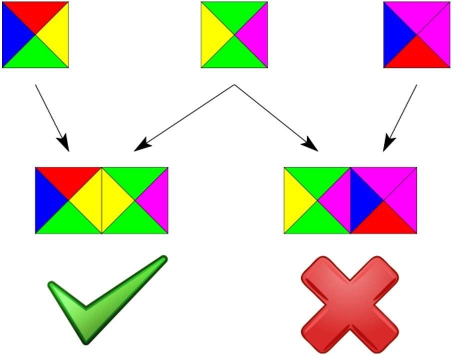
(halts after 119×10^{15} steps, with 374×10^{6} non-zero cells)

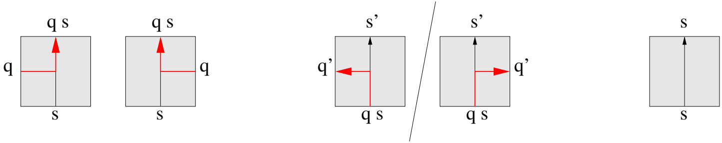
Wang tiles (1961)



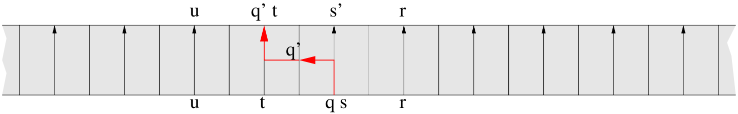
Wang tiles: colored squares; match along edges of the same color.

Simulating Turing machines by Wang tiles

Three tiles for each rule $(q, s, s', \leftrightharpoons, q')$, one for each symbol s:



Rows of a tiling of the plane $\simeq$ tape evolution of the machine:



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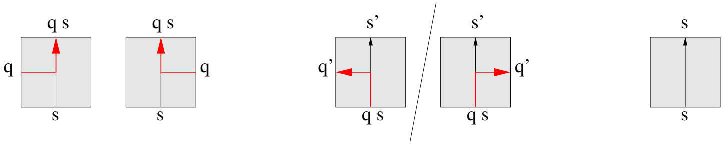

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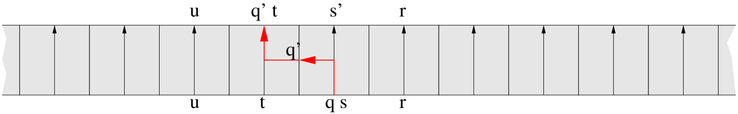
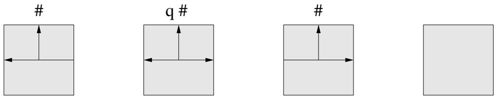


No proper computation initialization.

The Completion problem

Additional tiles to start a computation on a empty imput:



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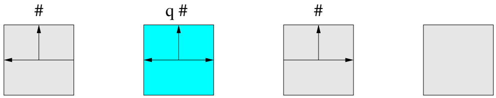
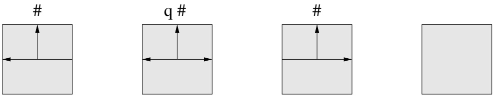


Undecidability of Halting problem bis then yields the one of:

Completion problem: given a finite tileset and a seed tile, is it possible to extend this seed tile to a tiling of the whole plane?

The Completion problem

Additional tiles to start a computation on a empty imput:



Undecidability of Halting problem bis then yields the one of:

Completion problem: given a finite tileset and a seed tile, is it possible to extend this seed tile to a tiling of the whole plane?

And without seed?

The Domino problem

Domino problem: does a given finite tileset tile the whole plane?

To prove undecidability: forbid translational order (as seeds do)?

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Theorem (Wang, 1961)

If any finite tileset which tile the plane does admit a periodic tiling, then the domino problem is decidable.

Proof: just try to tile larger and larger squares till finding a period.

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Domino problem: does a given finite tileset tile the whole plane?

To prove undecidability: forbid translational order (as seeds do)?

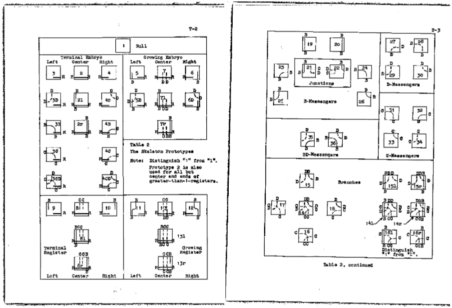
Theorem (Wang, 1961)

If any finite tileset which tile the plane does admit a periodic tiling, then the domino problem is decidable.

Proof: just try to tile larger and larger squares till finding a period.

Does exist finite tilesets which tile the plane only non-periodically?
Wang conjectured that there are no such so-called aperiodic tileset.

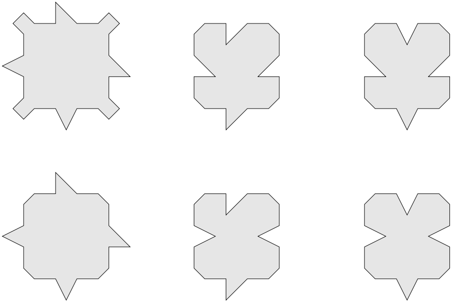
Undecidability (Berger, 1964)



Berger proved the undecidability of Domino problem in his thesis.

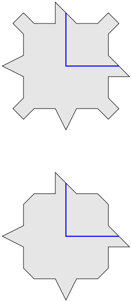
In particular, he constructed the first aperiodic tileset: 20426 tiles!

Robinson tiles (1971)



Six bumped and dented tiles which can be rotated or reflected.
Two corners (bumpy and dented, left) and four arms (right).

Robinson tiles (1971)



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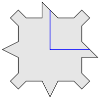
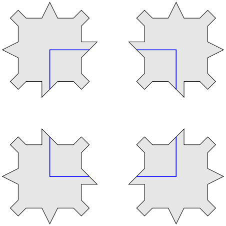
Bumpy corners



Order n bumpy corners: recursively defined squares of side $2^n - 1$.

Bumpy corners



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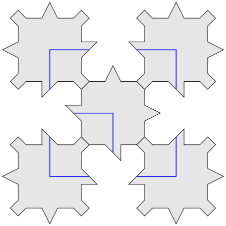

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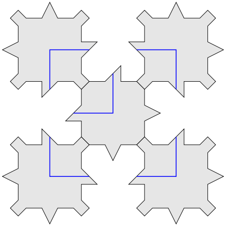

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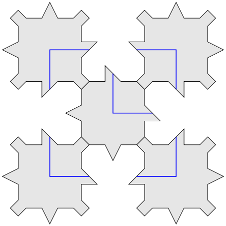

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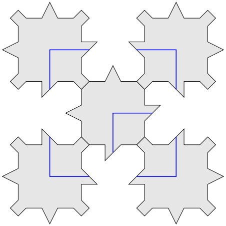

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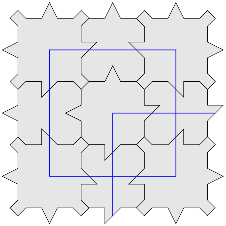

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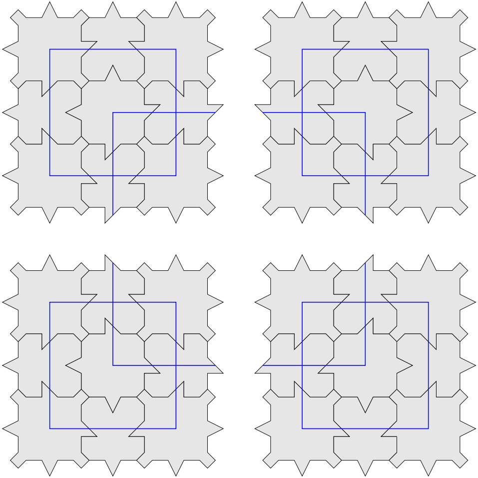

