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CIMPA Research School

Tilings and Tessellations

Isfahan University of Technology Isfahan Mathematics House

Foreword

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Chapter 1

Tileability

A *tiling* is a way of covering a given region using a given set of tiles completely and without any overlap. More precisely here we consider the following problem: is a finite figure drawn on a plane grid (*i.e.*, a set of cells of $\mathbb{Z} \times \mathbb{Z}$ tilable with a given set of tiles? We refer the reader to [2] for an introduction to the subject. We first follow a combinatorial approach, then turn to an algebraic approach based on *Gröbner basis*.

1.1 Tiling with dominoes

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical dominioes, that is 1×2 and 2×1 bars ?

The problem can be transformed into the search of a perfect matching (*i.e.*, a set of edges without common vertices that cover all vertices) in a non-directed graph in the following way. The vertices of the graph are the cells of the region we want to tile and there is a non-directed edge between each pair of (horizontally or vertically) adjacent cells. Then there exists a tiling of the region with dominoes if and only if there exist a *perfect matching* in the corresponding graph. The augmenting path algorithm solves the problem in quadratic time.

Theorem 1.1 *A figure of surface n can be tiled by 1×2 and 2×1 tiles, or proved not to be tileable, in quadratic time in n.*

Moreover in the case of tiling with dominoes, we can easily give a proof of the fact that there is no tiling (when it is the case). In order to do that, we transform the graph into a bipartite graph: we color in black and white alls cells of the figure in such a way that two adjacent cells have different colors. Then a domino always cover two cell of different colors. By Hall's theorem [41], there exists a perfect matching in the graph is and only there are the same number of white and black cells in the figure and each subset of k white cells have at least k black neighbors (in other words sufficently many adjacent vertices).

Finally, from a theorem independently due to Kasteleyn [55] and Temperley and Fisher [98] (see Theorem 2.1 of the chapter Random tiling), the number of tiling of a figure by dominoes is known.

The statement of this result in the particular case of tilings of rectangles is the following

Theorem 1.2 (Kasteleyn, Temperley-Fisher) *The number of tilings of a $2m \times 2n$ rectangle with $2mn$ dominoes is equal to*

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

1.2 Tiling with bars

Can we tile a region of $\mathbb{Z}^2$ with horizontal or vertical bars, at least one of them having a length greater than two? The answer basically depends on the existence of holes in the figure.

Tiling a polygon. In [58] the authors present a linear, w.r.t. the surface of the polygon, time algorithm for deciding if a polygon can be tiled with $1 \times \ell$ and $k \times 1$ tiles (and giving a tiling when it exists). This algorithm generalizes a domino tiling algorithm due to Thurston [99] based on ideas of Conway and Lagarias [20] which rely on geometric group theory and the notion of height function.

Theorem 1.3 (Kenyon-Kenyon [58]) *A polygon of surface n can be tiled by $1 \times \ell$ and $k \times 1$ tiles, or proved not to be tileable, in time linear in n.*

A quadratic algorithm for the same problem when the tile types are$\ell \times k$ and $k \times \ell$ is also given in [58].

Tiling an arbitrary region. In the genaral case, the result is completly different.

Theorem 1.4 (Beauquier-Nivat-Rémila-Robson [4]) *Deciding whether a figure can be tiled with $1 \times \ell$ and $k \times 1$ bars is an NP-complete problem as soon as k or ℓ is greater than 2.*

The result is based on a reduction of this problem to a classical NP-complete problem.

1.3 Tiling with rectangles

Tiling a rectangle with another rectangle. When can an $m \times n$ rectangle be tiled with $a \times b$ rectangles (in any orientation)?

For example, can a 5×9 rectangle be tiled with 2×3 rectangles? This is clearly impossible, because each 2×3 rectangle contains 6 cells, while the number of cells in a 5×9 rectangle is 45, which is not a multiple of 6. For a tiling to be possible, the

number of cells of the large rectangle must be divisible by the number of cells of the small rectangle. Is this condition enough?

If we try to tile a 11×20 rectangle with 4×5rectangles, the number of tiles needed is 11. But if we manage to cover the 11 cells of the first column with 4×5 tiles, we will have written 11 as a sum of 4s and 5s, which is impossible. So it may be impossible to cover the first row or column, because either m or n cannot be written as a sum of as and bs.

Is it then possible to tile a 10×15 rect- angle using 1×6 rectangles? 150 is in fact a multiple of 6, and both 10 and 15 can be written as a sum of 1s and 6s. However, this tiling problem is still impossible!

Theorem 1.5 (de Bruijn-Klarner 1969 [63]) *An $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if:*

- *The first row and first column can be covered (i.e., m and n can be expressed in the form $ax + by$ with $x, y \geq 0$).*
- *m or n is divisible by a, and m or n is divisible by b.*

The key ingredient of the proof is the following property: an $m \times n$ rectangle can be tiled with $1 \times b$ rectangles if and only if b divides m or n. The proof is then based on the use of this property together with a coloring argument.

Tiling the square with similar rectangles. Can a square be tiled with finitely many rectangles similar to a $1 \times x$ rectangle (in any orientation)? In other words, can a square be tiled with finitely many rectangles, all of the form $a \times ax$ (where a may vary)?

When $x = \frac{p}{q}$ is a positive rational number, a tiling a the unit square is obtained with pq tiles $\frac{1}{p} \times 1q$ that are similar to the $1\times = \frac{p}{q}$ rectangle some of the tiles.

It can easily be shown that if x is one of the two roots of $5x^2 - 5x + 1 = 0$, that is $x = \frac{5+\sqrt{5}}{10}$ or $x = \frac{5-\sqrt{5}}{10}$, the unit square can be tiled with one $1 \times x$ rectangle together with five $(1 - x) \times x(1 - x)$ rectangles.

But the following result, simultaneoulsy established by Freiling-Rinne [36] and Laczkovich-Szekeres[66], proves that it is not possible to obtain such a kind of tiling when $x = \sqrt{2}$ for example.

Theorem 1.6 (Freiling-Rinne [36], Laczkovich-Szekeres[66], 1995) *Let x be a positive real number. The three following statements are equivalent*

1. *It is possible to tile a square with rectangles. similar (up to a rotation) to the $1 \times x$ rectangle if and only if:*
2. *There exist rational positive numbers $c_1, \ldots, c_n$ such that*

$$c_1 x + \cfrac{1}{c_2 x + \cfrac{1}{\ddots + \frac{1}{c_n x}}} = 1.$$

3. The real x is algebraic and every (complex) conjugate of x has positive real part.

The Statement 2 easily relies tiling and number theory. On one hand, Statement 2 is the natural characterization of the tileabilty of the square by rectangles similar to $1 \times x$ and a tiling of the square can be directly obtained from this continued fraction.

Take a unit square. Cut off a rectangle of ratio (that is, the length of the horizontal side divided by the length of the vertical one) $c_1 x$ from the square by a vertical cut. The remaining part is a rectangle of ratio

$$1 - c_1 x + = \cfrac{1}{c_2 x + \cfrac{1}{\ddots + \frac{1}{c_n x}}}.$$

Now cut off a rectangle of ratio $\frac{1}{c_2 x}$ from the remaining part by a horizontal cut. We get a rectangle of ratio

$$c_3 x + \cfrac{1}{c_4 x + \cfrac{1}{\ddots + \frac{1}{c_n x}}}$$

Continue this process alternating vertical and horizontal cuts. Statement 3 guarantees that after $(n-1)$ step we get a rectangle of ratio $c_n x$. Since all c_k are rational one can chop the tiling into rectangles similar to the $1 \times x$ rectangle.

On the other hand Statement 2 is derived from Statement 3 by a theorem due to Wall (see Theorem A of [100] or Theorem 47.1 of [101]). The proof of the converse is more technical and basically use arguments from algebraic number theory.

Tiling a rectangle with distinct squares. Can a rectangle be tiled with finitely many square of disctinct sizes? When it is possible and the ling contains more one square, the tiling is said to be *perfect*.

An $m \times n$ rectangle, where m and n are integers, can be tiled by $m\dot{n}$ squares. Thus a rectangle with rational side ratio can be tiled by squares. The following result shows that this condition is sufficient:

Theorem 1.7 (Dehn [25] 1903) *A rectangle can be tiled by squares (not necessarily equal) if and only if the ratio of two sides of the rectangle is rational.*

After original complcated proof, many improvements have been made.

In [12], Brooks, Smith, Stone, and Tutte study perfect tilings of squares introducing a nice interpretation in terms of electrical networks. Roughly speaking, the study of squared rectangles is transformed into the study of certain flows of electricity in networks. More precisely, they constructed a directed graph whose vertices are the horizontal lines found in the rectangle. There is one edge for each square, which goes from its top horizontal line to its bottom horizontal line.

Each square of the tiling corresponds to a wire in the network and the current flowing through each wire is equal to the length of the corrsponding square. The "horizontal equations" for the side lengths of the squares are equivalent to the equations for conservation of currrent in this network, and the "vertical equations" are

equivalent to Ohm's law. Then from Kirchhoff's theorem: The flow in each wire is determined uniquely, once the potential difference between some two vertices (up to scaling) is known. Therefore the lengths of the squares can be calculated from determinants formed from the incidence matrix of the network. The correspondence between certains networks and squared rectangles can also be used to find all the squared rectangles of a given (small) order n.

To find tilings of squares instead of rectangles an additional linear equation is needed, stating that the vertical and horizontal side lengths of the rectangle are equal. In terms of the electrical network, this is equivalent to saying that the network has total resistance 1.

Using this technique Duijvestijn [28] with a computer showed that the smallest possible number of squares in a perfect tiling of a square is 21. We refer the reader to `http://www.squaring.net` for a survey and artwork.

Tiling a rectangle with a polyomino. Finally we conclude with the following related question: Given a polyomino P, does there exist a rectangle which can be tiled using copies of P?

First recall that roughly speaking an *undecidable* problem is a decision problem for which it is known to be impossible to construct a single algorithm that always leads to a correct yes-or-no answer, in other words any possible program would sometimes give the wrong answer or run forever without giving any answer.

Theorem 1.8 ([7]) *The general problem of whether a given arbitrary polyomino can tile a rectangle is undecidable.*

1.4 Gröbner bases

The theory of Gröbner bases was developed by Bruno Buchberger in 1965. It is a particular kind of generating set for a polynomial ideal, and it can be considered as a non-linear generalization of Gaussian elimination for linear systems. The method of Gröbner bases is a powerful technique that provides algorithmic solutions to a variety of problems in commutative algebra and algebraic geometry. This section gives a brief overview on the theory of Gröbner bases.

Let K be any field, such as a finite field $K = \mathbb{F}_p$, the rational numbers $K = \mathbb{Q}$, the real numbers $K = \mathbb{R}$, or the complex numbers $K = \mathbb{C}$. We write $R = K[x_1, \ldots, x_n]$ for the ring of polynomials in n variables x_i's with coefficients in K. Given a finite set $F = \{f_1, \ldots, f_k\} \subset R$ of polynomials, the ideal generated by F is the set $\langle F \rangle$ consisting of all polynomial combinations:

$$\langle F \rangle = \{h_1 f_1 + \cdots + h_k f_k \mid h_1, \ldots, h_k \in R\}.$$

We say that this ideal is generated by $\{f_1, \ldots, f_k\}$. By Hilbert's basis theorem, every ideal $I \subset R$ is generated by a finite set of polynomials.

To define the Gröbner basis for an ideal, we introduce the concept of *monomial ordering* on the set of all *monomials* in R. Recall that a monomial in R is a power product of the form $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Let us consider first a univariate polynomial ring $K[x]$. Then, to sort a finite set of monomials, we can consider the degree of the monomials. For example, in the set $\{1, x^2, x^{12}\}$, the monomial x^{12} is the largest one. Below, we generalize this concept to the ring of multivariate polynomials. A term ordering on R is a total order $\prec$ on the set of all monomials satisfying the following two properties:

- It is multiplicative: $x^\alpha \prec x^\beta$ implies $x^\alpha.x^\gamma \prec x^\beta.x^\gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.
- The constant monomial 1 is the smallest monomial; i.e. $1 \prec x^\alpha$ for all $\alpha \in \mathbb{N}^n$.

An example of a monomial ordering is the *lexicographical ordering*. We write $x^\alpha \prec_{lex} x^\beta$ if the left most non-zero entry of $\beta - \alpha$ is positive. If we fix a term ordering $\prec$ on R, then every polynomial $f \in R$ has a unique leading monomial. This is the largest monomial which occurs with non-zero coefficient in the expansion of f, and we denote it by $\mathrm{LM}(f)$. The coefficient of $\mathrm{LM}(f)$ is called the leading coefficient of f and is denoted by $\mathrm{LC}(f)$. The multiplication $\mathrm{LT}(f) = \mathrm{LC}(f)\mathrm{LM}(f)$ is the leading term of f. For instance, let us consider the polynomial ring $K[x_1, x_2]$ and $\prec_{lex}$ on this ring. Then, x_1^2 is the leading monomial of the following quadratic polynomial

$$f = 7\underline{x_1^2} - 3x_2^2 + 5x_1x_2 + 11x_1 + 13x_2 - 17$$

and $\mathrm{LT}(f) = 7x_1^2$. Suppose now that $I \subset R$ is an ideal and $\prec$ is a monomial ordering on R. Then its leading term ideal $\mathrm{LT}(I)$ is the ideal generated by the leading term of all the polynomials in I:

$$\mathrm{LT}(I) = \langle \mathrm{LT}(f) \mid f \in I \rangle.$$

Definition 1.1 *A finite subset $G \subset I$ is called a* Gröbner basis *for I w.r.t. $\prec$ if the leading terms of the elements in G suffice to generate the leading term ideal of I; i.e.*

$$\mathrm{LT}(I) = \langle \mathrm{LT}(g) \mid g \in G \rangle.$$

Note that a finite set $G \subset R$ is called a Gröbner basis if it is a Gröbner basis for $\langle G \rangle$. For example, we consider the polynomial ring $K[x, y, z]$ and the monomial ordering $z \prec_{lex} y \prec_{lex} x$. Let us consider the polynomials $f_1 = x - y - z$, $f_2 = x + y - z^2$ and $f_3 = x^2 + y^2 - 1$. We set $I = \langle f_1, f_2, f_3 \rangle$. We observe that $\mathrm{LT}(f_1) = x$, $\mathrm{LT}(f_2) = x$ and $\mathrm{LT}(f_3) = x^2$. On the other hand, $f_1 - f_2 = -2y - z + z^2$ and its leading term is $-2y$. Since $-2y \notin \langle x, x, x^2 \rangle$, then $\{f_1, f_2, f_3\}$ does not form a Gröbner basis for I. Thus, a generating set of an ideal is not necessarily a Gröbner basis for the ideal. We will see later in this chapter that a Gröbner basis for I is $\{2x - z^2 - z, 2y - z^2 + z, z^4 + z^2 - 2\}$.

Theorem 1.9 *Every ideal has a Gröbner basis w.r.t. a fixed monomial odering.*

Buchberger, in his PhD thesis, proved this theorem. For more details, we refer the reader to [6, 22, 21].

1.5 Buchberger's algorithm

In this section, we describe Buchberger's algorithm for computing a Gröbner basis of a given ideal. It should be noted that polynomial reduction is the corner stone in this algorithm. Suppose that $R = K[x_1, \ldots, x_n]$ and $\prec$ is a monomial ordering on R. Let $F = \{f_1, \ldots, f_k\} \subset R$ and $f \in R$. Then, f is *reducible* to g modulo F, and we write $f \longrightarrow_F g$, if there exist a monomial m appearing in the expression of f and an integer i with $\mathrm{LM}(f_i) \mid m$ and $g = f - auf_i$ for some element $a \in K$ and monomial $u \in R$ so that g does not contain m.

Definition 1.2 *A polynomial g is called a* normal form *of f modulo F if there exists a sequence $g_1, \ldots, g_t$ so that $f \longrightarrow_F g_1 \cdots \longrightarrow_F g_t \longrightarrow_F g$ and g is not reducible modulo F.*

We formalize the computation of a normal form of a polynomial as an algorithm.

Algorithm 1 NormalForm

Input: A set of polynomials $F = \{f_1, \ldots, f_k\}$ and $f \in R$
Output: A normal form of f w.r.t. F
$r := f$;
while r is reducible modulo F **do**
  $r := g$ where $r \longrightarrow_F g$
end while
return(r)

This is the essence of Buchberger's algorithm to compute Gröbner bases.

Example 1.1 *The normal form of a polynomial w.r.t. a given set of polynomials is not, in general, unique. Let us consider the polynomial ring $K[x, y]$ and $y \prec_{lex} x$. Suppose that $f_1 = y^2 - 1, f_2 = xy - 1$ and $f = xy^2 + x^2y + y^2$. Then, if we divide f by f_1, f_2 we get*

$$f = (x + 1)(y^2 - 1) + x(xy - 1) + 2x - 1$$

however if we divide f by f_2, f_1 we have

$$f = (x + y)(xy - 1) + (y^2 - 1) + x + y + 1.$$

Theorem 1.10 *A finite set G is a Gröbner basis iff for each $f \in R$ the normal form of f w.r.t. G is unique.*

An important consequence of this theorem is the ideal membership problem. Suppose that $I \subset R$ is an ideal and $f \in R$. Then, $f \in I$ iff the normal form of f w.r.t. a Gröbner basis of I is zero.

In order to propose an algorithm to compute Gröbner bases, let us look at some obstructions that may happen when a finite set cannot be a Gröbner basis. Let us consider the ring $K[x, y]$ and $y \prec_{lex} x$. Further, let $f_1 = xy - x, f_2 = x^2 - y$

and $I = \langle f_1, f_2 \rangle$. Then, we have $xf_1 - yf_2 = y^2 - x^2 \in I$. The normal form of this polynomial w.r.t. $\{f_1, f_2\}$ is $y^2 - y$ which lies in I and it has the new leading monomial $y^2 \notin \langle xy, x^2 \rangle$. Based on this construction, we describe the Buchberger's algorithm to construct Gröbner basis. In doing so, we define first a new fuction called *Spolynomial* to cancel the leading monomials.

Definition 1.3 *Let $\prec$ be a monomial ordering and f and g be two polynomials. Then, the* Spolynomial *of f and g is defined to be*

$$\mathrm{Spol}(f, g) = \frac{x^\alpha}{\mathrm{LT}(f)} f - \frac{x^\alpha}{\mathrm{LT}(g)} g$$

where $x^\alpha = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$.

For example, keeping the above notations, $\mathrm{Spol}(f_1, f_2) = xf_1 - yf_2 = y^2 - x^2$. We state now Buchberger's criterion which leads us to Buchberger's algorithm.

Theorem 1.11 *(Buchberger's criterion) A finite set $G \subset R$ of non-zero polynomials is a Gröbner basis iff the normal form of* $\mathrm{Spol}(g_i, g_j)$ *w.r.t. G is zero for each $g_i, g_j \in G$.*

Algorithm 2 BUCHBERGER

Input: A set of polynomials $F = \{f_1, \ldots, f_k\}$
Output: A Gröbner basis for $\langle f_1, \ldots, f_k \rangle$
$G := F$;
$P := \{\{f, g\} \mid f, g \in G\}$;
while $P \neq \{\}$ **do**
  select and remove $\{f, g\}$ from P;
  $r :=$NORMALFORM$(\mathrm{Spol}(f, g), G)$;
  if $r \neq 0$ **then**
    $P := P \cup \{\{r, h\} \mid h \in G\}$;
    $G := G \cup \{r\}$
  end if
end while
return(G)

This algorithm has been implemented in many systems such as COCOA, MACAULAY2, MAGMA, MAPLE, MATHEMATICA, or SINGULAR.

Example 1.2 *Let us consider the monomial ordering $y \prec_{lex} x$ on $K[x, y]$. Let $f_1 = xy - x$, $f_2 = x^2 - y$ and $I = \langle f_1, f_2 \rangle$. We set $G = \{f_1, f_2\}$ and $P = \{\{f_1, f_2\}\}$. We select and remove this pair, and we let r the normal form of* $\mathrm{Spol}(f_1, f_2)$ *w.r.t. G which is equal to $y^2 - y$. Then, we have $P = \{\{r, f_1\}, \{r, f_2\}\}$ and $G = \{f_1, f_2, r\}$. We observe that the Spolynomial of each of these two pairs reduces to zero, and therefore G is a Gröbner basis for I.*

At the end of this section, we illustrate an application of Gröbner bases to simplification of expressions. As we already mentioned, the normal form of a polynomial w.r.t. a given Gröbner basis is unique. This can be used to simplify mathematical expressions with respect to polynomial relations. In the following, we describe an example from the Dutch Mathematics Olympiad of 1991. Let a, b, c be real numbers such that $a+b+c = 3, a^2+b^2+c^2 = 9$ and $a^3+b^3+c^3 = 24$. Compute $a^4+b^4+c^4$. We compute first a Gröbner basis of the ideal

$$I = \langle a+b+c-3, a^2+b^2+c^2-9, a^3+b^3+c^3-24\rangle$$

w.r.t. the monomial ordering $c \prec_{lex} b \prec_{lex} a$. This basis is equal to

$$G = \{a+b+c-3, b^2+c^2-3b-3c+bc, c^3-3c^2+1\}.$$

and the normal form of $a^4+b^4+c^4$ is 69. This shows that $a^4+b^4+c^4-69 \in I$. Now, if we set $a+b+c = 3, a^2+b^2+c^2 = 9$ and $a^3+b^3+c^3 = 24$ in I then it becomes zero, and thus any polynomial in I would be zero. Thereby $a^4+b^4+c^4 = 69$.

1.6 Gröbner bases and tilings

In this section, we review the application of Gröbner bases in tiling problems. The main reference for this section is [10]. The main problem in this section is as follows: Given a *polyomino* (a polyomino is a finite set of unit sqares) P and a finite set F of polyominoes (the tiles). $\mathbb{Z}$-tiling problem consists of a finite number of translated tiles to cover P so that the sum of signs on P at each cell is $+1$. A *celle* in the square lattice is the set $c(i,j) = \{(x,y) \mid i \leq x < i+1, j \leq y < j+1\}$. Thus we call lable $c(i,j)$ by its lower left corner; the point (i,j). Thereby, we can associate to $c(i,j)$ the monomial $x^i y^j$, and we denote it by $\mathcal{P}_{c(i,j)}$. Since each polyomino (or tile) is the union of some celles, thus we can give the following definition.

Definition 1.4 *If the polyomino (or tile) T is a union of celles $c(i,j)$ for $(i,j) \in \Lambda$, then we define the asssociate polynomial of T to be $\mathcal{P}_T = \sum_{(i,j)\in\Lambda} \mathcal{P}_{c(i,j)}$.*

Fig. 1.6 illustrates this.

If we would like to cover a given polyomino P by a finite set F of tiles, then we shall consider all rotations of the tiles in F, say $\tilde{F}$. Now, for each tile $A \in \tilde{F}$, we consider $\mathcal{P}_A$. Thus, the $\mathbb{Z}$-tiling problem is equivalent to the fact that whether $\mathcal{P}_P \in \langle \mathcal{P}_A \mid A \in \tilde{F}\rangle$. However, we shall note that this ideal lies in the polynomial ring $\mathbb{Z}[x,y]$. Thus, to check this ideal membership we need to define and compute the Gröbner bases over the ring of integers. In the following, we give an overview on this topic. We mimic the constructions of Section 1.4 as much as we can. We keep the notations of the mention section. Let $R = \mathbb{Z}[x_1, \ldots, x_n]$, $\prec$ be a monomial ordering and $I \subset R$ be an ideal.

Definition 1.5 *A finite subset $G \subset I$ is called a* Gröbner basis *for I w.r.t. $\prec$ if*

$$\mathrm{LT}(I) = \langle \mathrm{LT}(g) \mid g \in G\rangle.$$

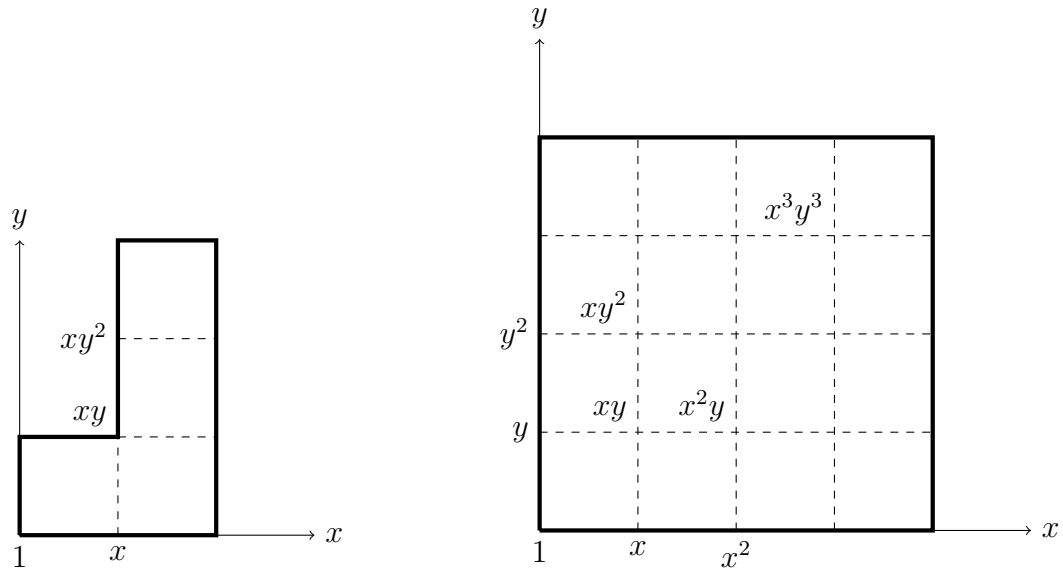


Figure 1.1: Left, the tile associated with the polynomial $1+x+xy+xy^2$. Right, a 4×4 square polyomino P whose associate polynomial is $\mathcal{P}_P = (1+x+x^2+x^3)(1+y+y^2+y^3)$.

For example, the set $\{2x\} \subset K[x]$ is a Gröbner basis for the ideal $I = \langle 2x, 3x\rangle$, however, it is not a Gröbner basis when we consider I in $\mathbb{Z}[x]$. Indeed, we note that $x \in I$ but $2x$ does not divide x in $\mathbb{Z}[x]$. As a more challenging example, let $I = \langle 4x-1, 6y+1\rangle \subset \mathbb{Z}[x, y]$ and $y \prec_{lex} x$. Then, we can see easily that if we consider this ideal in a polynomial ring over a field, then $\{4x-1, 6y+1\}$ is a Gröbner basis for I. However, that does not hold in $\mathbb{Z}[x, y]$. Indeed, we observe that $\mathrm{Spol}(4x-1, 6y+1) = -2x - 3y$ and $-2x \notin \langle 4x, 6y\rangle$. To state the Buchberger like algorithm, we need to define a new concept of reduction.

Definition 1.6 *Let us consider the polynomials $f, g \in R$ and $F = \{f_1, \ldots, f_k\} \subset R$. We say that f reduces to g modulo F and we write $f \longrightarrow_F g$ if $g = f - (c_1 m_1 f_1 + \cdots + c_k m_k f_k)$ for $c_i \in \mathbb{Z}$ and monomials $m_1, \ldots, m_k$ where $m_i \mathrm{LM}(f_i) = \mathrm{LM}(f)$ for all i such that $c_i \neq 0$ and $\mathrm{LT}(f) = c_1 m_1 \mathrm{LT}(f_1) + \cdots + c_k m_k \mathrm{LT}(f_k)$.*

Based on this definition, we can define the normal form of a polynomial modulo a given set of polynomials. Finally, we shall note that the normal form algorithm, Buchberger's criterion, Buchberger's algorithm and all the results in Section 1.4 hold for the polynomial ring over integers. Specially, using Gröbner bases over $\mathbb{Z}$ we can test the membership problem, and answer the $\mathbb{Z}$-tiling problem.

Chapter 2

Random Tilings

2.1 Dimer model and random tilings

The dimer model was introduced in the physics and chemists communities to represent the adsorption of di-atomic molecules on the surface of a crystal. It is first mentioned in a paper by Fowler and Rushbrooke [35] in 1937. Mathematicians studied related questions as for example the enumeration of plane partitions by Mac Mahon [74], the understanding of geometric and combinatorial properties of tilings of regions of the plane by dominos or rhombi. To the best of our knowledge, the latter problem was first introduced in a paper by David and Tomei [23]. A major breakthrough was achieved in the paper [99] Thurston, where the author interprets rhombus tilings as 2-dimensional interfaces in a 3-dimensional space. The goal of this section is to define the dimer model and equivalent random tilings.

2.1.1 The dimer model

A *graph* $G = (V, E)$ consists of:

- a set of *vertices* V,
- a set of (unoriented) *edges* E, where an *edge* is a pair of vertices $e = \{v, w\}$. The unoriented edge e is also denoted vw.

Vertices can be represented by points, and edges, as lines (or simple curves) connecting pairs of points. In what follows, we shall simply denote a graph by G.

We say that a vertex v is *incident* to an edge e if v is one of the vertices in the pair representing e. We say that v and w are *adjacent* if $\{v, w\}$ is an edge of G.

A *dimer configuration* of a graph G, also known as a *perfect matching*, denoted by D, is a subset of E such that every vertex $v \in V$ is incident to exactly one edge of D. Let $\mathcal{D}(G)$ denote the set of dimer configurations of a graph G.

Figure 2.1 gives an example of a dimer configuration when the graph G is a finite subgraph of the honeycomb lattice $\mathbb{H}$.

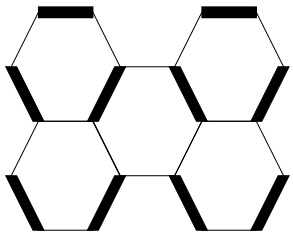


Figure 2.1: Dimer configuration of a subgraph of the honeycomb lattice.

Since a dimer of a perfect matching covers exactly two vertices, a necessary condition for the existence of a dimer configuration of a graph G is that the number of vertices of G is even.

In general, it is a difficult problem to compute the number of dimer configurations of a graph. We will see that if the graph is *planar*, *i.e.*, if the graph (vertices and edges) can be drawn in the plane in such a way that edges do not intersect except at vertices, this problem is much easier.

Dimer configurations represent di-atomic molecules adsorbed on the surface of a crystal. A given dimer configuration has a certain probability of occurring, where this probability is given by the Boltzmann measure. Suppose that the graph G is finite, and suppose that edges are assigned a positive *weight function* ν, that is every edge e of G has weight $\nu(e)$. The *Boltzmann measure* μ is a probability measure on the set of dimer configurations $\mathcal{D}(G)$, defined by:

$$\forall\, D \in \mathcal{D}(G), \quad \mu(D) = \frac{\nu(D)}{Z(G)},$$

where $\nu(D)$ is the *weight* of the dimer configuration defined as $\nu(D) = \prod_{E \in D} \nu(E)$. The term $Z(G)$ is the normalizing constant known as the *partition function*. It is the weighted sum of dimer configurations, that is,

$$Z(G) = \sum_{D \in \mathcal{D}(G)} \nu(D).$$

When $\nu \equiv 1$, the partition function counts the number of dimer configurations of the graph G, and the Boltzmann measure is simply the uniform measure on the set of dimer configurations.

2.1.2 Dimers on planar graphs and tilings

Let G be an infinite, connected planar graph, embedded in the plane. Cutting along the edges, we obtain several pieces that are homeomorphic to disks. These pieces are known as the *faces*[1] of the graph G.

[1]There may be several ways to embed G in the plane. Faces of G depend on the graph structure of G *and* on the choice of embedding.

From the embedded graph G, one define a new graph $G^* = (V^*, E^*)$, known as the *dual graph* of G: vertices of G^* correspond to faces of G, and there is a *dual edge* connecting two *dual vertices* if the corresponding faces are bordered by the same edge. There is a bijection between edges of G and edges of G^*. The dual of G^* can be identified with G. When the graph G is finite, we consider a version of the dual graph modified along the boundary, see Figure 2.2. With a slight abuse of notation, we also denote this modified dual by G^*.

The tiling model on the dual graph G^* is defined as follows. A *tile* of G^* is a polygon consisting of two adjacent inner faces of G^* glued together. A *tiling* of G^* is a covering of the graph G^* with tiles, such that there are no holes and no overlaps. Figure 2.2 gives an example of a tiling of a finite subgraph of the triangular lattice $\mathbb{T}$, the dual graph of the honeycomb lattice. Tiles of the triangular lattice are 60°-rhombi, and are also known as *lozenges* or *calissons*.

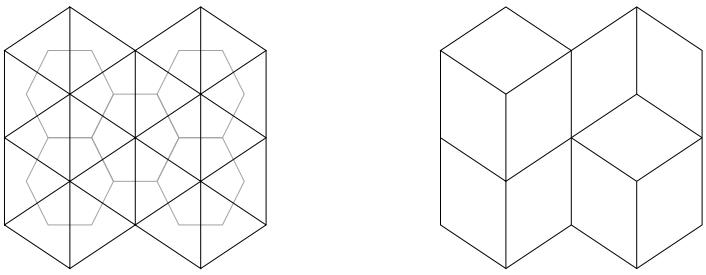


Figure 2.2: Dual graph of a finite subgraph of the honeycomb lattice (left). Tiling of this subgraph (right).

Another classical example is the tiling model on the graph $\mathbb{Z}^2$, the dual of the graph $\mathbb{Z}^2$. Tiles are made of rectangles consisting of two adjacent squares, and are known as *dominos*.

Dimer configurations of the graph G are in bijection with tilings of the graph G^* through the following correspondence, see also Figure 2.3: dimer edges of perfect matchings connect pairs of adjacent faces forming tiles of the tiling. As an exercise, prove that this indeed defines a bijection.

Using the bijection between dimers and tilings, the Boltzmann measure can be seen as a probability measure on tilings of the dual graph G^*.

2.1.3 Height function

By means of the height function, Thurston [99] interprets lozenge tilings of the triangular lattice as discrete surfaces in a rotated version of $\mathbb{Z}^3$ projected onto the plane. He gives a similar interpretation of domino tilings of the square lattice. This approach can be generalized to dimer configurations of bipartite graphs using flows. This yields an interpretation of the dimer model on a bipartite graph as a random interface model in dimensions $2 + 1$, and offers more insight into the model. In this section we exhibit Thurston's construction of the height function on lozenge tilings.

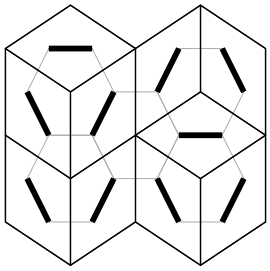


Figure 2.3: Bijection between dimer configurations of the graph G and tilings of the graph G^*.

Faces of the triangular lattice $\mathbb{T}$ can be colored in black and white, so that black faces (resp. white ones) are only adjacent to white ones (resp. black ones). This is a consequence of the fact that its dual graph, the honeycomb lattice, is bipartite. Orient the black faces counterclockwise, and the white ones clockwise, see Figure 2.4 (left). Consider a finite subgraph X of $\mathbb{T}$ which is tileable by lozenges, and a lozenge tiling T of X. Then the height function h^T is an integer valued function on vertices of X, defined inductively as follows:

- Fix a vertex v_0 of X, and set $h^T(v_0) = 0$.
- For every boundary edge uv of a lozenge, $h^T(v) - h^T(u) = +1$ if the edge uv is oriented from u to v, implying that $h^T(v) - h^T(u) = -1$ when the edge uv is oriented from v to u.

The height function is well defined, in the sense that the height change around any oriented cycle is 0. An example of computation of the height function is given in Figure 2.4 (right).

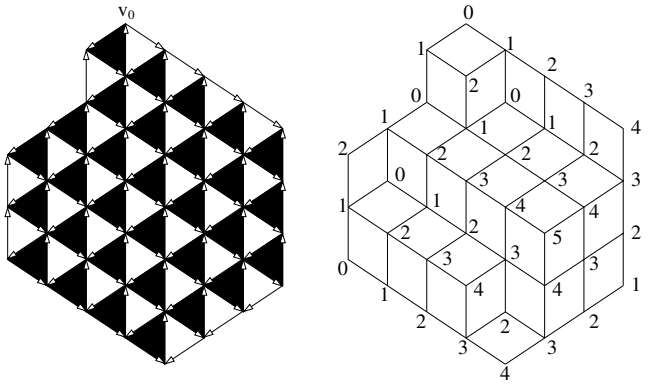


Figure 2.4: Orientation of faces of the triangular lattice (left). Height function corresponding to a lozenge tiling (right).

As a consequence, lozenge tilings are interpreted as stepped surfaces in $\widetilde{\mathbb{Z}^3}$ projected onto the plane, where $\widetilde{\mathbb{Z}^3}$ is $\mathbb{Z}^3$ rotated so that diagonals of the cubes are orthogonal to the plane. The height function is then simply the "height" of the surface (*i.e.* third coordinate). This construction gives a mathematical sense to the intuitive feeling of cubes sticking in or out, which strikes us when watching a picture of lozenge tilings.

Height functions characterize lozenge tilings as stated by the following lemma.

Lemma 2.1 *Let X be a finite simply connected subgraph of the triangular lattice $\mathbb{T}$, which is tileable by lozenges. Let h be an integer valued function on the vertices of X, satisfying:*

- *$h(v_0) = 0$, where v_0 is a fixed vertex of X.*
- *$h(v) - h(u) = 1$ for any boundary edge uv of X oriented from u to v.*
- *$h(v) - h(u) = 1$ or -2 for any interior edge uv of X oriented from u to v.*

Then, there is a bijection between functions h satisfying these two conditions, and tilings of X.

Thurston uses height functions to determine whether a subgraph of the triangular lattice can be tiled by lozenges. More details are to be found in the paper [99].

2.2 Kasteleyn/Temperley and Fisher's theorem

The explicit computation of the partition function is due to Kasteleyn [56, 57] and independently to Temperley and Fisher [98]. The goal of this section is to state their theorem in the case where the graph G is planar and bipartite. To this purpose, we first introduce the Kasteleyn matrix of the graph G. Note that the statement in the planar, non-bipartite case, is similar in spirit, but involves a Pfaffian instead of a determinant.

A graph $G = (V, E)$ is *bipartite* if the set of vertices V can be split into two subsets $W \cup B$, where W denotes white vertices, B black ones, and vertices in W are only adjacent to vertices in B. We suppose that $|W| = |B| = n$.

An orientation of the edges the graph G is said to be *clockwise odd* if all cycles bounding faces are clockwise odd, meaning that when traveling clockwise along such a cycle, there is an odd number of co-oriented edges. Such an orientation exists when the graph is embedded in the plane, and we fix one.

Now, label the white vertices $w_1, \dots, w_n$ and the black ones $b_1, \dots, b_n$. Then the corresponding *oriented, weighted, adjacency matrix* is the $n \times n$ matrix K whose lines are indexed by white vertices, whose columns are indexed by black ones, and whose entry $K(w_i, b_j)$ is:

$$K(w_i, b_j) = \begin{cases} \nu(w_i b_j) & \text{if } w_i \sim b_j, \text{ and } w_i \to b_j \\ -\nu(w_i b_j) & \text{if } w_i \sim b_j, \text{ and } w_i \leftarrow b_j \\ 0 & \text{if the vertices } w_i \text{ and } b_j \text{ are not adjacent.} \end{cases}$$

The matrix K is known as a *Kasteleyn matrix*. Then, Kasteleyn [56] and Temperley and Fisher [98] prove the following theorem.

Theorem 2.1 ([56, 98]) *Let G be a finite planar graph. Let K be a Kasteleyn matrix of G. Then*

$$Z(G) = |\det K| .$$

In particular, when the weight function $\nu \equiv 1$, this formula gives the number of perfect matchings of the graph G.

2.3 Explicit expression for natural probability measures on dimers

From Kasteleyn's theorem and Jacobi's formula, one can deduce and explicit expression for the Boltzmann measure μ when the graph is finite. This is the subject of Section 2.3.1. Then, in Section 2.3.2, we give a hint at what happens in the case where the graph G is infinite.

2.3.1 The finite case

Let us first state Jacobi's formula. For a matrix $A = (a_{i,j})_{1\leq i,j\leq n}$, and I, J two subsets of $[1..n]$, we denote by A_I^J the sub-matrix extracted from A with rows (resp. columns) indexed by I (resp. J).

Theorem 2.2 (Jacobi's formula) *Let $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$ two subsets of $[1..n]$ of size k. If A is invertible,*

$$\det A_I^J = (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} \det(A) \det(A^{-1})_{\bar{I}}^{\bar{J}}.$$

Using Jacobi's formula, it is possible to write a compact expression for the probability that a given edges appear in a dimer configuration sampled according to the Boltzmann measure μ.

Theorem 2.3 ([60]) *Suppose that the graph G is planar and bipartite, with at least one dimer configuration. Let $E = \{e_1, \dots, e_k\}$ be a subset of edges, such that for all j, $e_j = \{w_j, b_j\}$. Then:*

$$\mathbb{P}(\textit{edges of } E \textit{ are dimers}) = \left(\prod_{j=1}^{k} K(w_j, b_j) \right) \times \det_{1\leq i,j\leq k} K^{-1}(b_i, w_j).$$

2.3.2 A hint at infinite volume Gibbs measures

Since dimers represent di-atomic molecules, one is typically interested in very large graphs. It is actually easier to study infinite graphs rather than very large ones. In this case, the Boltzmann measure is not well defined, since one divides by infinity. The notion of Boltzmann measure is thus replaced by that of *Gibbs measure*, which is a probability measure on $\mathcal{D}(G)$ satisfying the DLR [2] conditions: if the perfect matching in an annular region of G is fixed, matchings inside and outside of the annulus are independent, and the probability of any interior matching is proportional to the product of its edge-weights.