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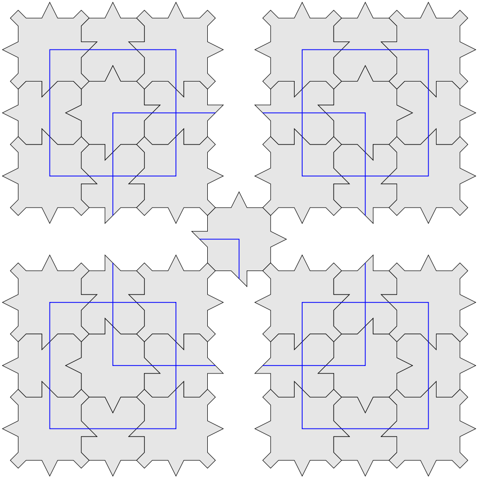

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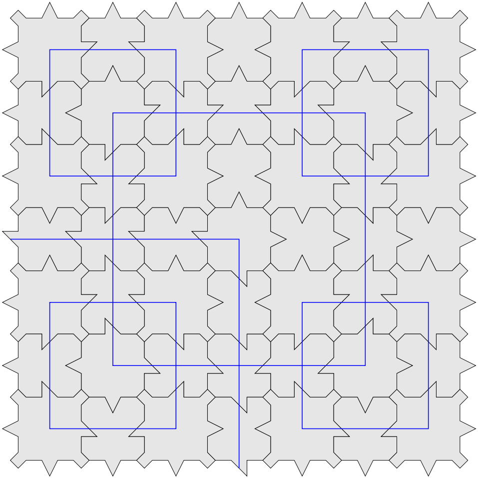

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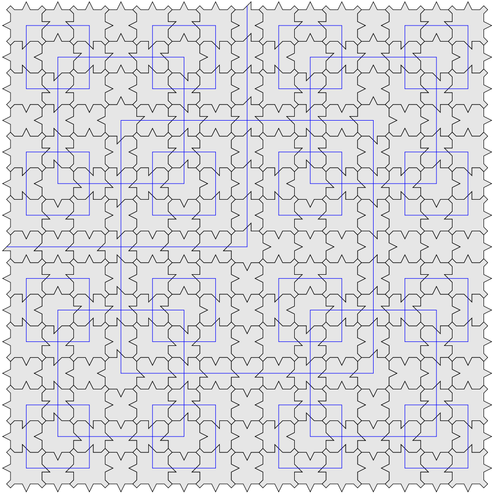

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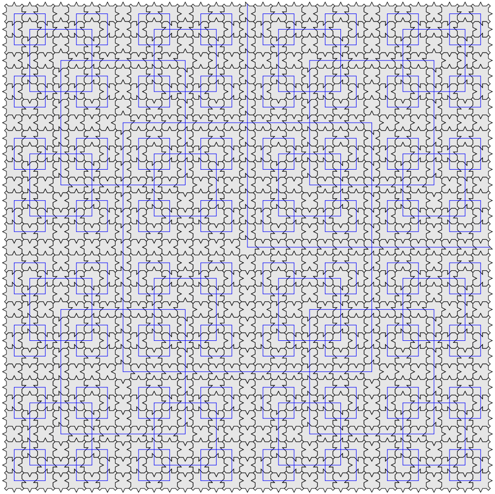

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Tiling the whole plane

Lemma

The Robinson tileset does tile the plane.

Proof: infinite spiral-growing increasing sequence of bumpy corners.

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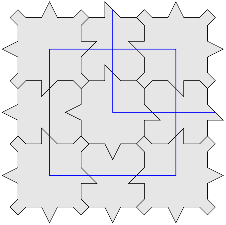
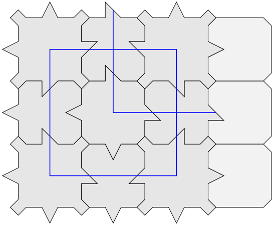
Proof 2: Kőnig lemma on an infinite growing sequence of patches.

Nested bumpy corners



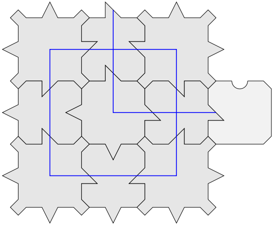
Assume a tiling has an order n bumpy NE-corner.

Nested bumpy corners



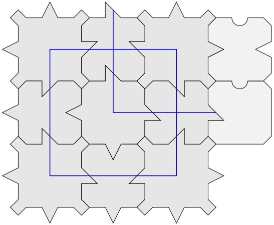
The tiles along the east side can only be arms.

Nested bumpy corners



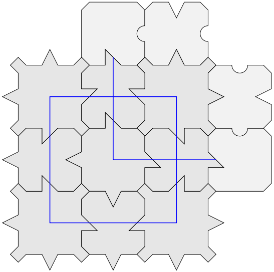
The middle one is S or E. Both have an inwards N-arrow.

Nested bumpy corners



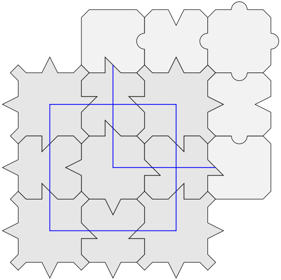
This forces northern arms to be S-arms.

Nested bumpy corners



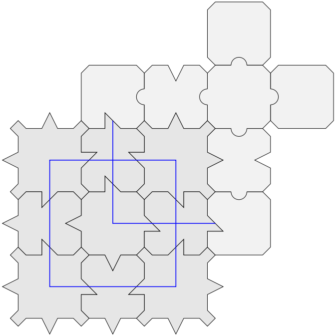
Symmetrically north.

Nested bumpy corners



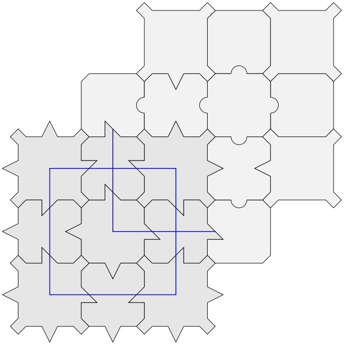
This forces a corner,

Nested bumpy corners



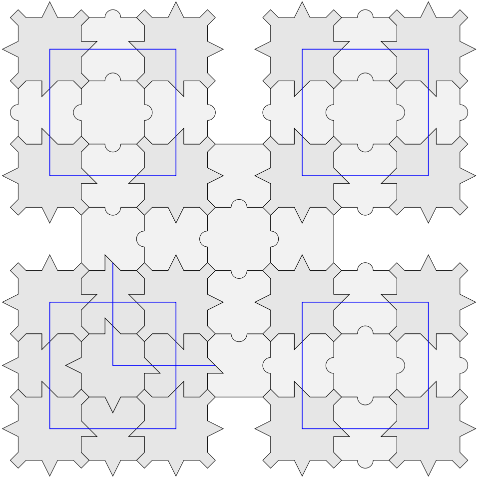
This forces a corner, two arms,

Nested bumpy corners



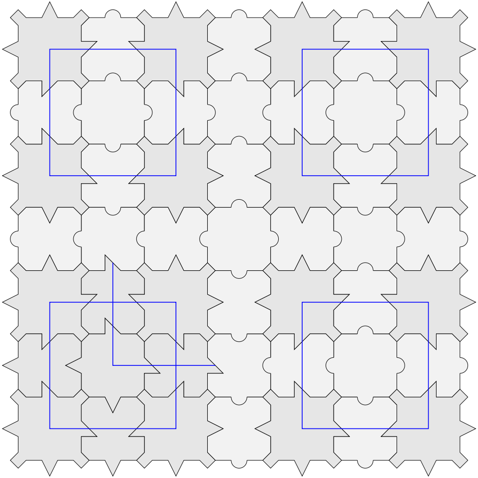
This forces a corner, two arms, and three order 1 bumpy corners.

Nested bumpy corners



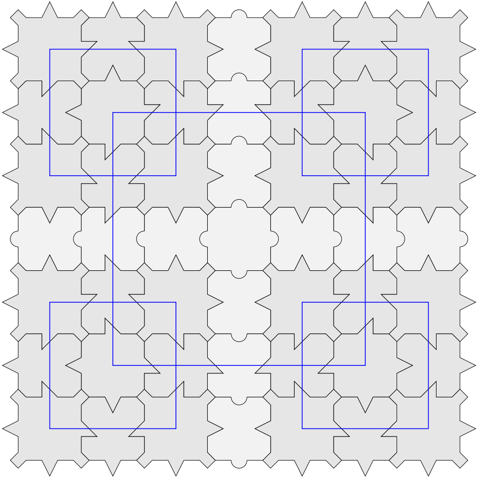
By induction, they appear in order n bumpy corners.

Nested bumpy corners



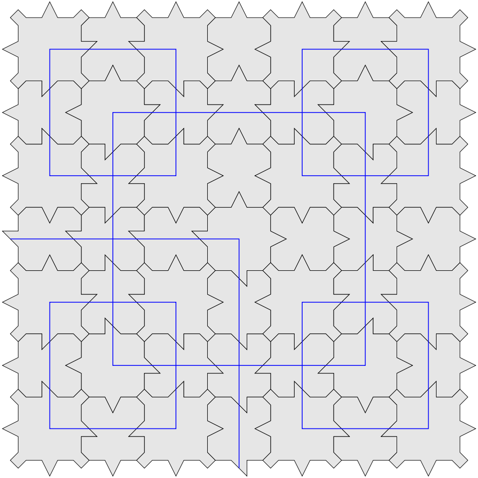
Gaps must be filled by arms oriented away from the central corner.

Nested bumpy corners



This fixes the orientation of all the order n bumpy corners.

Nested bumpy corners



The central corner orientation fixes the arrow types of all the arms.

Aperiodicity

Lemma

The Robinson tileset cannot tile periodically.

Proof:

- tiling $\rightsquigarrow$ infinite sequence of nested bumpy squares;
- such a sequence forms arbitrarily large blue squares;
- no finite translation can leave them all invariant.

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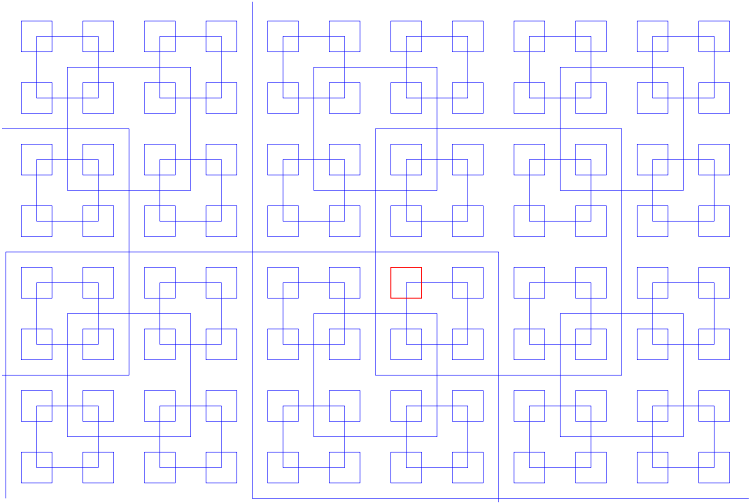
Theorem (Robinson, 1971)

The Robinson tileset is aperiodic.

How many tilings (up to translation)?



Tiling τ and square $\square$ $\rightsquigarrow$ infinite four-arrow sequence $\tau(\square)$.

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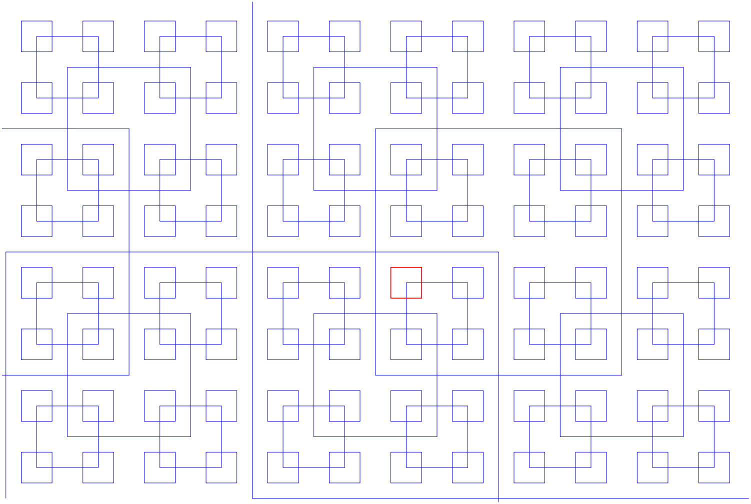

$\tau(\square) =$

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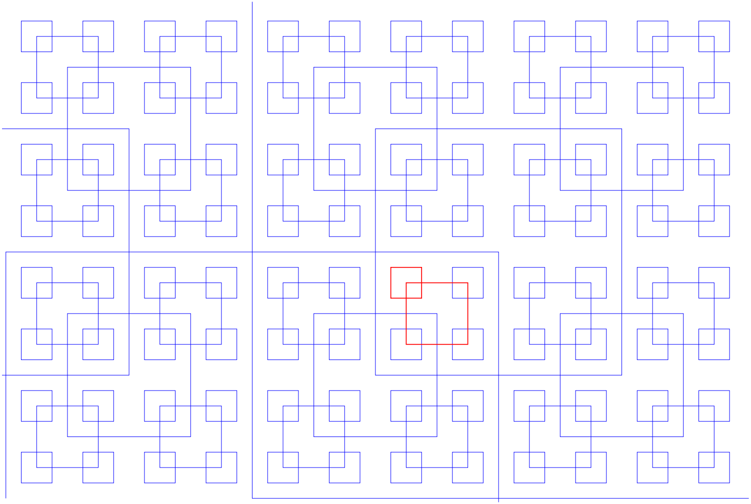

$\tau(\square) = \searrow$

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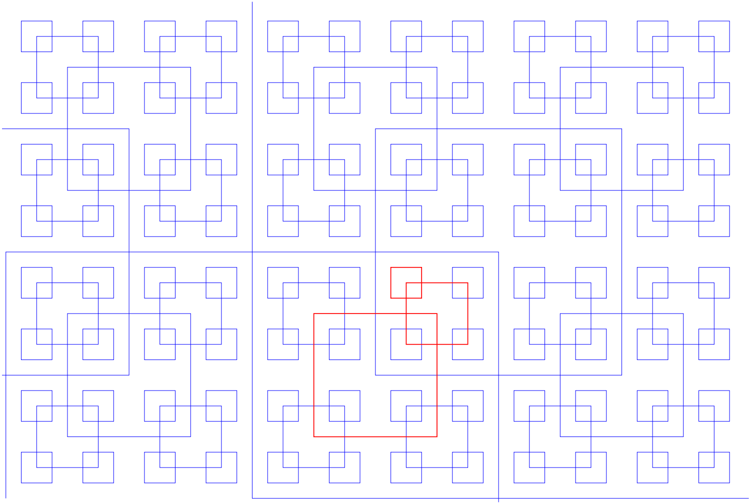