Suppose that the graph G is infinite, $\mathbb{Z}^2$-periodic, and bipartite. Then, there is a natural exhaustion of the infinite graph G by toric graphs $\{G_n\}_{n\geq 1}$, where $G_n = G/n\mathbb{Z}^2$. A natural way of constructing a Gibbs measure is to take the limit of the Boltzmann measures on *cylinder sets* of $\mathcal{D}(G_n)$, where a cylinder set consists of all perfect matchings containing a fixed subset of edges of G_n.

Kenyon, Okounkov and Sheffield [62] prove that there is a two-parameter family of translation invariant, ergodic Gibbs measures defined on the infinite graph G. The two parameters come from the fact that the graph is $\mathbb{Z}^2$-periodic, and from the height function defined on dimer configurations of planar, bipartite graphs.

2.4 Computation of the number of tilings : a few examples

In some specific cases, there are techniques, other than the one of Kasteleyn, for computing the number of dimer configurations. In this section, we present two such types of results.

2.4.1 Tilings of a hexagon and non intersecting paths

Let $\mathbb{H}_{a,b,c}$ the subgraph of the hexagonal lattice whose dual is a hexagon cut in the triangular lattice with side-length a, b, c. See Figure 2.5.

A dimer configuration of $\mathbb{H}_{a,b,c}$ (or equivalently a tiling of $\mathbb{H}^*_{a,b,c}$ with rhombi) can be encoded by a family of c left to right non-intersecting paths of length $a+b$, connecting points $\{L_i = (i-\frac{1}{2}, 0), 1 \leq i \leq c\}$ on the left and $\{R_i = (i+b-\frac{1}{2}, a+b), 1 \leq i \leq c\}$ on the right. During one step to the right, the height of a path of this family either stays the same, or increases by one.

The number of non-intersecting families of paths can be written as a determinant:

Proposition 2.1 (Lindström-Gessel-Viennot's lemma [37]) *Let $N_{i,j}$ be the number of lattice paths connecting L_i to R_j. Then the number of ways to connect $\{L_i\}$ to*

[2] DLR stands for Dobrushin, Lanford and Ruelle.

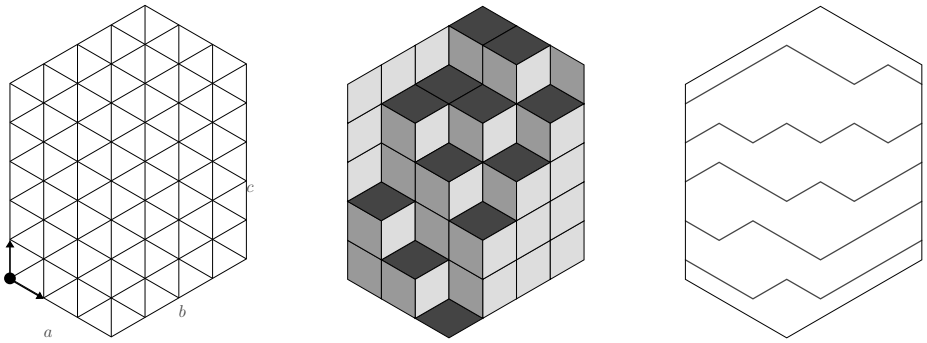


Figure 2.5: Left: the hexagon $\mathbb{H}^*_{3,4,5}$ and the three types of tiles. Coordinates are defined, with an origin at the lower leftmost corner, and axes with directions indicated by the arrows. Middle: a tiling of hexagon $\mathbb{H}^*_{3,4,5}$. Right: the corresponding family of intersecting paths.

$\{R_j\}$ with a collection of c paths without intersection is

$$Z = \det_{1\leq i,j\leq c} (N_{i,j})\,. \tag{2.1}$$

In this particular case, $N_{i,j}$ has a simple form:

$$N_{i,j} = \binom{a+b}{b+j-i}.$$

. The corresponding determinant (2.1) can be computed exactly using method similar to the one used for the Vandermonde determinant [64]. One gets then the beautiful Mac Mahon's formula, giving the number of non intersecting path families connecting the left side and the right side of $\mathbb{H}^*_{a,b,c}$ (and thus the number of tilings of the hexagon $\mathbb{H}^*_{a,b,c}$ with rhombi):

$$Z_{a,b,c} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

2.4.2 Tilings of the Aztec diamond

Let A_n be the subgraph of the square lattice (shifted by one-half in both directions) whose vertices are the points of the plane $(x, y) \in \mathbb{Z}^2+(1/2, 1/2)$, such that $|x|+|y| \leq n$. The graph A_n is called the *Aztec diamond* of size n. See Figure 2.6

Theorem 2.4 ([31]) *The number of dimer configurations of the Aztec diamond of size n is $2^{n(n+1)/2}$.*

We now explain how to generate a uniform domino tiling of the Aztec diamond of size n with a random algorithm which allows one to grow step by step. This

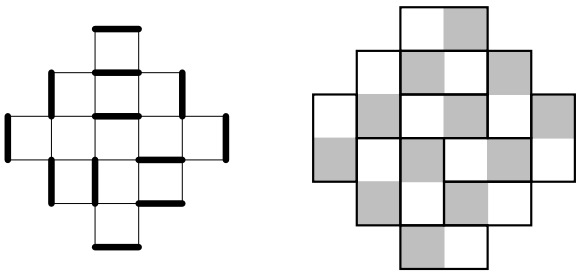


Figure 2.6: Left: a dimer configuration of the Aztec diamond A_3. Right: the corresponding domino tiling.

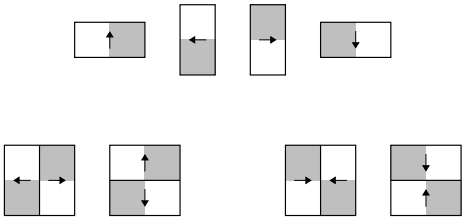


Figure 2.7: The four types of dominos and their associated direction. Below, the two types of good blocks (left) and the two types of bad blocks (right).

algorithm, introduced in [31] to compute the number of dimer configurations of A_n, is called the *domino shuffling* algorithm.

An iteration of the algorithm to go from the Aztec diamond of size n to that of size $n+1$ can be decomposed into 3 steps. In order to perform these steps, we need to associate to each domino a direction, according to its orientation and the checkerboard coloring. See Figure 2.7.

deletion Remove dominos forming *bad blocks*, *i.e.*, 2×2 squares filled with dominos sharing a long edge, with arrows pointing to the center of the square.

migration Move all remaining dominos one step in the direction of their arrow. (and switch at the same time the checkerboard coloring).

creation Fill the remaining space, formed by 2×2 squares, by *good blocks*, *i.e.*, dominos sharing a long edge, with arrows pointing toward the exterior of the square. For each square, there are two possibilities. Pick one at random with probability 1/2, independently for each square.

All the randomness of the algorithm is located in the last step of the iteration.

2.5 Arctic curves

When generating several uniform tilings of a large Aztec diamond, something interesting happens. Whereas the tiling is still random at a microscopic level, all the

generated samples look almost the same at large scale, with the presence of ordered regions close to the corner, and a disordered region in the middle.

The *North polar region* of a domino tiling of the Aztec diamond is the connected component of (horizontal) dominos with a north going arrow touching the top edge of the Aztec diamond. Define similarly the South, East and West polar regions.

The union of the polars region is called the *arctic* (or *frozen*) region. Its complement in the Aztec diamond is called the *temperate* (or *liquid*) region. The curve separating the temperate region from the arctic region is called the *arctic curve*.

If we rescaled the edges by a factor $\frac{1}{n}$, G_n is contained in the square $S = \{(x, y) \in \mathbb{R}^2 \ ; \ |x| + |y| \leq 1\}$. Let D be the disk inscribed inside S, centered at the origin, with radius $\frac{1}{\sqrt{2}}$. The following theorem states that in the scaling limit, the arctic curve for the Aztec diamond converges to a circle:

Theorem 2.5 (Arctic circle theorem [49]) *For all $\varepsilon > 0$, the probability that the liquid region rescaled by a factor $\frac{1}{n}$ contains $(1 - \varepsilon)D$ and is contained in $(1 + \varepsilon)D$ converges to 1 as n goes to infinity.*

This result is a kind of geometric law of large numbers.

Similar results hold for a more general class of domains. For tilings with rhombi of polygonal shape with sides parallel to directions of the triangular lattices, Kenyon and Okounkov [61] showed that the arctic curve is in the scaling limit an algebraic curve inscribed in the domain: an ellipse when the domain is a hexagon, a cardioid if the domain is a hexagon with a corner removed. See Figure 2.8. Many aspects of their work apply also to the case of domino tilings. For more general domain, the arctic curve, and more generally, the *limit shape* (*i.e.*, the deterministic scaling limit of the height function) is given by the solution of a variational principle [19].

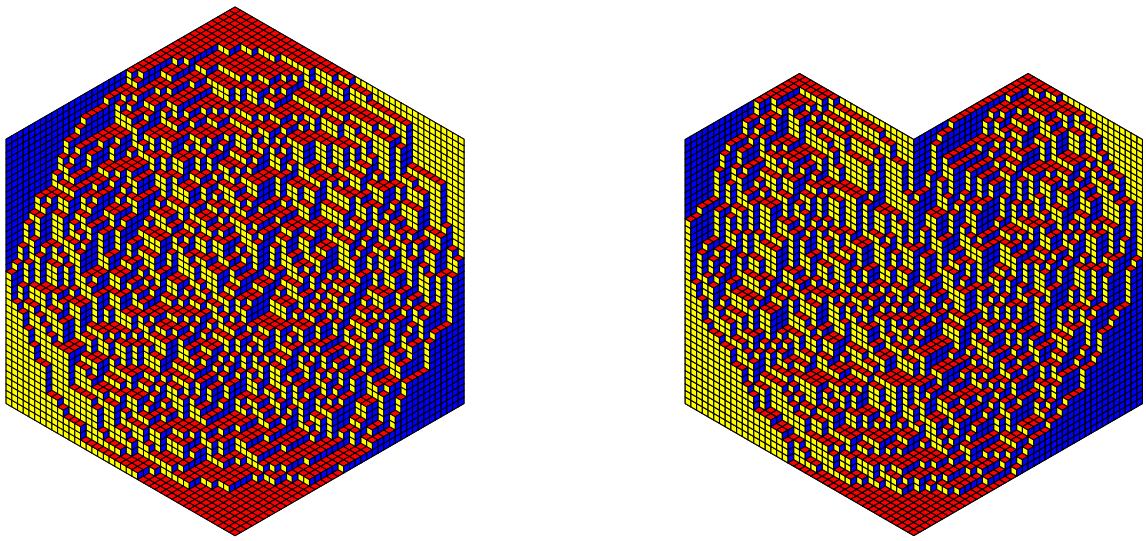


Figure 2.8: Tiling by rhombi of a hexagon (left) and with a corner cut off (right).

Chapter 3

Symbolic dynamics

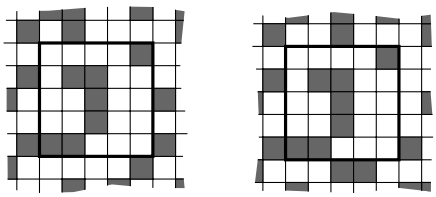
3.1 Preliminaries

3.1.1 Symbolic spaces

Consider a finite **alphabet** $\mathcal{A}$ of at least two **letters**, and some free Abelian group $\mathbb{Z}^d$. We note $\mathcal{B}_r := [\![-r, r]\!]^d$. A **pattern** is a map $p \in \mathcal{A}^{\mathbb{U}}$ for some finite **support** $\mathbb{U} \subset_{\text{finite}} \mathbb{Z}^d$. A **configuration** is a map $x \in \mathcal{A}^{\mathbb{Z}^d}$. We note by $x_{|\mathbb{U}}$ its restriction to some finite **support** $\mathbb{U} \subset_{\text{finite}} \mathbb{Z}^d$. A pattern p **appears** in a configuration x if it is equal to $x_{|\mathbb{U}+\mathbf{i}}$ for some finite support $\mathbb{U} \subset_{\text{finite}} \mathbb{Z}^d$ and some $\mathbf{i} \in \mathbb{Z}^d$; we note $p \sqsubset x$.

The set $\mathcal{A}^{\mathbb{Z}^d}$ of **configurations** can be endowed with the product of the discrete topology, which makes it a Cantor (compact, metrizable, perfect, totally discontinuous) space, by the Tychonov theorem (see [65] for background on topology). A basis for this topology is the set of (clopen) **cylinders** $[\![p]\!] := \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \middle| x_{|\mathbb{U}} = p \right\}$ for $p \in \mathcal{A}^{\mathbb{U}}$ and $\mathbb{U} \subset_{\text{finite}} \mathbb{Z}^d$. Actually any Cantor space is homeomorphic to $\mathcal{A}^{\mathbb{Z}^d}$.

Example 3.1 *We often use* $\mathcal{A} = \{0, 1\}$ *and represent* 0 *as* □ *and* 1 *as* ■.



These two configurations are in the same cylinder $[\![p]\!]$*, where* $p \in \mathcal{A}^{\mathcal{B}_2}$.

3.1.2 Topological dynamics

A (topological, discrete) **dynamical system** is a continuous map $F : \mathbf{X} \to \mathbf{X}$ from some compact space $\mathbf{X}$ into itself. The **orbit** of point $x \in \mathbf{X}$ is the sequence $(F^t(x))_{t \in \mathbb{N}}$ (or sometimes the corresponding set). x is called **periodic** if its orbit is finite, **transitive** if it is dense (in which case the system is called **transitive** as well).

A **subsystem** is the restriction of F to some closed set $\mathbf{Y} \subset \mathbf{X}$ such that $F(\mathbf{Y}) \subset \mathbf{Y}$. The closure of any orbit is a **transitive** subsystem. A system which has no proper nonempty subsystem is called **minimal**. Equivalently, all orbits are dense. It is clear that a minimal system which is not a single finite orbit is **aperiodic**: it has no periodic orbit. Zorn's lemma allows all dynamical system to admit a minimal subsystem.

A continuous onto map $\Phi : \mathbf{X} \twoheadrightarrow \mathbf{Y}$ is called a **factor map** from $F : \mathbf{X} \to \mathbf{X}$ to $G : \mathbf{Y} \to \mathbf{Y}$ if $\Phi F = G\Phi$. In that case, G is called a **factor** of F. Φ is a **conjugacy** if it is also one-to-one. In that case, by compactness, Φ^{-1} is also a conjugacy, and F and G are called **conjugate**.

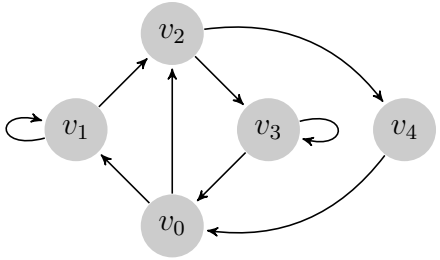
Example 3.2 *Let* $\mathbf{X}$ *be the circle* $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$*, and* $F : x \mapsto x + \theta \bmod 1$ *the rotation by some angle* $\theta \in \mathbb{S}^1$*. It is known that if* θ *is rational, then all orbits are periodic, whereas if* θ *is irrational, then all orbits are dense.*

3.1.3 Graphs

A (finite directed) **graph** is a pair (V, E) of finite sets, the set of **vertices** and the set of **edges**, endowed with two maps $s, t : E \to V$, representing the **initial** and **terminal** vertices of each edge. A **path** is a sequence $(e_i)_{i \in I}$ of edges such that $s(e_{i+1}) = t(e_i)$ whenever $i, i+1 \in I$, and where $I = [\![0, n[\![$ for some **length** $n \in \mathbb{N}$ (finite path) or $I = \mathbb{N}$ (infinite path) or $I = \mathbb{Z}$ (biinfinite path).

If V is $[\![0, n[\![$ for some $n \in \mathbb{N}$ and E is considered up to bijection, then a graph corresponds to a $n \times n$-matrix with nonnegative entries: the entry i, j stands for the number of edges from vertex i to vertex j. The number of paths of length k from vertex i to vertex j is then given by the entry i, j of the matrix to the power k.

Example 3.3 *Here is how we represent the graph with* $V := \{v_0, v_1, v_2, v_3, v_4\}$*,* $E := \{e_0, \dots, e_8\}$*,* $s(e_0) = s(e_1) = v_0 = t(e_7) = t(e_8)$*,* $s(e_2) = s(e_3) = v_1 = t(e_0) = t(e_2)$*,* $s(e_4) = s(e_5) = v_2 = t(e_1) = t(e_3)$*,* $s(e_6) = s(e_7) = v_3 = t(e_4) = t(e_6)$*,* $s(e_8) = v_4 = t(e_5)$*.*

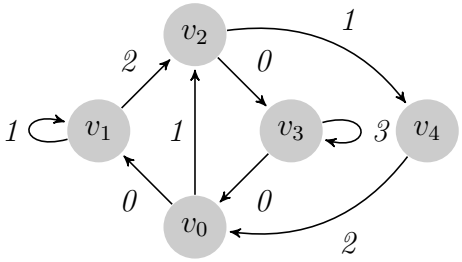


The corresponding matrix is:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A **labeled graph** is a graph endowed with a map l from E to an alphabet $\mathcal{A}$, representing a **label** for each edge. The **label** of a path $(e_i)_{i\in I}$ is the sequence $(l(e_i))_{i\in I} \in \mathcal{A}^I$.

Example 3.4 *Here is how we represent the labeled graph supported on the graph of Example 3.3 and with labels* $l(e_0) = l(e_4) = l(e_7) = 0$, $l(e_1) = l(e_2) = l(e_5) = 1$, $l(e_3) = l(e_8) = 2$, $l(e_6) = 3$.



The graph is strongly connected (all vertices are pairwise connected by a path) if and only if the matrix is **irreducible**, *i.e.*, for every i, j, there exists $k \in \mathbb{N}$ such that $M^k_{i,j} \neq 0$. By the Perron-Frobenius theorem, such a matrix with nonnegative entries has a unique eigenvalue of maximal modulus, which is actually nonnegative.

3.2 Symbolic dynamics

A lot on topological and symbolic dynamics can be found in [65].

We are interested in the **shift** action, defined by $\mathfrak{S}^{\mathbf{i}}(x)_{\mathbf{j}} := x_{\mathbf{j}+\mathbf{i}}$ for any $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$, and $d \in \mathbb{N}$ being the dimension. Any subsystem corresponds to a set $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^d}$, called a **subshift**, which is closed and such that $\mathfrak{S}^{\mathbf{i}}(\mathbf{X}) \subset \mathbf{X}$ for any $\mathbf{i} \in \mathbb{Z}^d$. Equivalently, $\mathbf{X}$ is the set $\left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \middle| \forall p \sqsubset x, p \notin \mathcal{U} \right\}$ of configurations avoiding some forbidden family $\mathcal{U} \subset \bigcup_{\mathbb{U}\subset_{\text{finite}}\mathbb{Z}^d} \mathcal{A}^{\mathbb{U}}$ of patterns. Let us denote $\mathcal{L}_{\mathbb{U}}(\mathbf{X}) := \left\{ x_{|\mathbb{U}} \middle| x \in \mathbf{X} \right\}$.

A configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$ is **periodic** if there is a nontrivial vector $\mathbf{i} \in \mathbb{Z}^d \setminus (0,0)$, called a **period**, such that $\mathfrak{S}^{\mathbf{i}}(x) = x$.

Example 3.5 *Consider* $\theta \in \mathbb{S}^1$ *irrational, and* $\mathbf{Y} \subset \{0,1\}^{\mathbb{Z}}$ *the set of so-called* ***Sturmian*** *sequences* y *such that* $y_{\mathbf{i}} := \lfloor \mathbf{i}\theta + x \rfloor - \lfloor (\mathbf{i}-1)\theta + x \rfloor$ *for every* $\mathbf{i} \in \mathbb{Z}$ *and every* $x \in \mathbf{X} := \mathbb{S}^1$. *For every such* y *there is exactly one corresponding* $x = \Phi(y) \in \mathbf{X}$. $\mathbf{Y}$ *is a minimal subhift, of which the rotation in Example 3.2 is a factor, by the factor map* $\Phi : \mathbf{X} \to \mathbf{Y}$. Φ *is injective except over the countable set of configurations* $x \in \theta\mathbb{Z}$, *which have two preimages.*

Block maps. The following is sometimes called the *Hedlund theorem*, and justifies that we sometimes talk about *sliding block maps*.