$\tau(\square) = \searrow, \swarrow$

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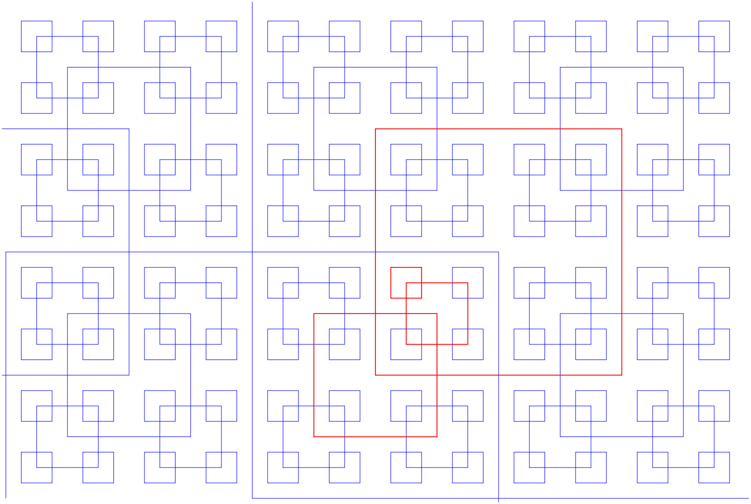

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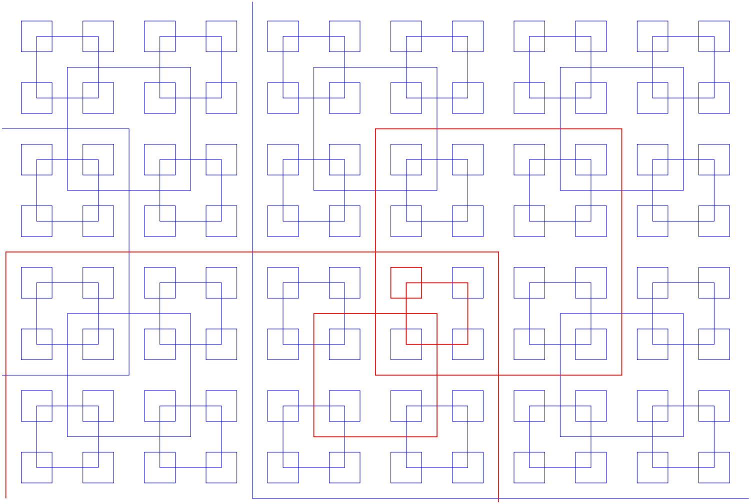

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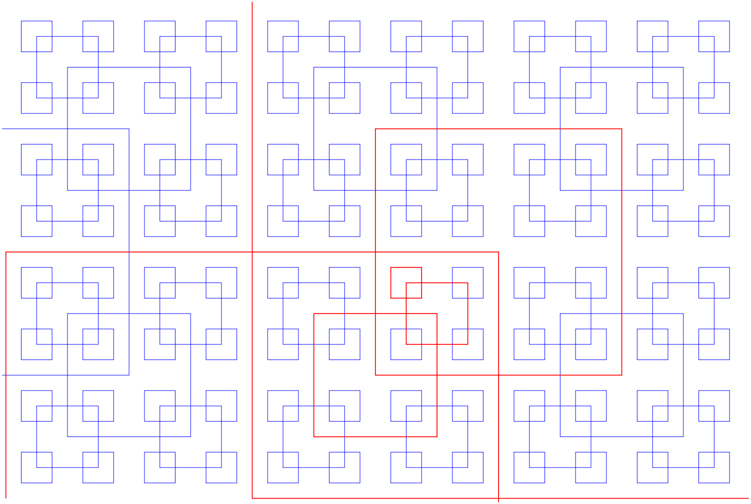

$\tau(\square) = \searrow, \swarrow, \nearrow, \swarrow, \nearrow, \ldots$

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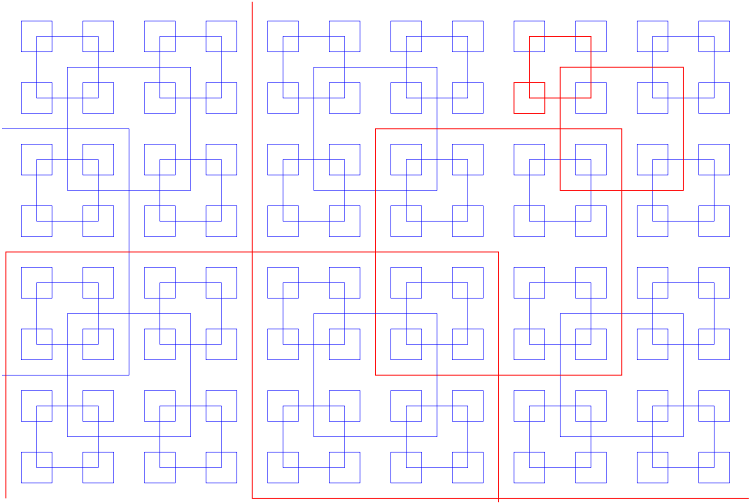
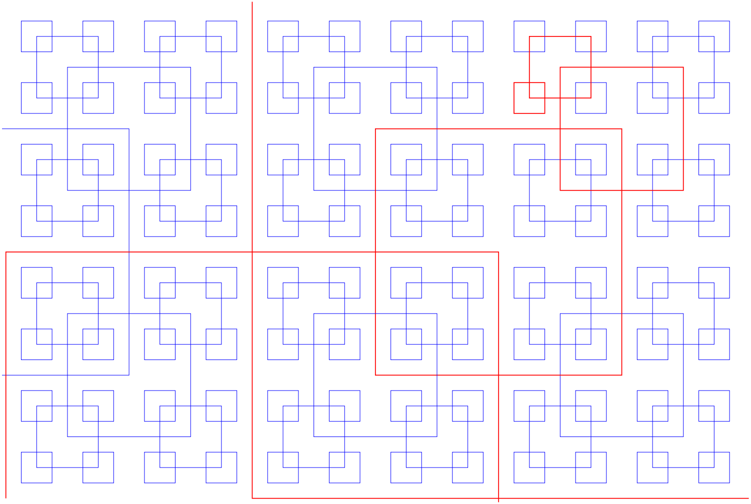


Translated square or tiling $\rightsquigarrow$ sequence with the same *tail end*.

How many tilings (up to translation)?



There are countably many such sequences (*heads* are countable).

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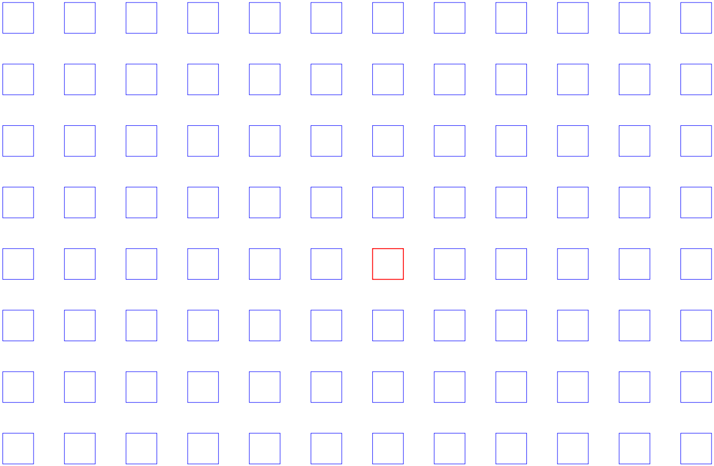

Conversely: square + infinite four-arrow sequence $\rightsquigarrow$ tiling.