Proposition 3.1 *Any factor map Φ from a subshift $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^d}$ onto a subshift $\mathbf{Y} \subset \mathcal{B}^{\mathbb{Z}^d}$ admits a **radius** $r \in \mathbb{N}$ and a **local map** $\phi : \mathcal{L}_r(\mathbf{X}) \to \mathcal{B}$ such that for any $\mathbf{i} \in \mathbb{Z}^d$ and any $x \in \mathbf{X}$, $\Phi(x)_{\mathbf{i}} = \phi(x_{|\mathbf{i}+\mathcal{B}_r})$.*

Subshifts of finite type. If $\mathbf{X} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \middle| \forall p \sqsubset x, p \notin \mathcal{U} \right\}$ for some *finite* forbidden family $\mathcal{U}$ of patterns, then $\mathbf{X}$ is called a subshift of **finite type** (SFT). In that case, it admits some **order** $k \in \mathbb{N}$, *i.e.*, all supports of patterns in $\mathcal{U}$ are included in $\mathcal{B}_k$. Any conjugate of a SFT is a SFT.

Sofic subshifts. If $\mathbf{X}$ is a subshift which is a factor of a SFT, then $\mathbf{X}$ is called a **sofic** subshift. Any factor of a sofic subshift is sofic.

Entropy. The (topological) **entropy** of subshift $\mathbf{X}$ is $\inf_{r \in \mathbb{N}} \frac{\log |\mathcal{L}_{\mathcal{B}_r}(\mathbf{X})|}{|\mathcal{B}_r|} \in [0, \log \mathcal{A}]$, which is known to be a limit when $r \to \infty$, by subadditivity and Fekete's lemma. Entropy cannot increase via factor maps and restrictions to subsystems.

3.3 1D subshifts

Assume here that $d = 1$. Most results in this section, and many more, can be found in [72].

Factorial extendable languages. Up to shifting the support, the **language** $\mathcal{L}(\mathbf{X}) := \bigsqcup_{n \in \mathbb{N}} \mathcal{L}_{\mathcal{B}_n}(\mathbf{X})$ is its set of patterns. It is **factorial**, in the sense that if the concatenation pq is in $\mathcal{L}(\mathbf{X})$, then both p and q also. It is also **extendable**, in the sense that for any pattern $p \in \mathcal{L}(\mathbf{X})$, there are letters $a, b \in \mathcal{A}$ such that $apb \in \mathcal{L}(\mathbf{X})$. Conversely, any factorial extendable language is the language of a unique subshift.

Edge subshifts. An **edge subshift** is the set of biinfinite paths in a graph. It is rather clear that edge subshifts are SFT, with order 2.

Proposition 3.2 *Every 1D SFT is conjugate to an edge subshift.*

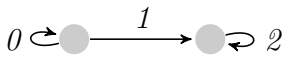
Proposition 3.3 *Let $\mathbf{X}$ be an edge subshift and M the corresponding matrix.*

1. *The entropy of $\mathbf{X}$ is the logarithm of the Perron-Frobenius eigenvalue of M.*
2. *$\mathbf{X}$ is transitive if and only if M is irreducible.*
3. *$\mathbf{X}$ is empty if and only if M^n is the zero matrix for some $n \in \mathbb{N}$.*
4. *The number of configurations with period $n \in \mathbb{N}$ in $\mathbf{X}$ is the trace of M^n.*
5. *$\mathbf{X}$ has no periodic configuration if and only if for any $n \in \mathbb{N}$, M^n has a diagonal of 0s.*

6. **X** *is empty if and only if it has no periodic configuration.*

As a consequence, all of these properties are decidable, *i.e.*, checkable by some computer program that takes the matrix as input. Note that there are always periodic configurations in every nonempty SFT (by the pigeon-hole principle, an infinite path involves a loop).

Example 3.6 *The* ***one-border subshift****, defined over alphabet* $\mathcal{B} := \{0, 1, 2\}$ *by the forbidden family* $\mathcal{U} := \mathcal{B}^2 \setminus \{00, 01, 12, 22\}$*, is represented below.*

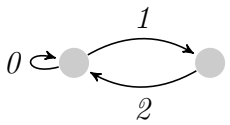


Its entropy is 0*, since the number of patterns of length* n *is* $n + 2$*. The corresponding matrix is:*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It is not transitive: the left-lowest entry of all powers of M *is always* 0*.*

Example 3.7 *The* ***golden mean edge subshift****, defined over alphabet* $\mathcal{B} := \{0, 1, 2\}$ *by the forbidden family* $\mathcal{U} := \mathcal{B}^2 \setminus \{00, 01, 12, 20\}$*, is represented by the graph below.*

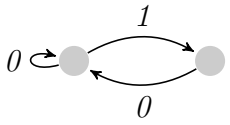


The corresponding matrix is:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Its entropy is the logarithm of the highest eigenvalue: $\log \frac{1+\sqrt{5}}{2}$ *(golden ratio). It is transitive: all entries of* M^2 *are positive.*

The ***golden mean subshift****, defined over alphabet* $\mathcal{A} := \{0, 1\}$ *by the forbidden family* $\mathcal{U} := \{11\}$*, can be represented by the graph below.*



It is not an edge subshift, but is conjugate to the subshift above by the conjugacy $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{B}^{\mathbb{Z}}$ *defined by* $\Phi(x)_{\mathbf{i}} = 2$ *if* $x_{\mathbf{i}-1} = 1$*,* $x_{\mathbf{i}}$ *otherwise, for every* $x \in \mathcal{A}^{\mathbb{Z}}$ *and* $\mathbf{i} \in \mathbb{Z}$*. Like its conjugate, it is transitive and has entropy* $\log \frac{1+\sqrt{5}}{2}$*.*

Label subshifts. A **label subshift** is the set of labels of biinfinite paths in a labeled graph.

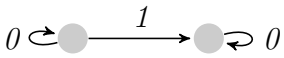
Proposition 3.4 *The 1D sofic subshifts are exactly the label subshifts.*

Said differently, 1D sofic subshifts are those whose language is regular, that is, recognizable by a finite automaton: the labeled graph corresponds to an automaton where no initial nor terminal state is distinguished, and the underlying graph corresponds to that of a SFT that factors onto the given sofic subshift.

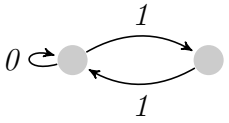
The results of the previous paragraph remain true for label subshifts (in particular, it has the same entropy as a SFT that factors onto it), if we see it on a graph which corresponds to the minimal automaton in some sense.

Example 3.8 *The set of configurations over alphabet $\mathcal{A} := \{0, 1\}$ that have exactly one 1 is shift-invariant but not closed.*

*The **at-most-one subshift** $\mathbf{X}$, consisting of those configurations over alphabet $\mathcal{A} := \{0, 1\}$ that have at most one 1, is sofic: it is the image of the border-one SFT by the factor map $\Phi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that $\Phi(x)_{\mathbf{i}} = x_{\mathbf{i}} \bmod 2$ for every $x \in \mathcal{B}^{\mathbb{Z}}$ and $\mathbf{i} \in \mathbb{Z}$. Its entropy is 0, since it cannot be more than the SFT of which it is a factor.*

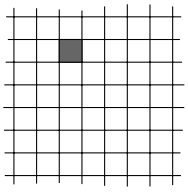


Example 3.9 *The **even subshift** is defined over alphabet $\mathcal{A} := \{0, 1\}$ as consisting of those configurations which have an even number of 1s between two consecutive 0s. It is sofic, as a factor of the golden mean edge subshift by the factor map $\Psi : \mathcal{B}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\Psi(x)_{\mathbf{i}} = \max(1, x_{\mathbf{i}})$ for every $x \in \mathcal{A}^{\mathbb{Z}}$ and $\mathbf{i} \in \mathbb{Z}$. Observe moreover that Ψ is one-to-one except in configuration $\ldots 11111 \ldots$, which has two preimages. Consequently, the entropy is the same as that of the golden mean subshift :* $\log \frac{1+\sqrt{5}}{2}$.

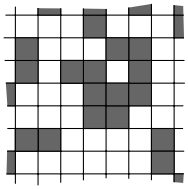


3.4 2D subshifts

Example 3.10 *The 2D **at-most-one subshift** $\mathbf{X}$, consisting of those configurations over alphabet $\mathcal{A} := \{0, 1\}$ that have at most one 1, is sofic. Its entropy is still 0: there are $n + 1$ patterns of size $n \times n$.*



Example 3.11 *The 2D* ***even subshift*** $\mathbf{X}$*, consisting of those configurations over alphabet* $\mathcal{A} := \{0, 1\}$ *in which all connected components of* 1*s have even cardinality, is sofic. The entropy is computable.*



While, in dimension 1, dynamical notions could be studied thanks to graph theory and linear algebra, in dimension 2, the relevant tool to shed light, instead, is computability theory.

Theorem 3.1 ([8, 39])

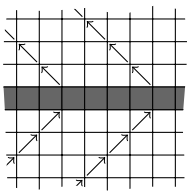
1. *There is a nonempty 2D SFT with no periodic configuration.*
2. *There is no algorithm deciding whether a 2D SFT is empty.*
3. *There is no algorithm deciding whether a 2D SFT admits a periodic configuration.*

Right-computable numbers. A real number x is **right-computable** if there is an algorithm which, for any input $n \in \mathbb{N}$, inputs a rational number x_n, such that the sequence $(x_n)_{n\in\mathbb{N}}$ decreases and converges to x. Note that one in general does not know how far the approximation is from x (if one does, then the real number is called **computable**).

Theorem 3.2 ([44]) *A real number is the entropy of a 2D SFT if and only if it is right-computable.*

Projective subdynamics. The **projective subdynamics** of some 2D subshift $\mathbf{X}$ is the 1D subshift of all columns appearing in its configurations: $\tau(\mathbf{X}) := \{(x_{0,i})_{i\in\mathbb{Z}} | \, x \in \mathbf{X}\}$.

Example 3.12 *The* ***signal mirror subshift*** *consists of configurations where north-est-going "signals" are turned, by a horizontal "mirror" state, into north-west ones. It is a SFT (with order* 2*) but its projective subdynamics is not sofic: the intersection of its language with the regular language* $\nearrow 0^* \blacksquare 0^* \nwarrow$ *is* $\{\nearrow 0^n \blacksquare 0^n \nwarrow | \, n \in \mathbb{N}\}$*, which is known to be nonregular.*

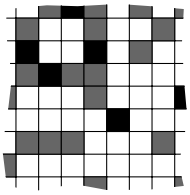


Effective subshifts. A subshift $\mathbf{X}$ is **effective** if it can be written as the set $\left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \middle| \forall p \sqsubset x, p \notin \mathcal{U} \right\}$ of configurations avoiding a *computable* forbidden family $\mathcal{U}$ of patterns (*i.e.*, described by some program). This class (strictly) includes SFT, and actually sofic subshifts. Note that one in general cannot compute whether a given pattern is in the language of $\mathbf{X}$ (all possible extensions of it might be forbidden very late by the program).

Theorem 3.3 ([43, 30, 3]) *A subshift is the factor of the projective subdynamics of some 2D SFT if and only if it is effective.*

Sofic subshifts are effective and have a strong constraint on how the number of finite patterns grows. It is not known whether there exist non-sofic subshifts with these two conditions.

Example 3.13 *Consider the subshift* $\mathbf{X}$ *defined over alphabet* $\mathcal{B} := \{0, 1, 2\}$ *as the set of configurations in which, in each column, there is at most one letter* 2*, and when there is one, what appears below is a mirror of what appears above. This 2D subshift is not sofic.*



More links between multidimensional symbolic dynamics and computability can be found in [77, 92, 48], as well as a nice survey in [47].

Chapter 4

Substitutions

4.1 Substitutions in dimension one

4.1.1 Definitions

We refer to [84] for this part. Consider a finite set $\mathcal{A}$, then denote $\mathcal{A}^*$ the set of finite words defined over $\mathcal{A}$. A **substitution** is a morphism σ of this monoid onto itself.

Fix a basis $(e_1 \dots e_d)$ of $\mathbb{R}^d$. There exists a map π from $\mathcal{A}^*$ into $\mathbb{Z}^d$ where d is the cardinality of $\mathcal{A}$ given by:

$$\pi(w_0 \dots w_n) = \sum_{k=0}^{n} e_{w_k}.$$

This allows to define a linear morphism of $\mathbb{Z}^d$ which commutes with π, σ: The morphism of $\mathbb{Z}^d$ can be defined by a matrix M_σ, called the **incidence matrix** of the substitution.

The substitution is said to be:

- **primitive** if there exists an integer k such that $M_\sigma^k > 0$.
- **irreducible** if the characteristic polynomial of M_σ is irreducible over $\mathbb{Z}$.
- **unimodular** if $det(M_\sigma) = \pm 1$.
- **Pisot** if the dominant eigenvalue is a Pisot number.

The substitution acts on $\mathcal{A}^*$ and it can be extended to an action on $\mathcal{A}^{\mathbb{N}}$.

A **fixed point** of σ is an element of $\mathcal{A}^{\mathbb{N}}$ such that $\sigma(u) = u$. A **periodic point** is an element such that $\sigma^k(u) = u$ for some $k > 0$.

The **language** of a substitution is the set of finite words which appear as a subword of some $\sigma^n(a)$ where $a \in \mathcal{A}$. The **subshift** associated to a substitution is the set of sequences such that every subword appear in the language of σ. It is denoted X_σ. We define S as the **shift map** which acts by left translation on the sequences. Let σ be a substitution and u be a periodic point. Then we define $X_u = \overline{\{S^n u, n \in \mathbb{N}\}}$.

Automaton

We refer to [84] or [1]. An **automaton** is a 5-uplet $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states.
- Σ is a finite set of symbols, called the alphabet.
- δ is a function $Q \times \Sigma \to Q$, called the transition function.
- $q_0 \in Q$ is the start state.
- F is the set of states, called the accept states.

An automaton reads a finite word $w = a_1 \dots a_n$ with $a_i \in \Sigma$ and a **run** of the automaton is a sequence of states $q_0 \dots q_n$ such that $q_i = \delta(q_{i-1}, a_i)$ for $0 < i \leq n$. The word w is **accepted** if q_n belongs to F.

Let k be an integer greatest or equal than one. One special class is given by the **k-automaton**. It is a directed graph defined by

- A finite set of vertices called S, and one initial vertex called i.
- k oriented edges from S to S denoted $0 \dots k-1$.
- A set Y and a map ϕ from S to Y called the output function.

A sequence $(u_n)_{n\in\mathbb{N}}$ is called **k-automatic** if we write $n = \sum_{i=0}^{j} n_i k^i$ and starting from the initial state we follow a path in the oriented graph defined by $n_0, \dots n_j$. At this point we are at vertex $a(n)$ and we have $u_n = \phi(a(n))$.

Complexity

The **complexity** of an infinite word is the function p defined over $\mathbb{N}$ such that $p(n)$ is the number of different words of length n which appear the infinite word. Consider a language and the set L_n of words of length n of this language. A word of L_n is said to be **right special** if it admits several right expansions in a word of L_{n+1}. By the same way we define a **left special word**. A word is **bispecial** if it is right and left special. We denote $s(n) = p(n+1) - p(n)$ for every integer n.

An infinite word is a **sturmian word** if the complexity of this word equals $n+1$ for every integer n. A substitution is a **sturmian substitution** if the image of every sturmian word is a sturmian word.

Recognizability

Theorem 4.1 (Mosse) *A substitution σ is aperiodic if and only if for every $u \in X_\sigma$, there exists a unique integer k and an unique $v \in X_\sigma$ such that $S^k\sigma(v) = u$.*

Automaton of prefixes-suffixes, see [29] and [18].

Consider an aperiodic substitution. Let $w \in X_\sigma$, then there exists an unique $v \in X_\sigma$ and an unique $k < |\sigma(v_0)|$ such that $w = S^k\sigma(v)$. We define a map

$$\theta : \begin{array}{ccc} X_\sigma & \to & X_\sigma \\ w & \mapsto & v \end{array}$$

Then consider

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A} | \exists b, \sigma(b) = p.a.s\}$$

Now define the application $\gamma : X_\sigma \to \mathcal{P}$ which sends w to (p, w_0, s) such that $\sigma(\theta(w)_0) = p.w_0 s$. The sequence $\gamma(\theta^i(w))_{i\in\mathbb{N}}$ is called the **development in prefix-suffixes**. Then we define an automaton such that

- The set of states is $\mathcal{A}$.
- The set of edges is $\mathcal{P}$.
- There is an edge from a to b if $\sigma(b) = p.a.s$. The edge is labelled by (p, a, s).

4.1.2 Tilings

This part is related to Section 2, and these tilings can be seen as substitutive tilings although the tiles are not polygons.

Projection and Rauzy fractal

We refer to [75] for the Tribonacci case and [86] for the seminal paper.

We restrict to a unitary Pisot substitution. Let us denote by H_e the expanding line, by H_c the contracting hyperplane and by p the projection on H_c along H_e.

$$\mathbb{R}^d = H_e \oplus H_c$$

Let u be a periodic point of σ. Then we define the **Rauzy fractal**:

$$\mathcal{R} = \overline{\{p \circ \pi(u_0 \dots u_n), n \in \mathbb{N}\}}$$

Then for each $i = 1 \dots d-1$ we define the tiles of the Rauzy fractal by:

$$\mathcal{R}_i = \overline{\{p \circ \pi(u_0 \dots u_n), u_n = i, n \in \mathbb{N}\}}.$$

We refer to Figure 4.1 (left).

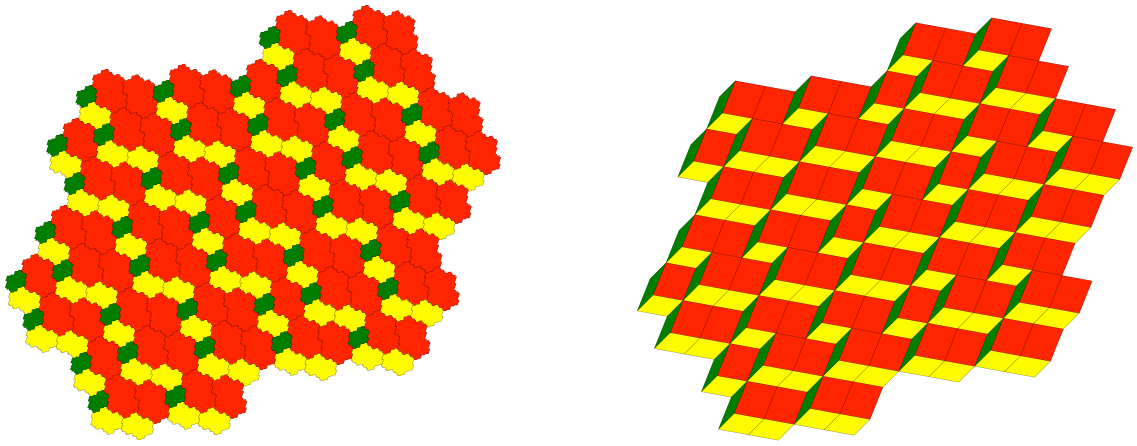


Figure 4.1: Rauzy fractal (left) and stepped surface (right)

Dual substitution

We only consider the case of Tribonacci. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the canonical basis of $\mathbb{R}^3$, which we will represent as follows in the rest of these notes.

Let $\mathbf{x} \in \mathbb{Z}^3$ and let $i \in \{1, 2, 3\}$. The **face** $[\mathbf{x}, i]^*$ of **vector** $\mathbf{x}$ and **type** i is a subset of $\mathbb{R}^3$ defined by

$$[\mathbf{x}, 1]^* = \{\mathbf{x} + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$
$$[\mathbf{x}, 2]^* = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$
$$[\mathbf{x}, 3]^* = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2 : \lambda, \mu \in [0, 1]\} = .$$

First we define

$$\mathbf{E} : \begin{cases} [\mathbf{x}, 1]^* \mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^* \\ [\mathbf{x}, 2]^* \mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{e}_3, 1]^* \\ [\mathbf{x}, 3]^* \mapsto \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{e}_3, 2]^*. \end{cases}$$

Now consider the set $\mathcal{F}$ of all the faces. Then we extend $\mathbf{E}$ to $\mathcal{F}$ by declaring that the image of a union of faces is the union of the images of these faces (the multiplicities of faces add up). Now consider $\mathcal{U} = [0, 1]^* \cup [0, 2]^* \cup [0, 3]^*$. We define the **stepped surface** as

$$H_c = \lim_{n \in \mathbb{N}} \mathbf{E}^n(\mathcal{U}).$$

We define a tiling of H_c, see Figure 4.1 (right), by

$$\{p([\mathbf{0}, i]^*) + p(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^3, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle, i \in \{1, 2, 3\}\}.$$

4.1.3 Dynamical properties

Definitions

We refer to [85]. We define the **dynamical system** associated to a substitution as the action of S on X_σ, equipped with the product topology. Recall that a dynamical system is **minimal** if every orbit is dense. An invariant measure μ is a measure on X such that $\mu(A) = \mu(S^{-1}A)$ for every set A. It is **uniquely ergodic** if there exists only one invariant measure for the action of S. Consider the **Koopman operator**

$$U : L^2(X_u, \mu) \mapsto L^2(X_u, \mu)$$

$$f :\to f \circ S$$

A function $f \in L^2(X, \mu)$ is an eigenfunction if there exists a complex number λ such that $Uf = \lambda f$. The set of eigenvalues form a countable subgroup of S^1. Now consider $V = \overline{< \text{eigenfunctions} >}$, this is the **spectrum**.

- If $V = L^2$, then we speak of **discrete** spectrum.
- If $V = \{\text{Const}\}$ we speak of **continuous** spectrum.
- If not, then we speak of **mixed** spectrum.

A map is **mixing** if it has non constant eigenfunctions. A map if **weak mixing** if every eigenfunction is constant almost everywhere.

A measure is **ergodic** if $S^{-1}A = A$ implies $\mu(A) = 0$ or $\mu({}^cA) = 0$. If there is only one invariant measure, the system is said to be **uniquely ergodic**.

The **spectral measure** of a function $f \in L^2$ is the measure $\mu_f \subset S^1$ such that $< f \circ S^n, f >= \int z^n d\mu_f$.

Perron Frobenius theory

Let F be an element of $\mathcal{M}_N(\mathbb{N})$. We associate a **graph** with N vertices, and edge from i to j if $F_{i,j} > 0$. A matrix F is called **irreducible** if one of the following properties holds:

- F does not have non-trivial invariant coordinate subspace.
- For every pair of indices i, j, there exists m such that $(F^m)_{i,j} \neq 0$.
- The graph associated to F is strongly connected.

Consider a matrix F with non negative coefficients. We define an equivalence relation by $i \sim j$ if $i = j$ or if there are two paths in the graph: one from i to j and one from j to i. The equivalence classes of the graph are denoted $\mathcal{E}_1, \ldots, \mathcal{E}_m$.

Let us denote these irreducible component F_α. The class β has **access** to the class α if $\beta = \alpha$ or there is a path from a vertex in the class $\mathcal{E}_\beta$ to the class $\mathcal{E}_\alpha$. We write $\beta > \alpha$ if β has access to α but $\alpha \neq \beta$. Let us denote λ_α the spectral radius of

F_α. Let us define $h_\alpha$the number of eigenvalues of F_α of modulus λ_α. It is the index of **imprimitivity**. Remark that the matrix is primitive iff $h_i = 1$. An irreducible component is **distinguished** if $\lambda_\alpha > \lambda_\beta$ for every class $\beta > \alpha$. A real numberλ is a **distinguished eigenvalue** if there exists a non negative eigenvector with $Fx = \lambda x$.

We define $C(F, q)$, for an integer q, as the cone generated by the distinguished eigenvectors of F^q and

$$Core(F) = \bigcap_{k \geq 1} F^k(\mathbb{R}_+^N).$$

In the exercises, the link with ergodic properties will be done.

4.1.4 List of classical examples

$\begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$	$\begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases}$	$\begin{cases} 0 \to 01 \\ 1 \to 02 \\ 2 \to 0 \end{cases}$	$\begin{cases} 0 \to 0010 \\ 1 \to 1 \end{cases}$
Fibonacci	**Thue − Morse**	**Tribonacci**	**Chacon**

4.2 Substitutive tilings

For this part, the most important reference is [87].

4.2.1 Tiling

We recall some definitions on tilings. This part is close to some other chapters.

Consider a finite number of polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^d$. A **tiling** of $\mathbb{R}^d$ is the given of

$$\bigcup_i T_i = \mathbb{R}^d$$

such that each T_i is a polytope translated from some P_j. The intersections of two polytopes P_i either empty or a face of the polytope. There P_i are called **tiles** of the tiling, and P_i is a **prototile**. The tiling is said to be **based** on the set of prototiles.

There are three classes of tilings which have been studied:

- Tilings by cut and projection: see Chapter 6.
- Tilings by local rules.
- Tilings by substitution.

Here we study the last one and some link with the second one.

A tiling $\mathcal{T}$ is said to be **non periodic** if we cannot find $u \in \mathbb{R}^d$ such that $u+\mathcal{T} = \mathcal{T}$. A tiling is said to be **strongly aperiodic** if it is aperiodic and we can not find a periodic tiling based on the same prototiles.

Example 4.1 *Consider one square and two right triangles which cut the square. Now consider a periodic tiling of the plane by the square. With probability $\frac{1}{2}$, cut a square in two triangles. We obtained a tiling. Almost every tiling obtained by this way is non periodic. Nevertheless it is not a strongly aperiodic tiling.*

An open question is: Can we find one tile in $\mathbb{R}^2$ such that every tiling based on this tile is aperiodic. This problem is called the **Einstein problem**.

Let $r > 0$, and $\mathcal{T}$ a tiling of $\mathbb{R}^d$. A **patch** of size r is the union of tiles included in a ball of radius r. Two patches are equivalent if there exists a translation which maps each tile of one to a tile of the other patch. It allows to define the notion of **protopatch**.

Remark 4.1 *In some example we add some labels on the tiles. This label allow to distinguish two polygons of the same shape and to simplify the examples. For example we can consider the tilings made by one white square and one black square.*

Let $P_1 \dots P_k$ be some polytopes in $\mathbb{R}^d$. We will define a subset of all the tilings based on these prototiles. Consider a finite set $\mathcal{F}$ of patches made upon these prototiles. A **subshift of finite type** is the set of tilings $\mathcal{T}$ based on these prototiles such that no patch of $\mathcal{T}$ belongs to $\mathcal{F}$.

4.2.2 Substitutive tiling

Definitions

Let $P_1, \dots, P_k$ be some polytopes and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map with all eigenvalues bigger than one. A **tile substitution** is the given of a map Ω defined as follow:

- For a prototile P_i, $\Omega(P_i)$ is a patch which can be tiled by the prototiles. In other words for every integer $1 \leq i \leq k$ there exists a finite set J_i and vectors $t_{i,j} \in \mathbb{R}^d$ such that
$$\Omega(T_i) = \bigcup_{j \in J_i} (T_j + t_{i,j}).$$
- For a tile translated by x we set $\Omega(P_i + x) = \varphi(x) + \Omega(P_i)$.
- Finally for a patch P we define $\Omega(P) = \bigcup_{T \in P} \Omega(T)$.

A fixed point of Ω is called a **substitutive tiling**. If φ is given for $d = 2$ by a similarity we speak of **self-similar tiling**.

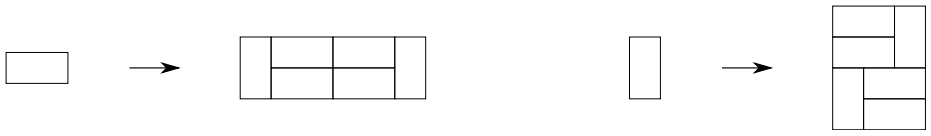
Denote by $M \in \mathbb{M}_d(\mathbb{N})$ be the incidence matrix defined by $M_{i,j} = card\{j \in J_i, t_{i,j} \neq 0\}$.

The tiling is said to be **primitive** if M is a primitive matrix. We denote by θ the Perron eigenvalue of M. The number $det(\varphi)$ is called the **expansion** of the tiling.

Proposition 4.1 *For every primitive tile substation, there exists n such that Ω^n admits a fixed point.*

A **tiling space** associated to a substitutive tiling $\mathcal{T}$ is the set of tilings $\mathcal{Q}$ such that every patch of $\mathcal{Q}$ is a patch of $\mathcal{T}$. We denote it $X_{\mathcal{T}}$.

Example 4.2 *Consider two rectangles and the substitution defined by*



The incidence matrix is $\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}$ *and the map* φ *has matrix* $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Rectangles and link with substitutions

One easy way to define a substitution is to use squares. The first reference is Mozes. Here is an example:

Consider now two substitutions σ, η of constant length on a two letters alphabet. It is easy to define a substitutive tiling. Define the image of the square a by a rectangle of size the lengths of the substitutions. The first line is made by the images of a by σ, and the column are the image by η of the letters of $\sigma(a)$. These substitutions are called **direct product substitutions.**

4.2.3 Expansion

Theorem 4.2 (Solomyak) *If M is a primitive matrix, then θ is equal to $|det(\phi)|$. Moreover the left eigenvector of A asociated to θ gives the volume of the prototiles.*

Theorem 4.3 (Lind) *[70] The set of expansive factors of all substitutive tilings of $\mathbb{R}$ is in bijection with the set of Perron numbers.*

In $\mathbb{R}^2$ we have the following result:

Theorem 4.4 (Thurston,Kenyon-Solomyak) *[99] Given the complex number λ,*

- *If there exists a substitutive tiling of the plane with expansion λ then λ is a complex Perron number.*
- *If λ is a complex Perron number, then there exists a self similar tiling with expansion λ^k for some integer k.*

A complex Perron number is a algebraic number such that all algebraic conjugates, other than its complex conjugates, have a strictly less modulus. In this theorem the tiles can be fractal sets, but they always are equal to the closure of its interior. Some generalization in $\mathbb{R}^d$ have been made by Kenyon and Solomyak [59].

4.2.4 Dynamics

First of all we make one remark: If the tiles are hypercubes, then we can consider $\mathbb{Z}^d$ action. In this case we refer to Chapter 3. We can also consider $\mathbb{R}^d$ action by translation. To study this dynamics we need to introduce a metric on the space of tilings.

Metric on the space

Consider a tiling $\mathcal{T}_0$ and the set of tilings based upon the same prototiles $X_{\mathcal{T}}$.

Definition 4.1 *We define a metric on this space by the following formula where x, y are two tilings, and $x[[K]]$ is the collection of patches x' of x such that $K \subset supp(x')$.*

$$d(x,y) = \inf\{\sqrt{2}/2\} \cup \{0 < r < \sqrt{2}/2, \exists x' \in x[[B_{\frac{1}{r}}]], y' \in y[[B_{\frac{1}{r}}]], g \in B_r, gx' = y'\}.$$

Two tilings are closed if, up to a small translation, they coincide on a big patch.

Proposition 4.2 (Rudolph, Radin-Wolff) *We have:*

- *The map d is a metric.*
- *The space $(X_{\mathcal{T}}, d)$ is a complete space.*
- *The map $\mathcal{T} \mapsto u.\mathcal{T}$ is continuous for every $u \in \mathbb{R}^d$.*
- *If $\mathcal{T}$ is a FLC tiling, then the space is compact.*

Dynamical results

We refer to the notes of Solomyak [94].

Theorem 4.5 (Pragastis, Solomyak) *Let $\mathcal{T}$ be a substitutive tiling, then the map $\Omega : X_{\mathcal{T}} \to X_{\mathcal{T}}$ is surjective. It is injective if and only if $\mathcal{T}$ is aperiodic.*

Theorem 4.6 (Pragastis, Solomyak) *If the substitution is primitive, then $X_{\mathcal{T}}$ is non empty and $(X_{\mathcal{T}}, \mathbb{R}^d)$ is minimal and uniquely ergodic.*

As in the first part we could also look at the eigenvalues. A vector is an eigenvalue $\alpha \in \mathbb{R}^d$ if there exists $f \in L^2(X_{\mathcal{T}}, \mu)$ such that for every $t \in \mathbb{R}^d$ we have for almost every $\mathcal{Q}$

$$f(t.\mathcal{Q}) = e^{2i\pi<\alpha.t>} f(\mathcal{Q})$$

If we exchange the condition $f \in L^2$ with f is continuous we speak of continuous eigenvalues. They have been characterized by Solomyak [95].

4.2.5 Link between different types of tilings

Cut and project tiling and substitutive tilings

There exists a link between cut and project tiling and the substitutive tilings.

A **quadratic expansion matrix** is a matrix M of $M_n(\mathbb{N})$ such that

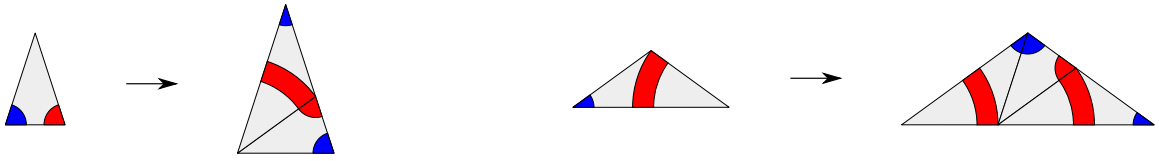
- $\mathbb{R}^n = E \oplus F \oplus R$
- Each space is stable by M.
- $dim E = dim F$
- The restriction of M to E is similar to λId.
- The restriction of M to F is similar to $\pm\lambda^{-1} Id$.
- The space R is rational: $R \cap \mathbb{Z}^n$ is a lattice with a rank equal to $dim R$.
- $E \cap \mathbb{Z}^n = F \cap \mathbb{Z}^n = \{0\}$.

Remark that the conditions force λ to be a quadratic unit.

Theorem 4.7 (Harris) *A canonical projection tiling is a substitutive tiling if and only if E, F are the eigenspaces of a quadratic expansion matrix with the vertex hierarchy property.*

The vertex hierarchy property is a technical assumption, always checked if $R = 0$.

Example 4.3 *Exemple of the Penrose substitution: The two tiles (up to rotation) and the substitution are given on the following figure, where the triangles are marked.*



SFT tilings and substitutive tilings

Consider two tilings spaces X, Y. A **factor map** is a continuous map $Q : Y \to X$ which is surjective and such that $Q(t+y) = t + Q(y)$ for all $t \in \mathbb{R}^d, y \in Y$. A factor is said to be **almost one to one** if there exists $x \in X$ such that $Q^{-1}(x)$ is of cardinal one.

Theorem 4.8 (Goodman-Strauss) *[38] Consider a substitutive tiling space X based on prototiles $\mathcal{T}$. Then there exists a* **marking** *$\mathcal{Q} \to \mathcal{T}$ and a subshift of finite type Y based on $\mathcal{Q}$ which is a factor of X for the forgetful mapping.*

Chapter 5

Cellular Automata

5.1 Turing machines and computability

A Turing machine is a dynamical system that acts on 1-dimensional configurations through a "head" that spots some position on the configuration and which can:

- read the content of the configuration at the current position;
- change the content of the configuration at the current position;
- move to an adjacent position.

Moreover, all these actions are prescribed by a finite automaton, *i.e.* a finite-memory program. Finally, when using Turing machines to define computability, we are only interested into *B-finite configuration*, *i.e.* configurations which are everywhere equal to some special symbol B, except on a finite part where they are filled with letters from some alphabet $\mathcal{A}$.

Definition 5.1 *A Turing machine is a 5-uple* $\big(\mathcal{A}, B, Q, q_i, q_f, \delta\big)$ *where:*

- $B \notin \mathcal{A}$ *is the blank symbol;*
- Q *is the finite* set of states*;*
- $q_i \in Q$ *is the* initial state*;*
- $q_f \in Q$ *is the* final state*;*
- $\delta : \mathcal{A} \times Q \to \mathcal{A} \times Q \times \{-1, 0, 1\}$ *is the* transition map.

A global state *of the machine is a triple* $(c, \mathbf{i}, q)$ *where* c *is a B-finite configuration,* $\mathbf{i} \in \mathbb{Z}$ *is the position of the head, and* $q \in Q$ *is the current state.*

The image of a global state $(c, \mathbf{i}, q)$ *by the machine is the global state* $(c', \mathbf{i} + m, q')$ *where*

$$c'(\mathbf{i}') = \begin{cases} a & \text{if } \mathbf{i}' = \mathbf{i} \\ c(\mathbf{i}') & \text{else.} \end{cases}$$

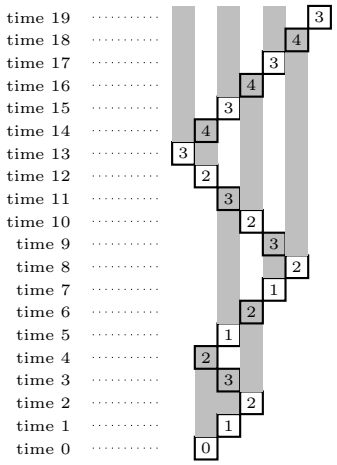
and $(q', a, m) = \delta(q, c(\mathbf{i}))$.

A run *of the machine on a global state* $(c, \mathbf{i}, q)$ *is the sequence of global states* $\big((c_n, \mathbf{i}_n, q_n)\big)_{n\in\mathbb{N}}$ *such that* $(c_0, \mathbf{i}_0, q_0) = (c, \mathbf{i}, q)$ *and* $(c_{n+1}, \mathbf{i}_{n+1}, q_{n+1})$ *is the image by the machine of* $(c_n, \mathbf{i}_n, q_n)$*, for all* $n \geq 0$.

The above definition describes Turing machine as dynamical systems that act on global states. We are now interested in defining a partial map from finite words to finite words associated to a Turing machine. The idea is to start a run of the machine from a B-finite configuration containing the word and consider the two cases:

- either the state q_f is reached at some point in the run and we consider that the *computation halts*, and has *produced as result* the finite word contained in the B-finite configuration at that time and pointed by the head;
- or the state q_f is never reached and we consider that the *computation doesn't halt* and has produced no result.

Example. Let $\mathcal{A} = \{\square, \blacksquare\}$ with $B = \square$ and $Q = \{0, 1, 2, 3, 4, 5, 6\}$ with $q_i = 0$ and $q_f = 6$.



$\mathcal{A}$	Q	$\mathcal{A}$	Q	move
$\square$	0	$\blacksquare$	1	$\rightarrow$
$\blacksquare$	0	$\blacksquare$	4	$\leftarrow$
$\square$	1	$\blacksquare$	2	$\rightarrow$
$\blacksquare$	1	$\blacksquare$	5	$\rightarrow$
$\square$	2	$\blacksquare$	3	$\leftarrow$
$\blacksquare$	2	$\square$	1	$\rightarrow$
$\square$	3	$\blacksquare$	4	$\rightarrow$
$\blacksquare$	3	$\square$	2	$\leftarrow$
$\square$	4	$\blacksquare$	0	$\leftarrow$
$\blacksquare$	4	$\square$	3	$\rightarrow$
$\square$	5	$\blacksquare$	6	$\rightarrow$
$\blacksquare$	5	$\blacksquare$	2	$\rightarrow$
$\square$	6	$\square$	6	$\leftarrow$
$\blacksquare$	6	$\square$	6	$\leftarrow$

Definition 5.2 *Given a Turing machine* $M = \big(\mathcal{A}, B, Q, q_i, q_f, \delta\big)$*, we define a partial map on finite words* $\phi_M : D_M \subseteq \mathcal{A}^* \to \mathcal{A}^*$ *in the following way. For* $u \in \mathcal{A}^*$*, let* c_u *be the* B*-finite configuration*

$$c_u(\mathbf{i}) = \begin{cases} u_{\mathbf{i}} & \text{if } 0 \leq \mathbf{i} < |u| \\ B & \text{else.} \end{cases}$$

Consider the run $\big(c_n, \mathbf{i}_n, q_n\big)_{n\in\mathbb{N}}$ *of* M *starting from the global state* $(c_0, \mathbf{i}_0, q_0) = (c_u, 0, q_i)$*. There are two cases:*

- *either* $q_n \neq q_f$ *for all* n *and then* ϕ_M *is* undefined *on* u, i.e. $u \notin D_M$*;*

- *or, for some* n, $q_n = q_f$ *and* $q_m \neq q_f$ *if* $m < n$*; then we define* $\phi_M(u)$ *as the empty word if* $c(\mathbf{i}_n) = B$*, and else,*

$$\phi_M(u) = c_n(\mathbf{i}_n) \cdots c_n(\mathbf{i}_n + k - 1)$$

where $k \geq 0$ *is the minimal integer such that* $c_n(\mathbf{i}_n + k) = B$*.*

So to any Turing machine M we associate a (partial) map ϕ_M on finite words on the alphabet $\mathcal{A}$ of the machine. Now given some partial map $\phi : D \subseteq \mathcal{A}^* \to \mathcal{A}^*$, we say that it is *computable* if there is some machine M with $D = D_M$ and $\phi_M = \phi$.

More generally, any partial map ϕ from $\mathcal{A}_1^*$ to $\mathcal{A}_2^*$ is *computable* if there is some Turing machine M over some alphabet $\mathcal{A}'$ containing both $\mathcal{A}_1$ and $\mathcal{A}_2$ such that the restriction of ϕ_M to $\mathcal{A}_1^*$ is exactly ϕ.