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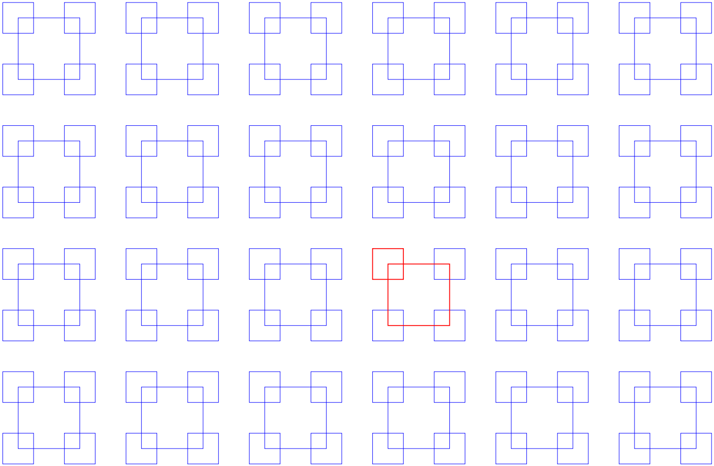
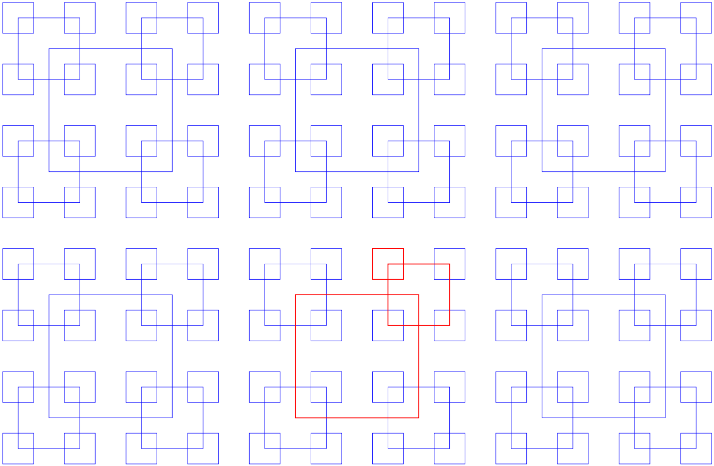


Conversely: square + infinite four-arrow sequence $\rightsquigarrow$ tiling.

How many tilings (up to translation)?



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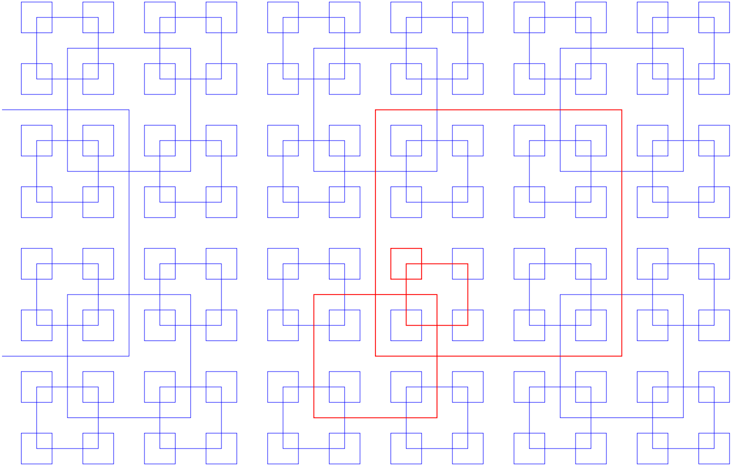

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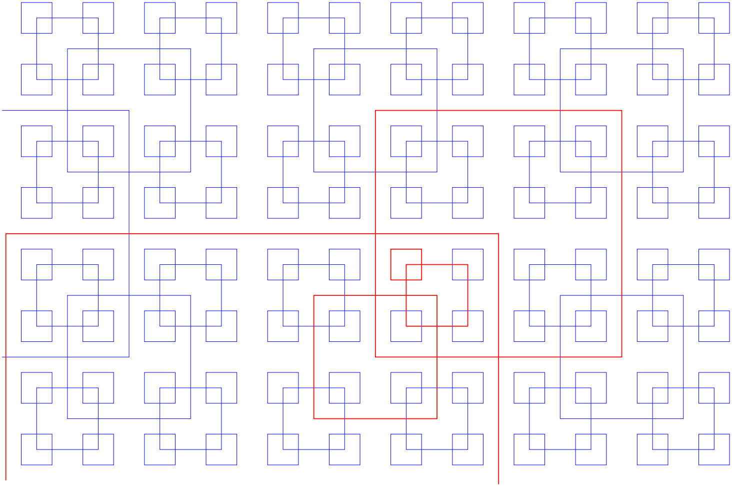

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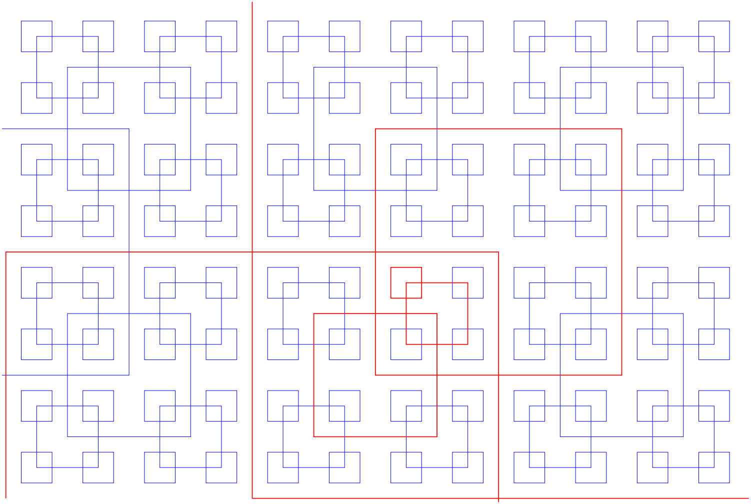

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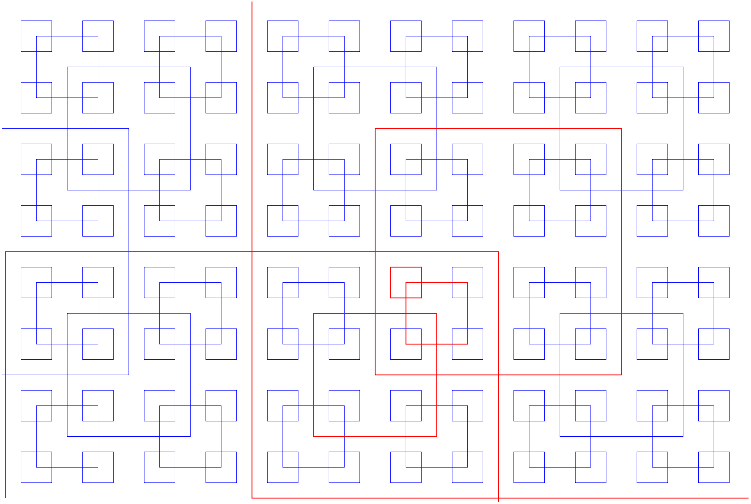
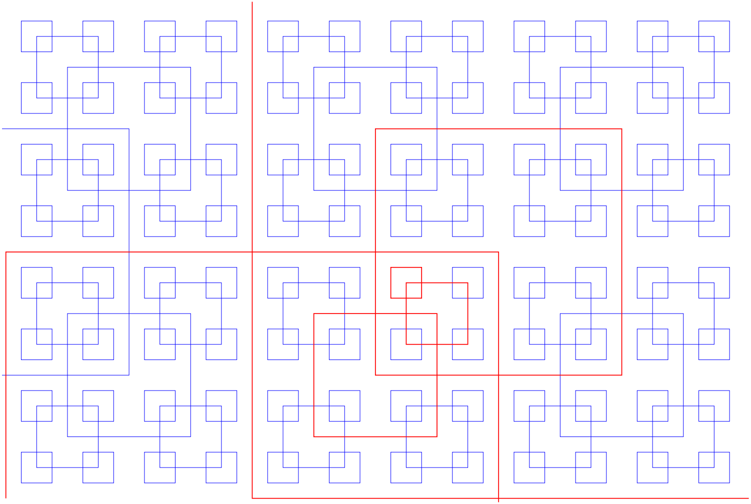


There are uncountably many infinite four-arrow sequences.