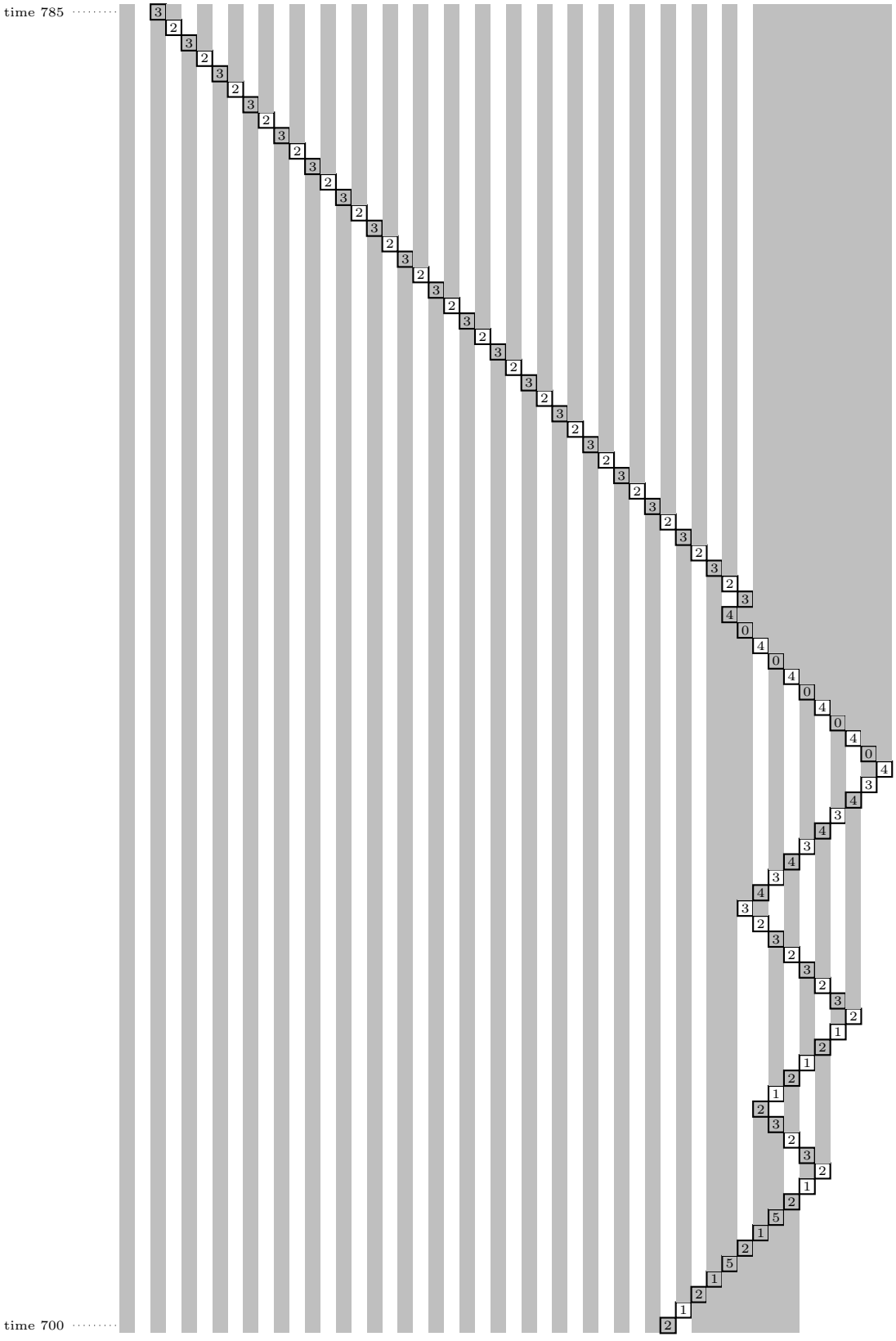
Definition 5.3 *Given some alphabet* $\mathcal{A}$ *and some set of finite words* $L \subseteq \mathcal{A}^*$*, often called a* language*, we say that* L *is* decidable *if its characteristic function* $\phi : \mathcal{A}^* \to \{0, 1\}$ *(*i.e. $\phi(u) = 1$ *for* $u \in L$ *and* $\phi(u) = 0$ *for* $u \notin L$*) is computable.*

Like for any family of finitely describable objects, we can fix a simple encoding of Turing machines into finite words over alphabet $\{0, 1\}$ in a straightforward way: if we adopt the convention that any alphabet is a subset of integers, then each component of the 5-uple defining a Turing machine is either an integer, or a list of integer (the alphabet), or a list of 5-uples of integers (the transition rule). The details of the particular encoding we choose don't matter. We is important however, is that basic operations on Turing machines correspond to computable maps over their encoding. There is a rigorous way of presenting this through the notion of *admissible numbering systems* [88] that we skip here.

From now, we identify a Turing machine M with its encoding over $\{0, 1\}$ so that $M \in \{0, 1\}^*$.

Theorem 5.1 (Undecidability of halting problem) *Let* L *be the set of Turing machines* M *such that* ϕ_M *is well-defined on the empty word. Then* L *is undecidable.*

Going back to the example of Turing machine given earlier, here are some configurations reached after many steps starting from the empty word:


time 785
time 700

(Long) Exercise. Show that this machine halts on the empty word, but only after more than 10^{36534} steps (*note: machine found by Pavel Kropitz in 2010*).

Theorem 5.2 (Rice theorem) *Let $\mathcal{P}$ be a property of partial maps from finite words to finite words. Suppose that $\mathcal{P}$ is* non-trivial *for computable maps,* i.e *there is a computable map having property $\mathcal{P}$, and also a computable map not having property $\mathcal{P}$. Let L be the set of Turing machines M such that ϕ_M has property $\mathcal{P}$. Then L is undecidable.*

5.2 Definition of cellular automata

Definition 5.4 *A* cellular automaton *of* dimension d *and* alphabet $\mathcal{A}$ *is a map $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ defined by a finite set called* neighborhood $\mathbb{U} \subseteq \mathbb{Z}^d$ *and a* local transition map $f : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ *in the following way:*

$$F(x)(\mathbf{i}) = f(x_{|\mathbf{i}+\mathbb{U}})$$

Example. In dimension $d = 1$, with $\mathcal{A} = \{0, 1\}$, let $\mathbb{U} = \{-1, 0\}$ and f such that $f(a, b) = a + b \bmod 2$. Then an example of evolution of the cellular automaton F given by these specifications is:

$F^8(x)$	0	1	0	0	0	0	0	0	0
$F^7(x)$	0	1	1	1	1	1	1	1	1
$F^6(x)$	0	1	0	1	0	1	0	1	0
$F^5(x)$	0	1	1	0	0	1	1	0	0
$F^4(x)$	0	1	0	0	0	1	0	0	0
$F^3(x)$	0	1	1	1	1	0	0	0	0
$F^2(x)$	0	1	0	1	0	0	0	0	0
$F(x)$	0	1	1	0	0	0	0	0	0
x	0	1	0	0	0	0	0	0	0

From the definition above, it appears that a d-dimensional cellular automaton is two things at the same time:

- a *dynamical system*, *i.e.* a global map F acting on infinite configurations;
- a *finite description* of it given by $(\mathcal{A}, \mathbb{U}, f)$ *i.e.* dimension, alphabet, neighborhood, transition map.

Of course different descriptions can correspond to the same dynamical system. In general, we are interested in properties of the dynamical systems and only mention the map F. The Curtis-Lyndon-Hedlund theorem [42] can serve as an alternative definition of cellular automata that doesn't mention the finite description.

For each $\mathbf{i} \in \mathbb{Z}^d$ we define the map $\mathfrak{S}_\mathbf{i} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ by

$$\mathfrak{S}_\mathbf{i}(x)(\mathbf{j}) = x(\mathbf{i} - \mathbf{j})$$

Theorem 5.3 (Curtis-Lyondon-Hedlun, [42]) *A map $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is a cellular automaton if and only if it satisfies the two following conditions:*

1. *it is continuous (Cantor topology);*
2. $\mathfrak{S}_{\mathbf{i}} \circ F = F \circ \mathfrak{S}_{\mathbf{i}}$ *for any* $\mathbf{i} \in \mathbb{Z}^d$

Corollary 5.1 *If a cellular automaton F is a bijection, then its inverse F^{-1} is also a cellular automaton. We call such a F a* reversible *cellular automaton.*

5.3 Garden of Eden theorem

wo configuration x and y are said *asymptotically equal*, denoted $x \stackrel{\infty}{=} y$, if they differ only on a finite set, *i.e.* if the set $\{\mathbf{i} : x(\mathbf{i}) \neq y(\mathbf{i})\}$ is finite.

Definition 5.5 *A cellular automaton F is* pre-injective *if, for any pair of configuration $x \stackrel{\infty}{=} y$, we have*

$$x \neq y \Rightarrow F(x) \neq F(y)$$

Theorem 5.4 (Moore-Myhill,[76, 78]) *A cellular automaton F is surjective if and only if it is pre-injective*

Corollary 5.2 *All injective cellular automata are surjective.*

5.4 First order theory

Surjectivity or injectivity are properties than can be expressed as a first-order formula using the global map F and quantifying over configurations:

- surjectivity: $\forall y, \exists x : F(x) = y$
- injectivity: $\forall x, \forall y : F(x) = F(y) \Rightarrow x = y$

We can consider all possible first-order formulas, but let's add two more examples:

- constant: $\forall x, \forall y : F(x) = F(y)$
- fixed-point: $\exists x : F(x) = x$

When we are interested in *decision problems*, we are usually given as input the finite description $(\mathcal{A}, \mathbb{U}, f)$ and must determine from that if the cellular automaton has some property or not.

Formally, if PROP is some property (about cellular automata), then we define the decision problem PB-PROP by:

- **input:** a finite description of cellular automaton $(\mathcal{A}, \mathbb{U}, f)$

- **question:** does the global map F associated to this finite description have the property PROP?

In any dimension d, the problem PB-CONSTANT is easy.

Fixed-points of cellular automata are exactly the same sets as subshifts of finite type.

Proposition 5.1 *Let d be a fixed dimension. It holds:*

1. *if $\mathbf{X}$ is a d-dimensional subshift of finite type, then there is a cellular automaton F with:*
$$X = \{x : F(x) = x\}.$$
2. *if F is a d-dimensional cellular automaton, then the set $\{x : F(x) = x\}$ is a d-dimensional subshift of finite type.*

Corollary 5.3 *The problem* PB-FIXED-POINT *is undecidable in dimension $d \geq 2$ and decidable in dimension* 1.

When we consider first-order properties, we can expect to have undecidability in dimension $d \geq 2$ for a lot of properties due to the possibility of encoding tile sets in the cellular automaton. Not all properties are undecidable in dimension $d \geq 2$ though (*e.g.* PB-CONSTANT).

On the contrary, for dimension 1, there is a tool that allows to decide *any* first-order property: Büchi automata. Their are finite automata recognizing semi-infinite words (*i.e.* elements of $\mathcal{A}^{\mathbb{N}}$).

Definition 5.6 *A Büchi automaton is a 5-uple $(\mathcal{A}, Q, q_0, F, \delta)$ where*

- *$\mathcal{A}$ is the* alphabet*;*
- *Q is the set of* states*;*
- *$q_i \in Q$ is the* initial state*;*
- *$F \subseteq Q$ is the set of* accepting states*;*
- *$\delta : \mathcal{A} \times Q \to 2^Q$ is the (non-deterministic)* transition map*.*

Given a semi-infinite word $\omega \in \mathcal{A}^{\mathbb{N}}$, a run *of the automaton over ω is a sequence of states $(q_n)_{n\in\mathbb{N}}$ such that $q_0 = q_i$ and, for any $n \in \mathbb{N}$, $q_{n+1} \in \delta(w_n, q_n)$.*

We say that the word ω is accepted *by the automaton if there exists a valid run $(q_n)_{n\in\mathbb{N}}$ and an accepting state $q_f \in F$ that occurs infinitely many times in the run, i.e.*
$$\{n : q_n = q_f\} \text{ is infinite.}$$

The language *accepted by the automaton is the set of all accepted (semi-infinite) words.*

For a modern exposition of the theory of automata on infinite words in general, and the following theorem in particular, see [82].

Theorem 5.5 (J. R. Büchi, [17])

1. *it is decidable to know whether a given Büchi automaton accepts a non empty language.*
2. *for any Büchi automaton accepting some language, there exists a Büchi automaton accepting exactly the complement.*

The following theorem was first proved by K. Sutner [97], then extended by O. Finkel [34].

Theorem 5.6 *For any first-order property* PROP*, the problem* PB-PROP *is decidable in dimension* $d = 1$.

Going back to dimension at least 2, two other important first-order properties are known to be undecidable and thus show a complexity gap between dimensions 1 and 2.

Theorem 5.7 (J. Kari, [54]) *The problems* PB-SURJECTIVITY *and* PB-INJECTIVITY *are both undecidable in dimension* $d \geq 2$.

5.5 Topological dynamics

For a survey of topological dynamics of cellular automata, see [65]. Here we will just consider some long-term dynamical properties, as opposed to the "one-step" kind of properties of the previous section.

Definition 5.7 *The* limit set *of a cellular automaton* F *is the set:*

$$\Omega_F = \bigcap_t F^t\big(\mathcal{A}^{\mathbb{Z}^d}\big)$$

Proposition 5.2 Ω_F *is the set of configuration with an "*infinite history*", i.e.* $x \in \Omega_F$ *if and only if there is a an infinite sequence* $(x_n)_{n\in\mathbb{N}}$ *such that* $x_0 = x$ *and* $F(x_{n+1}) = x_n$.

To any d-dimensional cellular automaton F given by the local description $(\mathcal{A}, \mathbb{U}, f)$ we can associate the $(d+1)$-dimensional subshift of finite type $\mathbf{X}_F$ defined by:

$$x \in \mathbf{X}_F \Leftrightarrow \forall \mathbf{i} \in \mathbb{Z}^d, x\big(\mathbf{i} + (0, \ldots, 0, 1)\big) = f(x_{|\mathbf{i}+\mathbb{U}_0})$$

where $\mathbb{U}_0 = \{(u_1, \ldots, u_d, 0) : (u_1, \ldots, u_d) \in \mathbb{U}\}$.

In $\mathbb{Z}^{d+1}$, for any $t \in \mathbb{Z}$ the *t-slice*, denoted S_t, is the set of positions $\mathbf{i} \in \mathbb{Z}^{d+1}$ whose ultimate coordinate is t.

Proposition 5.3 *$x \in \mathbf{X}_F$ if and only if for any $t \in \mathbb{Z}$:*

- $x_{|S_t} \in \Omega_F$;
- $x_{|S_{t+1}} = F\big(x_{|S_t}\big)$.

The special case $\Omega_F = \mathcal{A}^{\mathbb{Z}^d}$ is equivalent to surjectivity of F. In that case, $\mathbf{X}_F$ contains all the possible orbits of the dynamical system F. In the general case (non-surjectivity), this is not true.

Note that Ω_F is never empty, so, at the extreme opposite of surjectivity (Ω_F is the full space), we have *nilpotency*: Ω_F is a singleton.

Proposition 5.4 *F is nilpotent if and only if there is some t such that F^t is a constant map.*

A property of limit sets is a property such that if two cellular automata with the same limit set, then either both or none have the property. Nilpotency is a property of the limit set. Note that surjectivity is not: $\mathcal{A}^{\mathbb{Z}^d}$ can be the limit set of a surjective cellular automaton of alphabet $\mathcal{A}$ or the limit set of a non-surjective cellular automaton with an alphabet strictly bigger than $\mathcal{A}$.

Theorem 5.8 (J. Kari, [52]) *For any dimension d the problem* PB-NILPOTENCY *is undecidable.*

A property of limit sets is *non trivial* if there is a cellular automaton that verifies the property and another that do not verify the property.

Theorem 5.9 (J. Kari, [53]) *For any dimension d, any non-trivial property of limit sets is undecidable.*

5.6 Ergodic dynamics

In this section we consider only dimension 1.

We denote by $\mathcal{M}(\mathcal{A}^{\mathbb{Z}^d})$ the set of Borel probability measures on $\mathcal{A}^{\mathbb{Z}^d}$. By Carathéodory extension theorem, Borel probability measures are characterized by their value on cylinders whose domain is an interval. Therefore a measure is given by a function μ from cylinders to the real interval $[0, 1]$ such that

1. $\mu(\mathcal{A}^{\mathbb{Z}^d}) = 1$ and
2. for all pattern $p : \mathbb{U} \to \mathcal{A}$ with $\mathbb{U}$ an interval, we have:

$$\sum_{a \in \mathcal{A}} \mu([p \cdot a]) = \sum_{a \in \mathcal{A}} \mu([a \cdot p]) = \mu([p])$$

where $p \cdot a$ and $a \cdot p$ are the patterns extending p one cell to the right or to the left (respectively) and taking value a on that new cell.

A measure μ is said to have *full support* if for every pattern p we have $\mu([p]) > 0$.

A measure μ is said to be *translation invariant* if for any measurable set E and any $\mathbf{i}$ we have $\mu(E) = \mu(\mathfrak{S}_{\mathbf{i}}(E))$.

The *uniform measure* μ_0 is defined on any cylinder $[p]$ of domain $\mathbb{U}$ by:

$$\mu_0([p]) = |\mathcal{A}|^{-|\mathbb{U}|}.$$

More generally a *Bernoulli measure* is a product measure, *i.e.* a measure defined by a probability vector $(\pi_a)_{a\in\mathcal{A}}$ over the alphabet and extended to any cylinder p of domain $\mathbb{U}$ by product:

$$\mu([p]) = \prod_{\mathbf{i}\in\mathbb{U}} \big(\pi_{p(\mathbf{i})}\big).$$

Given a measure μ and a cellular automaton F, we define the measure $F\mu$ by:

$$F\mu([p]) = \mu\big(F^{-1}([p])\big).$$

Theorem 5.10 *A cellular automaton F is surjective if and only if it preserves the uniform measure: $F\mu_0 = \mu_0$.*

One thing we are interested in is the sequence of measures $\big(F^t\mu\big)_{t\in\mathbb{N}}$ starting from a given initial measure μ. In particular, what is the evolution of the probability of a given pattern.

Theorem 5.11 (D. Lind, [71]) *There exists a cellular automaton F, such that, for any Bernoulli measure μ of full support, we have:*

$$\frac{1}{T}\sum_{1\le t\le T} F^t\mu \to_T \mu_0.$$

At the opposite, we can have for some cellular automata that the sequence $\big(F^t\mu\big)_{t\in\mathbb{N}}$ does not converge to the uniform measure μ_0, even when started from $\mu = \mu_0$. We say that a pattern p *disappears* for μ if

$$F^t\mu([p]) \to_t 0.$$

If a pattern does not disappear, we say it is *μ-persistent.*

Definition 5.8 *Given a cellular automaton F and a measure μ, the* μ-limit set *of F is the subshift $\Omega_{F,\mu}$ made of configurations containing only μ-persistent patterns.*

We say that a CA is μ-nilpotent if $\Omega_{F,\mu}$ is a singleton.

Theorem 5.12 (Boyer-Poupet-Theyssier, [11]) *The problem* PB-μ-NILPOTENCY *is undecidable.*

Theorem 5.13 (M. Delacourt, [26]) *Let* PROP *be any non-trivial property of μ_0-limit sets (where μ_0 is the uniform measure), i.e. there is a cellular automaton F_1 such that Ω_{F_1,μ_0} has the property and another cellular automaton F_2 such that Ω_{F_2,μ_0} doesn't have the property. Then* PB-PROP *is undecidable.*

Chapter 6

Cut and Projection

6.1 Planar tilings

Cut and projection is one of the main ways to obtain non-periodic tilings. We here focus on *canonical* cut and projection. Here is the simplest case:

1. in the Euclidean plane, consider a line D such that $D \cap \mathbb{Z}^2 = \emptyset$;
2. translate the unit square along D to obtain the "slice" $D + [0,1]^2$;
3. select (*cut*) all the unit segments of $\mathbb{Z}^2$ that lies within this slice;
4. *project* these unit segments orthogonally onto D.

This yields a tiling of D by projections of horizontal and vertical unit segments of $\mathbb{Z}^2$ (see Fig. 6.1). The tiling is aperiodic if and only if D is irrational, and seen as a two-letter sequence it is then a Sturmian word.

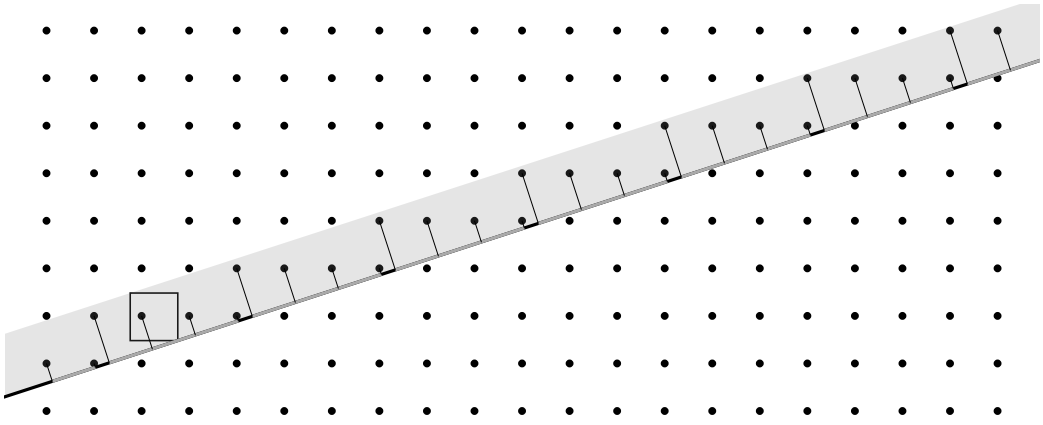


Figure 6.1: The canonical cut and project method in the $2 \to 1$ case.

In other terms, we cut a two-dimensional space (the Euclidean plane) by a one-dimensional affine space (the line D). The obtained tiling is therefore called a *planar* $2 \to 1$ *tiling*. This extends to higher dimensions:

Definition 6.1 (Planar tiling) *Let E be a d-dimensional affine space in $\mathbb{R}^n$ such that $E \cap \mathbb{Z}^n = \emptyset$. Select the d-dimensional faces of the unit hypercubes of $\mathbb{Z}^n$ lying in $E + [0,1]^n$ and project them orthogonally onto E. This yields a tiling of E and, by identifying E with $\mathbb{R}^d$, a so-called* planar $n \to d$ tiling.[1]

The space E is called the *slope* or *strain* of the tiling. The cases of planar $3 \to 1$ tilings (*billard words*) and planar $3 \to 2$ tilings (*discrete planes*) are rather intuitive, but this holds in any dimensions (see, *e.g.*, Fig. 6.2, right).

6.2 Dualization of multigrids

Definition 6.2 (Multigrid) *The* multigrid *with shifts $s_1, \ldots, s_n$ in $\mathbb{R}$ and* grid vectors *$\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^d$ is the set of n families of equally spaced parallel hyperplanes*

$$H_i := \{\vec{x} \in \mathbb{R}^d \mid \langle \vec{x} | \vec{v}_i \rangle + s_i \in \mathbb{Z}\},$$

where at most d hyperplanes are assumed to intersect in a point (this is generic).

Such a multigrid divides $\mathbb{R}^d$ into cells. Its *dualization* is obtained by associating with each cell a vertex in $\mathbb{R}^d$ and connecting vertices associated with adjacent cells, with the vertices corresponding to two cells adjacent along a hyperplane in H_i differing by $\vec{v}_i$. This yields a tiling of $\mathbb{R}^d$ (see Fig. 6.2). Introduced in [24], this method has been proven to be equivalent to the canonical cut and projection method:

Theorem 6.1 ([40]) *Multigrid dualizations are planar tilings, and conversely.*

One can moreover explicitly swap between the two methods. The planar tiling obtained by cut and projection from a d-dimensional affine plane E is indeed obtained (up to the identification of E with $\mathbb{R}^d$) by the dualization of the multigrid whose hyperplanes are the intersection of E with

$$G_i := \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x} | \vec{e}_i \rangle \in \mathbb{Z}\}.$$

Thus, if E is generated by d orthonormal vectors of $\mathbb{R}^n$, with u_{ij} being the j-th entry of the i-th vector, then the i-th entry of the j-th grid vector is u_{ij} divided by $\sum_i u_{ij}^2$ (which is not necessarily equal to one: mind the summation indice).

[1] In order to get a tiling of $\mathbb{R}^d$, the identification of E with $\mathbb{R}^d$ must map the orthogonal projections onto E of the standard basis of $\mathbb{R}^n$ onto vectors such that the skew product of any d vectors has the same sign as the skew product of their images.

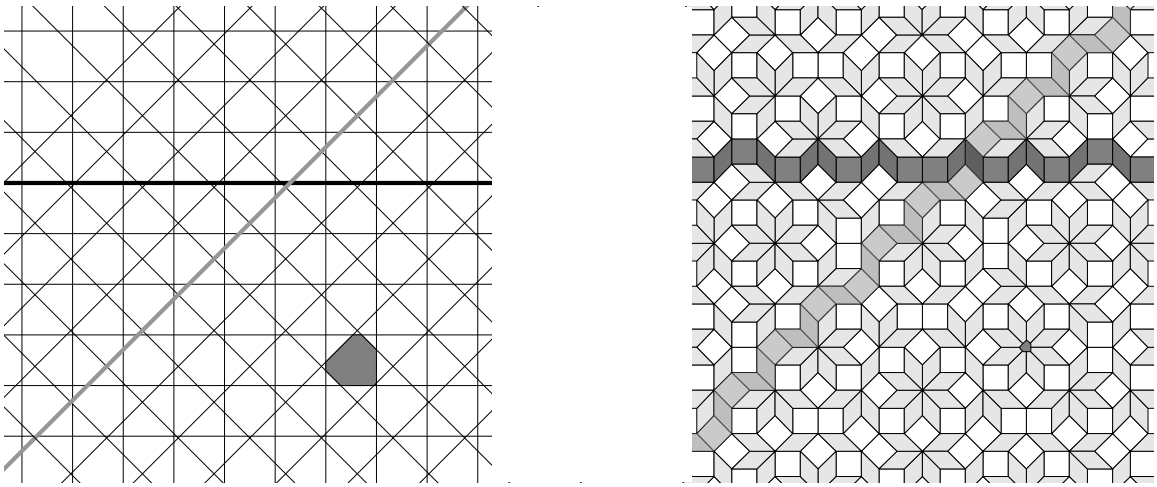


Figure 6.2: A multigrid with four line families (left) and the dual planar $4 \to 2$ tiling (right). The intersections on the black (resp. dark grey) thick line (left) correspond to the black (resp. dark grey) tiles (right). The shaded octagonal cell (left) corresponds to the shaded vertex of degree 8 (right).

6.3 Grassmann coordinates

It is sometimes convenient to express the slope of a planar tiling in terms of *Grassmann coordinates* (see, *e.g.*, [45], chap. 7, for a detailed account).

Definition 6.3 (Grassmann coordinates) *Let E be a d-dimensional vector space of $\mathbb{R}^n$ generated by $\vec{u}_1, \ldots, \vec{u}_d$. Its* Grassmann coordinates *are the $d \times d$ minors of the $n \times d$ matrix whose i-th column is $\vec{u}_i$.*

In codimension 1, ($d = n-1$), Grassmann coordinates just form the normal vector of E. They are unique up to a common multiplicative factor. A non-zero $\binom{n}{d}$-tuple $(G_{i_1,\ldots,i_d})$ of real numbers are the Grassmann coordinates of some vector space if and only if they satisfy, for any $1 \leq k \leq n$ and any two d-tuples of indices, the so-called *Plücker relation*:

$$G_{i_1,\ldots,i_d} G_{j_1,\ldots,j_d} = \sum_{l=1}^{d} \underbrace{G_{i_1,\ldots,i_d} G_{j_1,\ldots,j_d}}_{\text{swap } i_k \text{ and } j_l}.$$

Actually, up to a renormalization, $|G_{i_1,\ldots,i_d}|$ gives the proportion of tiles generated by $\vec{v}_{i_1}, \ldots \vec{v}_{i_d}$ in the planar tilings of slope E, where $\vec{v}_i$ is the orthogonal projection on E of the i-th vector of the standard basis of $\mathbb{R}^n$.

6.4 Patterns

A *pattern* of a tiling is a finite subset of the tiles of this tiling. A pattern formed by the tiles which intersect a closed ball of radius r is called a *r-map*. The set of r-maps of a tiling is called its *r-atlas*. The notion of *window* is useful to study patterns:

Definition 6.4 (Window) *The* window *of a planar $n \to d$ tiling with slope E is the compact set obtained by orthogonally projecting $E + [0,1]^n$ on $E^\perp$.*

For example, any planar tiling can be recovered from the projection $\vec{x}$ of a point of $\mathbb{Z}^n$ in its window. Let indeed $\vec{e}_i$ denotes the i-th vector of the standard basis of $\mathbb{R}^n$ and π_E (resp. $\pi_{E^\perp}$) the orthogonal projection onto E (resp. $E^\perp$). We then simply proceed edge by edge starting from $\vec{x}$: any jump by $\pi_{E^\perp}(\vec{e}_i)$ in the window corresponds to a jump by $\vec{e}_i$ in the slice $E + [0,1]^n$, that is, by definition, to an edge of the tiling directed by $\pi_E(\vec{e}_i)$.

Further, one can associate with any pattern P of a planar tiling of slope E a region of its window, such that whenever the orthogonal projection onto $E^\perp$ of a vertex of $\mathbb{Z}^n$ falls into this region, its orthogonal projection onto E falls into a pattern equal to P up to a translation. This is indeed the (polygonal) region starting from which any sequence of jumps corresponding to an edge path of P leads to a point in the window. This is used to compute the (asymptotic) *complexity* of planar tilings:

Theorem 6.2 ([50]) *Generic planar $n \to d$ tilings have an r-atlas of size $\Theta(r^{d(n-d)})$.*

Planar tilings are moreover *repetitive*, that is, whenever a pattern occurs somewhere, it reoccurs at uniformly bounded distance from any point. Even better, patterns have *frequencies*. Formally, denote by W the closure of $\pi_E(\mathbb{Z}^n)$ in the window and by μ the Lebesgue measure on W. Then, for any measurable subset S of W and any $\vec{y} \in \mathbb{R}^n$:

$$\lim_{r\to\infty} \frac{\mathrm{Card}\{\vec{x} \in \mathbb{Z}^n \cap B(\vec{y}, r) \mid \pi_{E^\perp}(\vec{x}) \in S\}}{\mathrm{Card}\{\vec{x} \in \mathbb{Z}^n \cap B(\vec{y}, r) \mid \pi_{E^\perp}(\vec{x}) \in W\}} = \mu(S),$$

where $B(\vec{y}, r)$ is the ball of center $\vec{y}$ and radius r. In particular, if S is the region associated with a pattern, then $\mu(S)$ is the frequency of this pattern. By looking how the set W changes by modifying the slope, we can also prove

Proposition 6.1 ([67, 69]) *If two planar tilings have parallel slopes which belong to the same rational subspace, then they have the same finite patterns.*

6.5 Local rules

Planar $n \to d$ tilings (Def. 6.1) are specific tilings of $\mathbb{R}^d$ by $\binom{n}{d}$ tiles. But there are many other tilings of $\mathbb{R}^d$ by the same tiles: those are called *$n \to d$ tilings*. The *lift* of

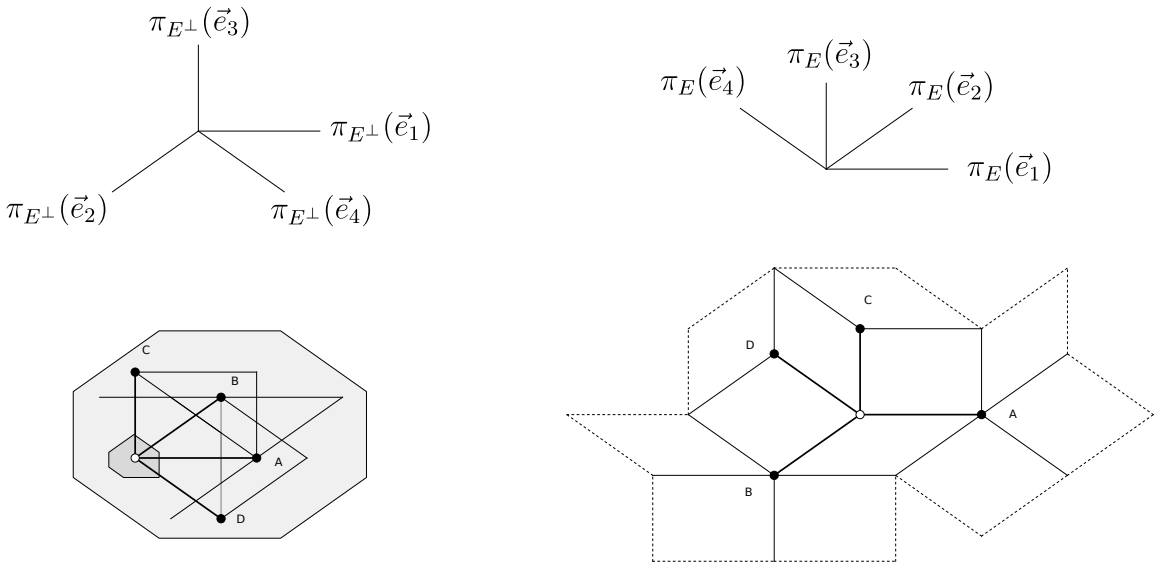


Figure 6.3: The projections of the standard basis of $\mathbb{R}^n$ on E (top-right) and $E^\perp$ (top-left) are used to associate with any pattern of a planar tiling of slope E (bottom-right) a polygonal regions in the window (bottom-left, shaded).

such a tiling is the d-dimensional surface of $\mathbb{R}^n$ obtained by mapping the n vectors which define the edges of the tiles onto the standard basis of $\mathbb{R}^n$ (the lift is unique up to a translation). The planar tilings are exactly those whose lift lies in a slice $E + [0,1]^n$.

A question raised by the study of *quasicrystals* is to characterize, among the planar tilings, those characterized only by local constraints. Planar tilings are indeed commonly used to model the long-range order of quasicrystals, and the above question is related to how finite range interactions can stabilize quasicrystals. Formally, we follow [69]:

Definition 6.5 (Local rules) *A planar $n \to d$ tiling of slope E is said to have* local rules *of* thickness t *and* diameter r *if any $n \to d$ tiling whose r-atlas is less or equal (for inclusion) has a lift which lies in the slice $E + [0,t]^n$.*

In dynamical terms, local rules define a tiling space of finite type, whose tilings are "almost" planar with the same slope E. Local rules are said do be *strong* if they have thickness $t = 1$, *weak* otherwise. Strong local rules are said to be *perfect* if they define tilings which have all the same finite patterns, that is, a *minimal* tiling space.

6.6 Sufficient conditions

We are here interested in sufficient condition on the slope of a planar tiling for having local rules. The easiest case is the one of rational slopes:

Proposition 6.2 *Any rational planar tiling has perfect local rules.*

Indeed, such a tiling is made of a fundamental pattern repeated along a lattice, and local rules whose diameter is some times larger than this fundamental pattern shall enforce it to repeat as in along this lattice. Irrational cases are more challenging. A celebrated example are *Penrose tilings* (Fig. 6.4 and 6.5):

Definition 6.6 (Penrose tiling) *A* Penrose tiling *is a planar* $5 \to 2$ *tiling whose slope is parallel to the plane generated by* $(\cos(\frac{2k\pi}{5}))_{0\leq k<5}$ *and* $(\sin(\frac{2k\pi}{5}))_{0\leq k<5}$ *and contains a point with entries that are all equal and whose sum is an integer.*

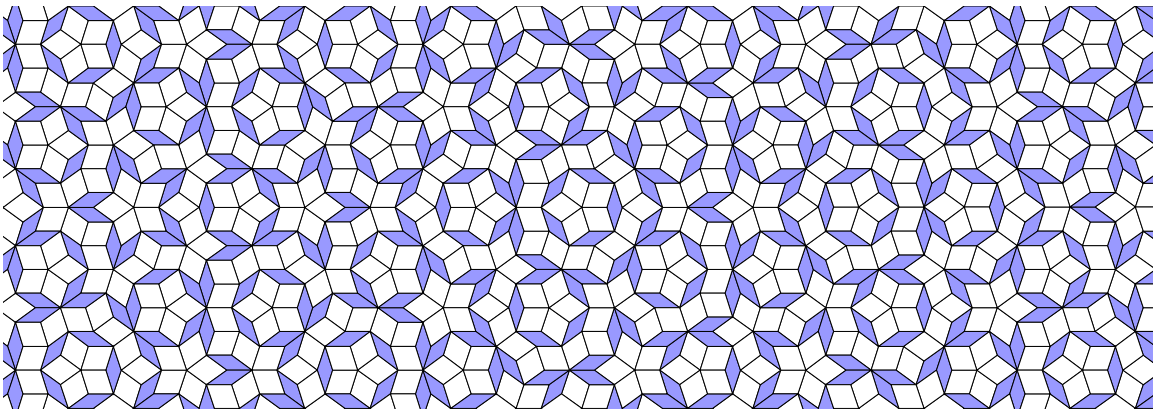


Figure 6.4: A Penrose tiling.

Theorem 6.3 *[24] The Penrose tilings have perfect local rules of diameter* 0.

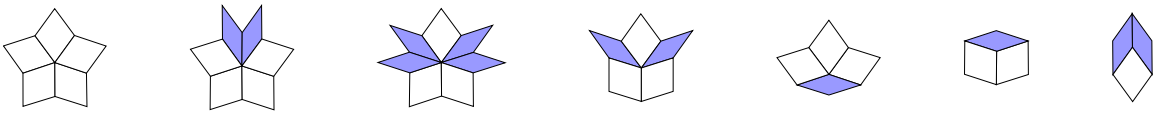


Figure 6.5: The 0-atlas (or *vertex-atlas*) of Penrose tilings. Any tiling whose tiles arrange in one of these seven way around every vertex is necessarily a Penrose tiling.

If the slope of a Penrose tiling does not contains a point with equal entries whose sum is in $\mathbb{Z}$, this yields a so-called *generalized Penrose tiling*. They have weak local rules (of positive diameter) and no perfect local rules. They are conjectured to have strong local rules. Penrose and generalized Penrose tilings actually belong to the family of n-fold tilings (for $n = 10$):

Definition 6.7 (n-fold tiling) *A* n-fold tiling *is a tiling of the plane which has the same finite patterns as its image under a rotation by* $2\pi/n$.

The planar n-fold tilings are those whose slope is generated by $(\cos(2k\pi/n))_k$ and $(\sin(2k\pi/n))_k$, $0 \leq k < n/2$ (n is necessarily even). Among them, not only the Penrose tilings have local rules:

Theorem 6.4 ([93]) *The n-fold tiling has weak local rules for $n \mod 4 \neq 0$.*

A key ingredient of the proof is that the slope is enforced by a simple *alternation condition*: in any "stripe", each tile appears in two orientations which must perfectly alternate. This condition translates into an equation of the type $G_{ij} = G_{ik}$ on the Grassmann coordinates of the slope. This is actually a particular case of *subperiod*:

Definition 6.8 (Subperiod) *Let I be a subset of $d+1$ elements of $\{1, \ldots, n\}$. A d-plane of $\mathbb{R}^n$ has a I-*subperiod *if there is a rational dependence between its $d+1$ Grassmann coordinates whose indices are all in I. By extension, we call subperiod of a planar tiling any subperiod of its slope.*

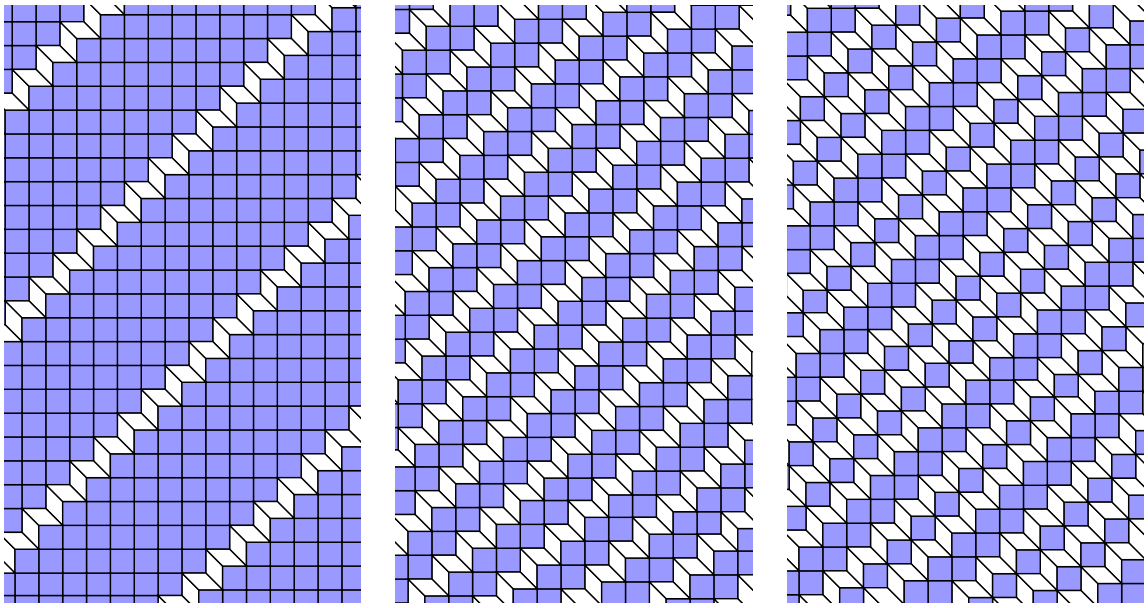


Figure 6.6: The respective projections on three basis vectors of the tilings of Fig. 6.7. Although different, they all have the same periodic direction (along the white stripes).