How many tilings (up to translation)?



There are thus uncountably many Robinson tilings.

Sketch

Proof of the undecidability of Domino problem:

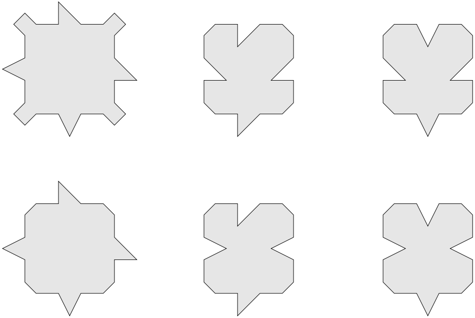
Given a Turing machine M:

- define Wang tiles that simulate M on an empty input;
- convert Robinson tiles in *equivalent* Wang tiles;
- extend Robinson tiles to start a computation in each square.

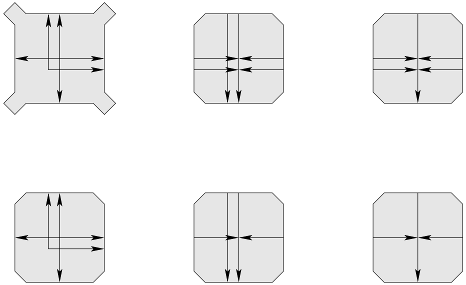
M halts $\Leftrightarrow$ there is a (big enough) square that cannot be tiled.

Halting problem bis undecidable $\Rightarrow$ Domino problem undecidable.

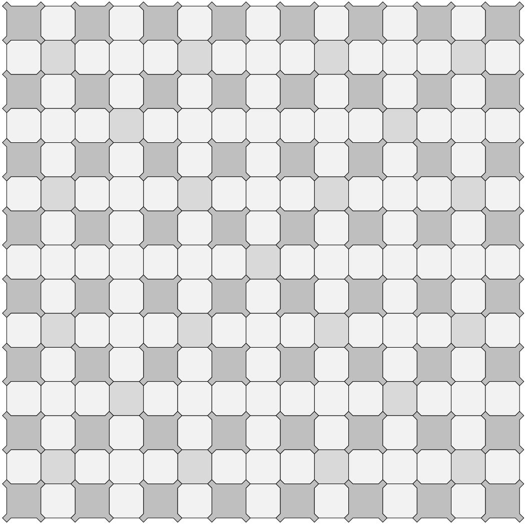
From Robinson tiles to Wang tiles



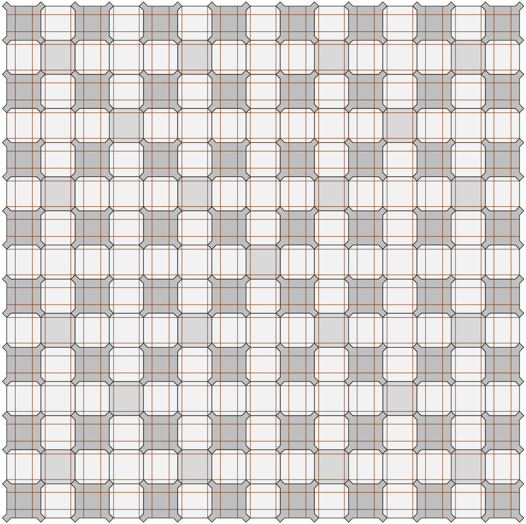
From Robinson tiles to Wang tiles



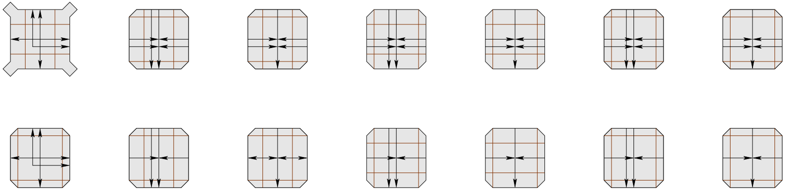
From Robinson tiles to Wang tiles



From Robinson tiles to Wang tiles



From Robinson tiles to Wang tiles



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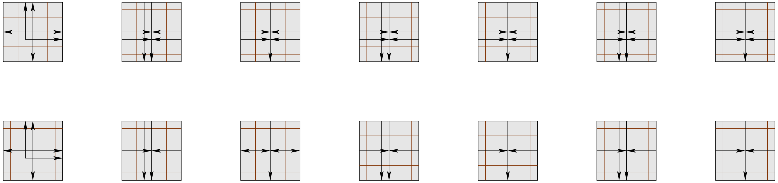

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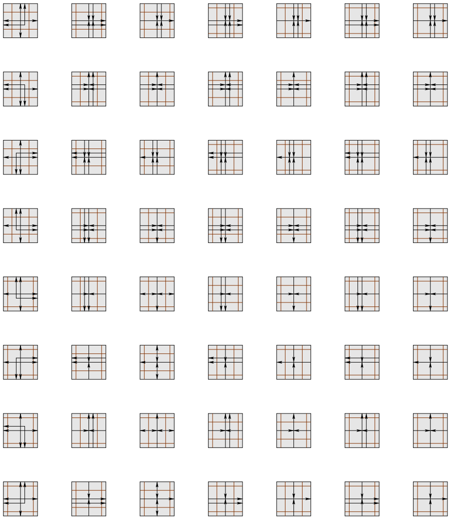

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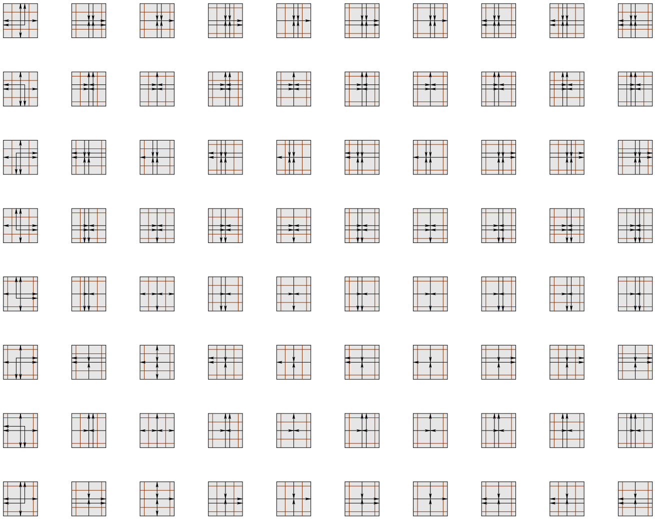

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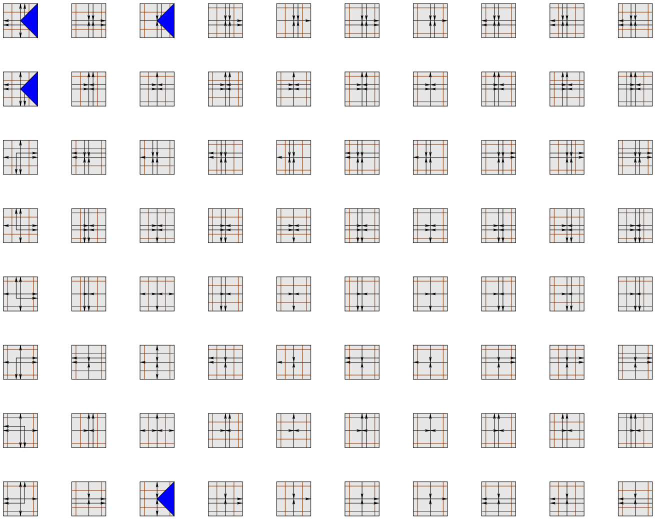

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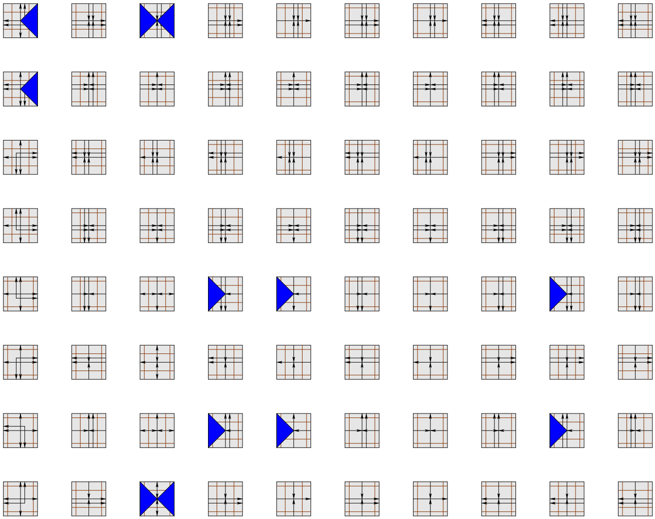

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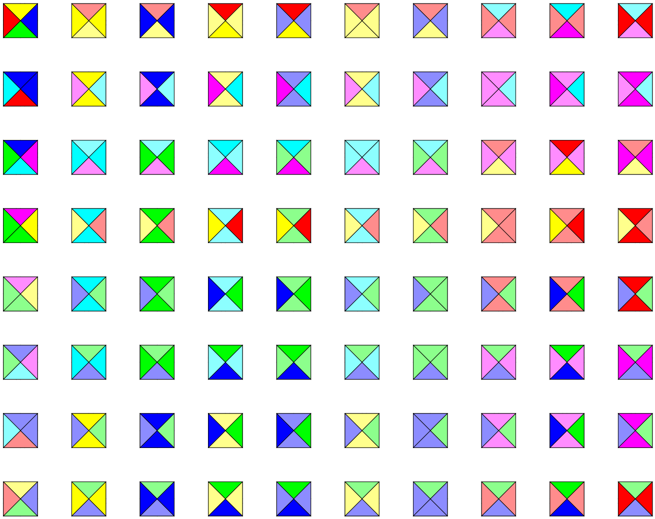

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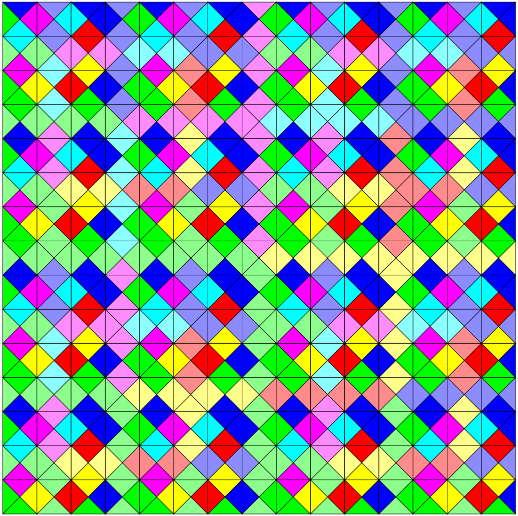

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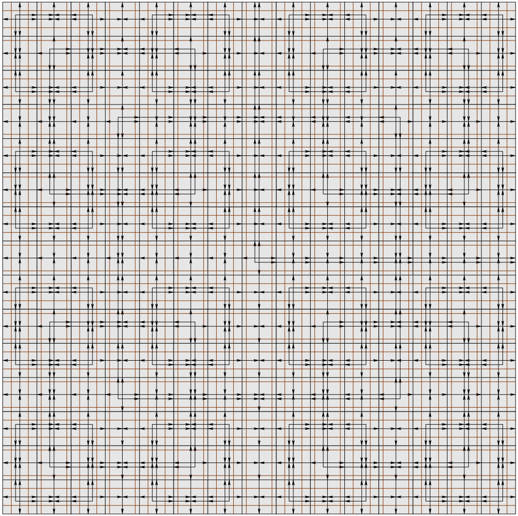

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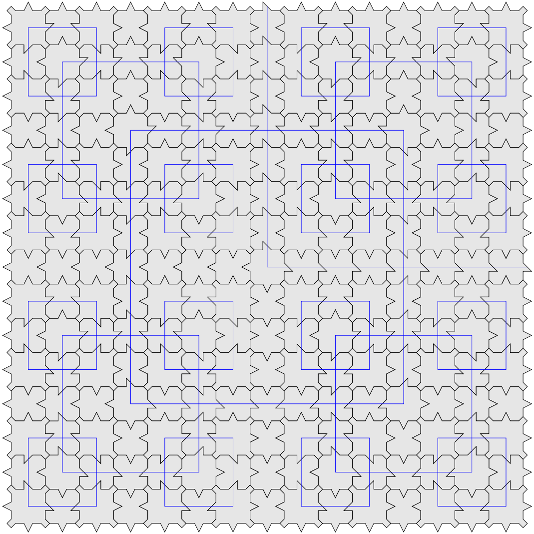
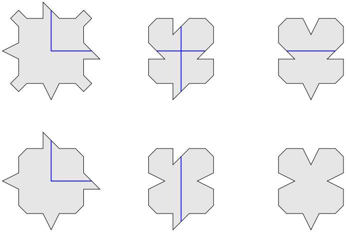


Alterning squares



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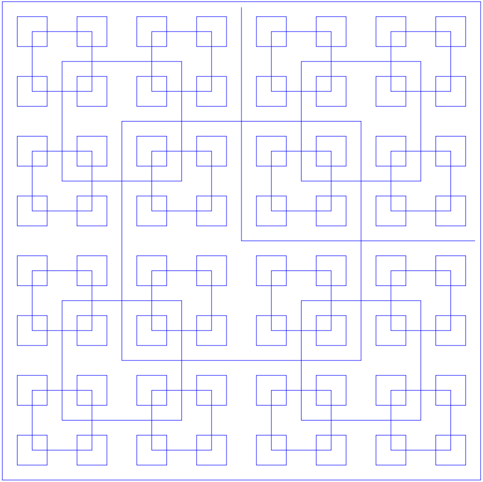

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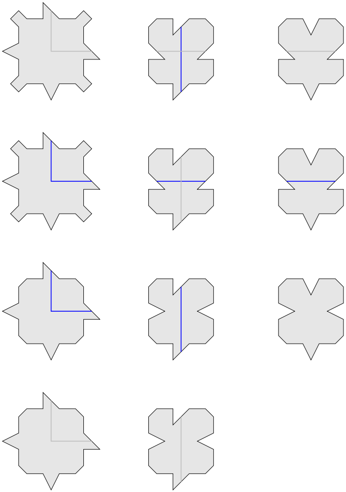

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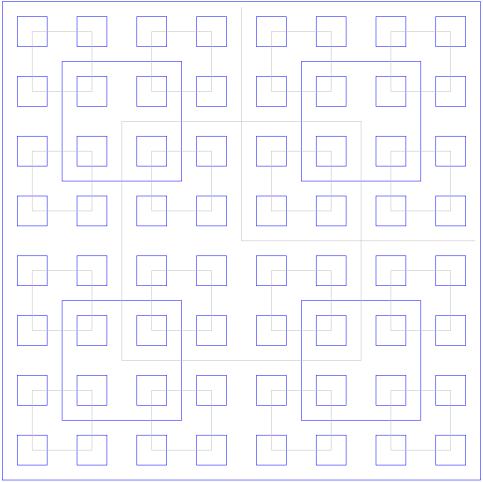

Computing in squares

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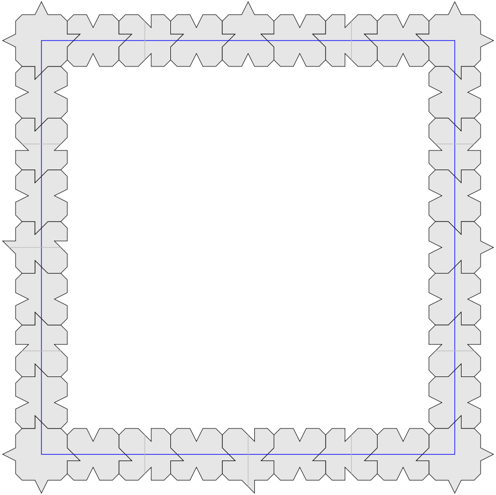
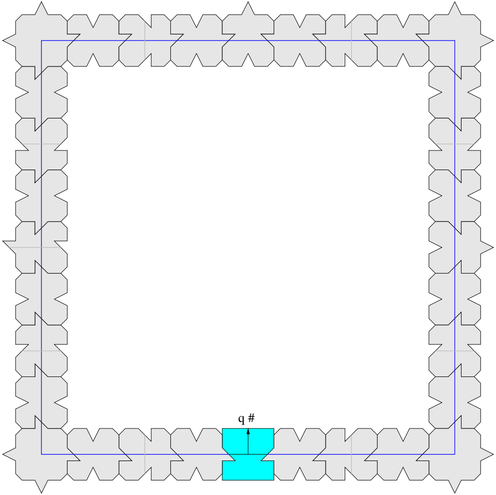


Computing in squares



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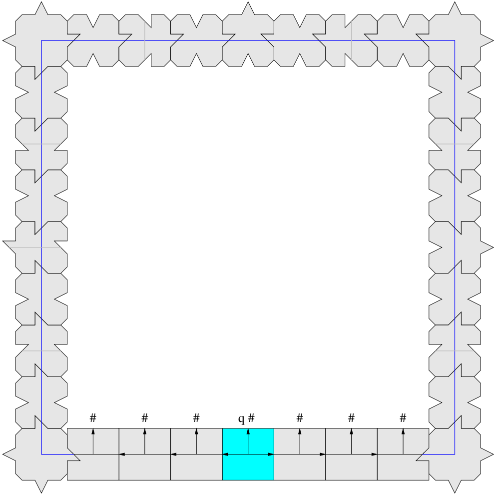

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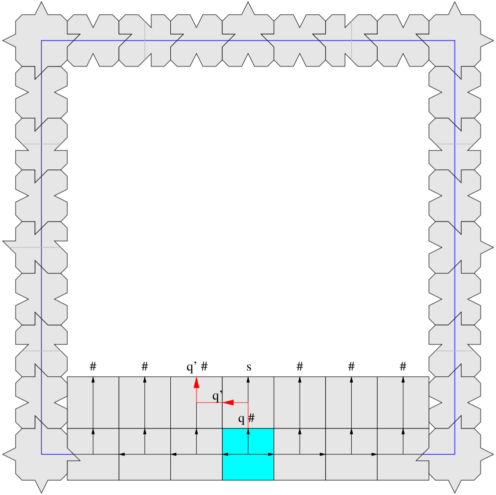
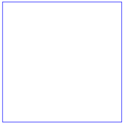
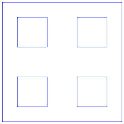


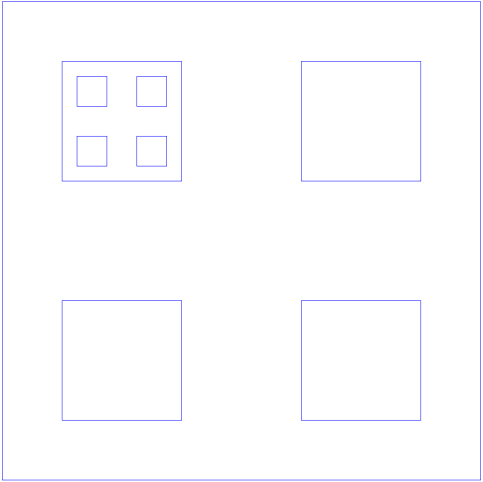
Obstruction signals



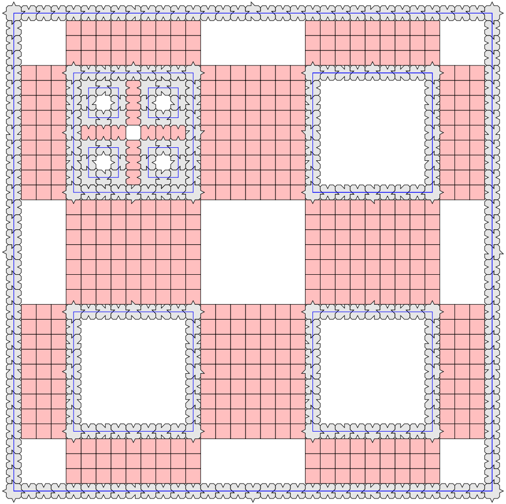
Obstruction signals



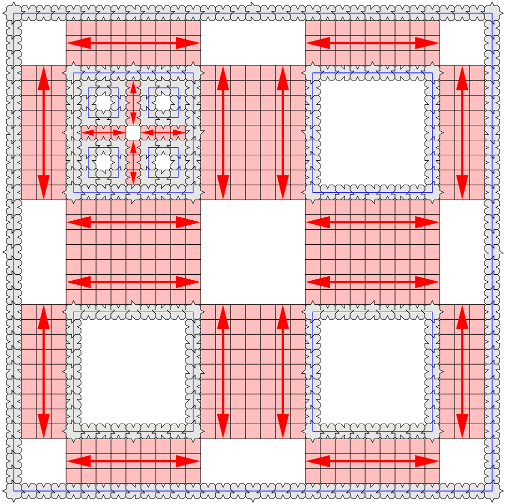
Obstruction signals



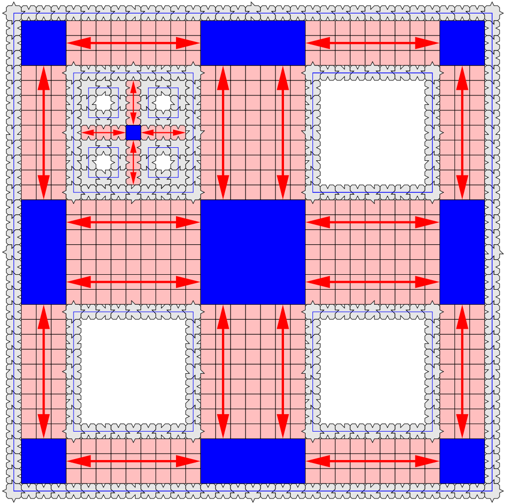
Obstruction signals



Transmission signals



Transmission signals



Some references for this lecture:

Raphael Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Inventiones Mathematicae **12** (1971).

Robert Berger, *The Undecidability of the domino problem*, PhD thesis, Harvard University, 1964.

Hao Wang, *Proving theorems by pattern recognition II*, Bell Systems technical journal **40** (1961).

These slides and the above references can be found there:

`http://www.lif.univ-mrs.fr/~fernique/qc/`