Equivalently, a planar $n \to d$ tiling has a I-subperiod if, by orthogonally projecting it onto the d-dimensional space generated by the vectors of the standard basis of $\mathbb{R}^n$ whose indices are in I, one gets a periodic tiling (Fig. 6.6). One shows that subperiods can be enforced by local rules. In particular, if a slope is characterized by its subperiods, then there are local rules to enforce the corresponding planar tiling *among planar tilings*. But it is generally unclear whether these local rules could allow *non-planar* tilings. Some cases are known (see [16]), in particular for $4 \to 2$ tilings:

Theorem 6.5 ([15]) *A planar $4 \to 2$ tiling has weak local rules iff its slope is characterized by its subperiods.*

6.7 Necessary conditions

Proposition 6.3 *Irrational planar $n \to n-1$ tilings do not have local rules.*

This can be proven by looking what happens in the window when the slope slightly varies. The first local rules appear for $4 \to 2$ tilings (this easily follows from Th. 6.5). However, the most celebrated $4 \to 2$ tilings, namely the 8-fold tilings (also referred to as Ammann-Beenker tilings) do not have local rules [13]. This is more general:

Theorem 6.6 ([14]) *The $4p$-fold tilings do not have local rules.*

The proof relies on the fact that the subperiods of such slopes do not characterize them. Instead, they characterize a one-parameter family of slopes, where two slopes have the same finite patterns of a given diameter if they are sufficiently close. Fig. 6.7 show some tilings in this family for $n = 8$.

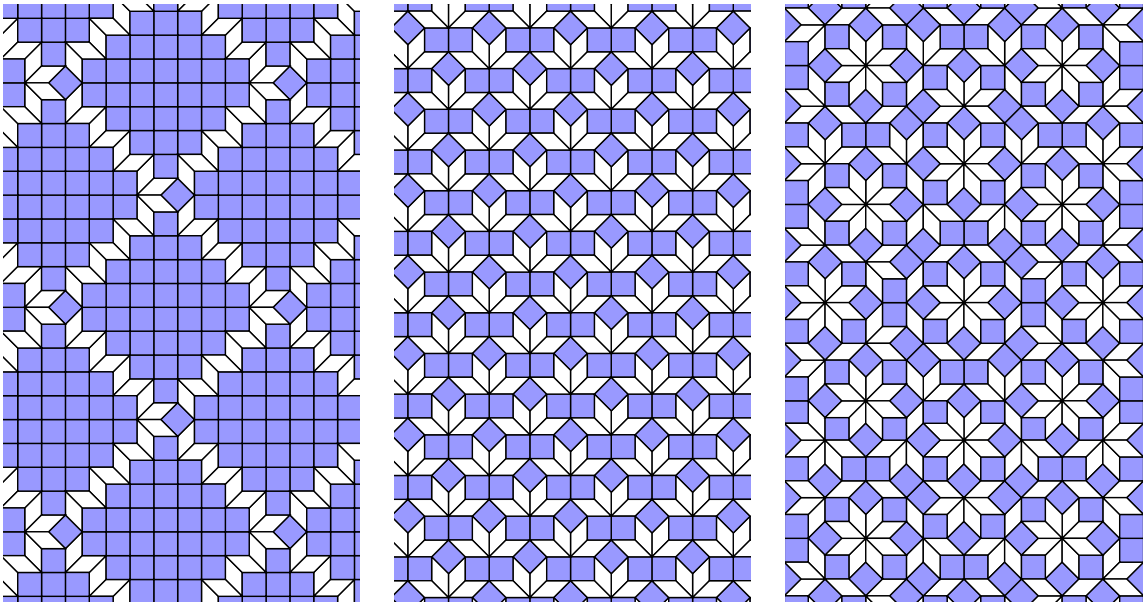


Figure 6.7: Planar tilings with the same subperiods as an 8-fold tiling.

Levitov has given a necessary condition for the existence of strong local rules. A plane is said to have *full subperiods* if it has an I-subperiod for any subset I of $d+1$ elements of $\{1, \ldots, n\}$.

Theorem 6.7 ([69]) *A planar tiling with strong local rules must have full subperiods.*

Otherwise, it is possible to slightly shift the slope in a suitable direction, so that the subperiods "miss" the resulting flips. As a corollary, Levitov obtained that n-fold tilings can have strong local rules only for $n \in \{4, 6, 8, 10, 12\}$. Since $n \in \{4, 6\}$ are periodic cases and $n \in \{8, 12\}$ are forbidden by Th. 6.6, the remaining question is: do the 10-fold tilings (*i.e.*, the generalized Penrose tilings) admit strong local rules? One also knows a general "algebraic obstruction" to the existence of local rules:

Theorem 6.8 ([67]) *The slope of a planar tiling with local rules must be algebraic.*

In order to have local rules, a planar tiling must indeed be such that any modification of its slope creates a new pattern (which does not appeared in the tiling). Seen in the windows of a planar $n \to d$ tiling, this means that the slope is characterized by so-called *coincidences*, where a coincidence is a point at the intersection of at least $n-d+1$ projections of $(n-d)$-dimensional facets of $\mathbb{Z}^n$ so that any modification of the slope "explodes" this point into a non-empty region corresponding to a new pattern in the tiling. The theorem follows by proving that such coincidences can characterize only algebraic slope. For planar $4 \to 2$ tilings, one moreover knows that the slope is at most *quadratic* (this follows from Th. 6.5). The maximal algebraic degree for a planar $n \to d$ tiling is unknown beyond this particular case (one conjectures $\lfloor n/d \rfloor$).

6.8 Colored local rules

We mentioned that, in dynamical terms, local rules define a tiling space of finite type. This can actually extend to *sofic tiling spaces*, that is, to allow tiles to be "decorated". Let us consider, for example, the one-parameter family of tilings with the same subperiods as the 8-fold tilings (see Theorem 6.6). Theses are actually exactly the tilings by a square and a rhombi with 45° acute angles such that in any "stripe" the rhombi appear in two orientations which perfectly alternate. This can easily be enforced by notching the tiles as depicted on Fig. 6.8 but not by the previously defined local rules (Definition 6.5), because the information about the orientation of a rhombus need to be carried arbitrarily far. Among several ways to formally "decorate" tiles, we use the following one.

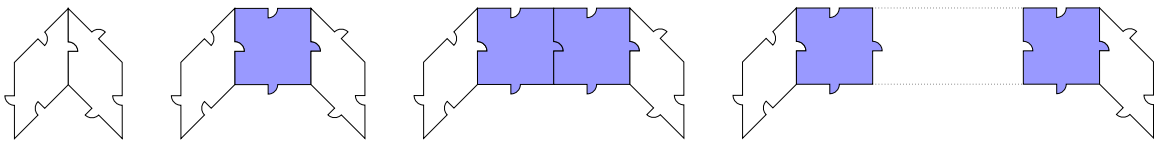


Figure 6.8: These notched square and rombus make consecutive rhombi in a stripe to alternate their orientation. Up to the notching, the possible tilings are exactly those with the same subperiods as the 8-fold tilings (recall Fig. 6.7).

Definition 6.9 (Colored local rules) *A planar $n \to d$ tiling of slope E is said to have* colored local rules *of* thickness *t if there is a finite set tiles with colored boundaries such that any tiling by these tiles where two adjacent tiles match along boundaries with the same colors is a $n \to d$ tiling with a lift in the slice $E + [0,t]^n$.*

When colored local rules are allowed, the algebraic obtruction of Theorem 6.8 transforms into a computability obstruction - and actually a complete characterization of the slopes that can be enforced by such local rules (at the price of a huge number of different tiles).

Definition 6.10 (Computable slope) *A d-dimensional linear space in $\mathbb{R}^n$ is* computable *if it is generated by vectors whose entries can be approximated by a rational within any desired precision by a finite, terminating algorithm.*

Theorem 6.9 ([33]) *A planar tiling has colored local rules iff its slope is computable.*

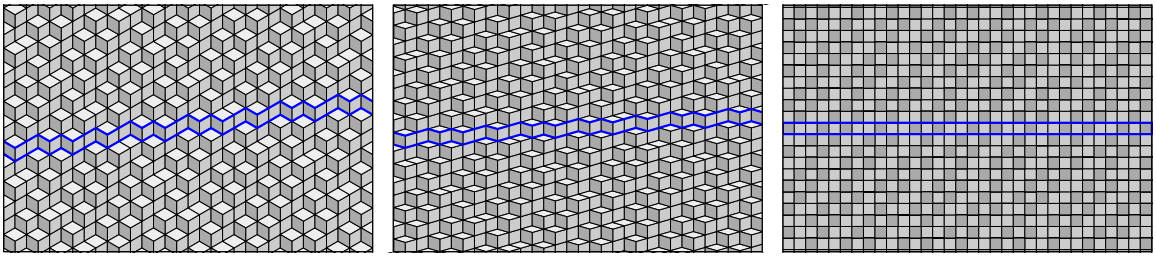


Figure 6.9: From a $3 \to 2$ tiling to a two-dimesional subshift.

The idea of the proof is to make a connection with the effective subshifts introduced in Chapter 3. Indeed, $n \to d$ tilings can be seen as intertwined d-dimensional subshifts, as suggested by Fig. 6.9. Then Theorem 3.3 (page 28) is used to check that the stripes of the $n \to d$-tilings correspond to the wanted slope.

Chapter 7

Self-Assembly

7.1 Abstract self-assembly

7.1.1 Introduction

The self-assembly model has been introduced by Adelman, Seeman et al. in [105], and later studied more formally, notably by Erik Winfree. It is a formalization of a model of DNA-based computation, using the attractive strength between complementary strands of DNA. This model, which has actually been implemented for instance in [90, 81], is quite close to Wang Tilings, a staple of theoretical computer science.

Resources are available online at `http://self-assembly.net`.

Preliminaries

We note the base vectors of $\mathbb{Z}^2$ as $N = (0, 1)$, $S = (0, -1)$, $E = (1, 0)$, $W = (-1, 0)$. For a set Σ, 4-tuples $x = (x_0, x_1, x_2, x_3)$ are identified as unit squares, and we note $N(x) = x_0$, $E(x) = x_1$, $S(x) = x_2$, $W(x) = x_3$ (i.e, going around x clockwise). We also have $-N = S$ and $-E = W$.

An edge of $\mathbb{Z}^2$ is a couple $(z, z + d)$ for $z \in \mathbb{Z}^2$ and $d \in \{N, S, E, W\}$, i.e., the link between two adjacent positions. A set E of edges is a *cut* if there are $z, z' \in \mathbb{Z}^2$ such that any path between z and z' contains at least an edge of E. For $U \subset \mathbb{Z}^2$, a cut of U is a set of edges with both extremities in U such that there are $z, z' \in U$ such that any path between z and z' *within* U contains at least an edge of E.

For any t and $x, y \in \mathbb{Z}^2$, the singleton pattern $\{t@(x, y)\}$ is the pattern p with domain $\{(x, y)\}$ defined by $p(x, y) = t$.

7.1.2 The model

Definitions

Self-assembly can be abstractly defined through the following characteristics:

1. a *high number*

2. of *simple* components,
3. *aggregating* together
4. through *local* interaction
5. to form *complex geometrical designs*.

Systems with these characteristics are frequent in nature, from the growth of crystals (and quasi-crystals), notably snowflakes, to coral reefs, viruses or galaxies. The aim here is to mimic these processes in an algorithmic, formal way.

These abstract characteristic can be turned into a formal model, ready for mathematical study. This model was first defined in [103]. First, the *high number* of particles can be represented, for simplicity's sake by an infinity of them. From now on, we will call these particles *tiles*. In contrast, since they need to be *simple*, they will be taken from a finite set of *meta-tiles*. As for *geometry*, the most general setting would be a Cayley Graph of some sort; in this introduction, the tiles will simply be squares; live in $\mathbb{Z}^2$.

The last abstract characteristic is local aggregative interactions. This is rendered as the ability for tiles to stick to each other or not when they are abutting; the tiles stick to each other with selective *glues*. The rest of this subsection is devoted to defining in detail this abtsract model of assembly, the *abstract Tile Assembly Model* (**aTAM**).

Definition 7.1 (Self-assembling tileset) *A* self-assembling tileset $\mathcal{T}$ *is composed of:*

- *a finite set of* glues $\mathcal{G}_\mathcal{T}$*;*
- *a* strength *function,* $s_\mathcal{T} : \mathcal{G}_\mathcal{T} \to \mathbb{N}$
- *a finite subset* $T_\mathcal{T}$ *of* meta-tiles*,* $T_\mathcal{T} \subset \mathcal{G}_\mathcal{T}^4$*;*

Figure 7.1 shows how tilesets are usually represented. The set $\mathcal{G}$ of glues is taken to be a set of words; each meta-tile is a square with a glue on each of its sides. Instead of representing s seperately, each glue is represented with a number of ticks equal to its strength. This means that for some glue a, if $s(a) = 2$, a will be represented with 2 ticks everywhere it appears. By abuse of notation, we will sometimes use the same symbol $\mathcal{T}$ to refer to $T_\mathcal{T}$, and say for instance "for any meta-tile in the tileset $\mathcal{T}$".

An instance of a meta-tile somewhere in the plane $\mathbb{Z}^2$ is called a *tile*, and a collection of tiles is a *pattern*. Most of this chapter on self-assembly is devoted to the study of self-assembly of finite patterns, but self-assembly of configurations on the whole plane $\mathbb{Z}^2$ has been studied too [80, 79], though less often.

A tileset only assembles into interesting production when put in some specific environment, forming a *self-assembling system*. If a tileset represents a set of molecules ready to self-assemble into some artefact, it is of no use leaving the molecules inert and separate: they need to be put into a test tube at the right temperature, with

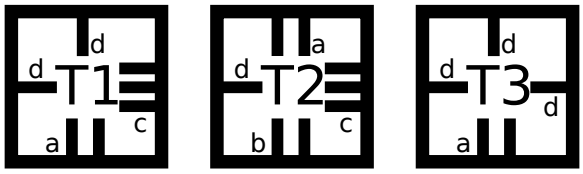


Figure 7.1: An example tileset, with glues a, b, c, d, three meta-tiles T_1, T_2 and T_3. s is defined as $s(a) = s(b) = 2$, $s(c) = 3$, and $s(d) = 1$.

a nucleation seed before anything happens. A self-assembling system represents the conditions in which the tileset will evolve.

Definition 7.2 (Self-assembling system) *A* self-assembling system $\mathcal{S}$ *is composed of:*

- *a self-assembling tileset* $\mathcal{T}(S)$*;*
- *an integer* $\tau(\mathcal{S})$*, the* temperature*;*
- *a pattern* $\sigma(\mathcal{S})$ *of tiles of* $\mathcal{T}(S)$*, the* seed

To avoid double indirection, the following notations are useful: $G(\mathcal{S})$ is the set of glues $G_{\mathcal{T}(\mathcal{S})}$ of the corresponding tileset, $T(\mathcal{S}) = T_{\mathcal{T}(\mathcal{S})}$ is the set of its tiles, and $g(\mathcal{S}) = g_{\mathcal{T}(\mathcal{S})}$ its temperature.

The dynamics

Stability The first thing to consider with a self-assembling system is the *stability* of patterns. Consider a pattern of tiles that has somehow grown up to some point in a test tube. The pattern is held together by attractive forces between the tiles. Stability is the answer to the following question: "will this pattern hold together for long?". In the aTAM, the attractive forces are defined as the link between two adjacent tiles. The link between two adjacent tiles is 0 if the glue on their matching sides is different, and $s(a)$, where s is the strength function, if they are the same glue a.

Definition 7.3 (link) *Let* $\mathcal{S}$ *be a self-assembling system, and let* p *be a pattern of tiles of* $T(\mathcal{S})$ *with domain* $U \subset \mathbb{Z}^2$*. Let* $(z, z + d)$ *be an edge of* $\mathbb{Z}^2$ *such that both* z *and* $z + d$ *are in* U*; the link between* z *and* $z + d$ *is defined as:*

- 0 *if* $d(p(z)) \neq -d(p(z + d))$
- $s_{\mathcal{S}}(a)$ *if* $d(p(z) = -d(p(z + d)) = a$*.*

This link represent the strength of the bound between two tiles along an edge. For a set C of U, the link along C is the sum of the links along all the edges of C.

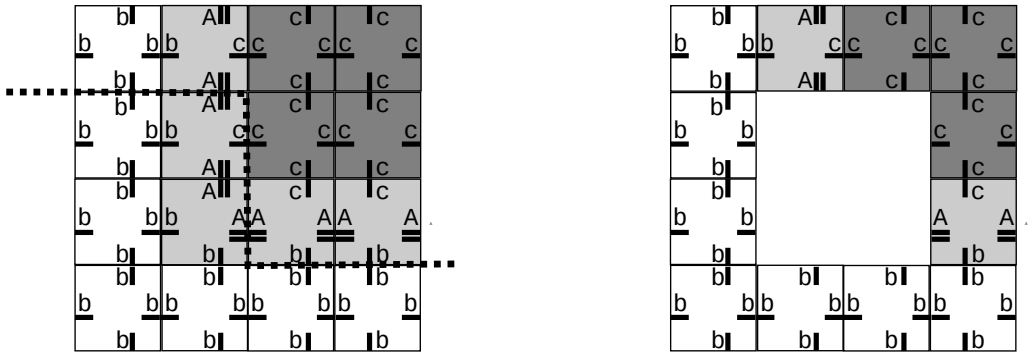


Figure 7.2: Left, an example of link along a cut; the total link along the cut here is 8. Right, a pattern stable at temperature 2 which cannot be grown from smaller stable patterns.

An example of a pattern, a cut and the corresponding link is represented on Figure 7.2 left.

Consider again the pattern in the test tube. This patterns holds as a whole, but thermal agitation threatens to take it apart. If it has a weakness in its fabric, it will be torn along that weakness. Thus, the definition of stability is a negative one: a pattern is *stable* at temperature t if there it has no cut with a link lower than the t.

Growth Taking a self-assembling system and looking at the set of its stable patterns is not very different from classical Wang Tilings. In particular, while stability is a local feature (it happens on an edge-by-edge basis), there is no sense of growth of patterns. In some cases, such as on Figure 7.2 right, some stable patterns cannot be obtained by progressive *growth*, as they have no stable subpatterns.

This progressive growth of patterns is represented by *transitions*.

Definition 7.4 (Transition) *Let $\mathcal{S}$ be a self-assembling system, and let p, p' be two patterns of $\mathcal{S}$. There is a transition $p \to_{\mathcal{S}} p'$ if:*

- *p is a subpattern of p'*
- *p' has exactly one more tile than p*
- *both p and p' are stable.*

Given a self-assembling system $\mathcal{S}$, a pattern q *can be obtained from* p if there exists a sequence of transitions $p = p_0 \to_{\mathcal{S}} p_1 \dots \to_{\mathcal{S}} p_n = q$. The set of *productions* of $\mathcal{S}$, noted $P(\mathcal{S})$ is the set of all patterns that can be obtained from the seed $\sigma(\mathcal{S})$. This focus on a seed is a little arbitrary, but [91] shows how to make sure that no pattern starting from anything else than the seed can grow much, in exchange of a slight increase of the size of the tileset.

It is useful to give a different view on how productions are obtained in the aTAM. Since a transition happens between two patterns which differ by exactly one tile, it

represents the addition of a tile to a stable pattern, with the condition that the link between the pattern and the new tile is at least equal to the temperature.

To be really qualified as *self-assembly*, the assembly process needs also to stop autonomously. A recipe such as "put the following molecules into a test tube, and extract any polymer that is exactly 19830309 atoms big" can hardly be called self-assembly, as weighting and extracting the polymer precisely at the right time is going to take a very precise control that self-assembly is supposed to make unnecessary. This control is not needed if the desired productions do not grow any bigger once they are formed. A *final production* of a self-assembling system $\mathcal{S}$ is production of $\mathcal{S}$ from which no transition is possible. $\mathcal{S}_\square$ is the set of final productions of $\mathcal{S}$.

The study of self-assembly is generally devoted to trying to understand how to get a self-assembling system which gives some target set of final productions.

From now on, we assume without loss of generality that there is only one glue of strength 0.

7.2 Assembly of shapes: squares and rectangles

The most elementary kind of control over self-assembly is getting the particles to form simple shapes such as squares, rectangles through the constructions of [5]. The study of how to assemble these simple shapes yields insights into general properties of self-assembly. It shows how synchronization and temperature 2 plays a primordial place in designing self-assembling system, and how to use randomness. It also introduces tile complexity, which is one of the most important performance measures for self-assembly.

7.2.1 Assembling rectangles

One rectangle versus all rectangles What does it mean to assemble rectangles? Let w, h be integers. The self-assembling system defined on figure 7.3 has one final production whose domain is a $w \times h$ rectangle.

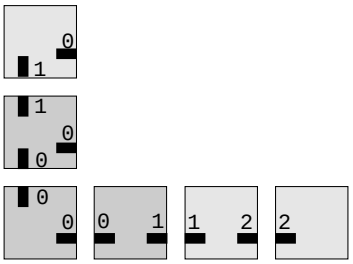


Figure 7.3: A self-assembling system for assemling a 4×3 rectangle; empty edges correspond to the (unique) strength-0 glue.

The assembly of this self-assembling system proceeds by concurrently assembling h rows of length w. This is the most basic technique of self-assembly: cutting the

target shape into parts and have independent sub-assemblies for each part. It can be readily seen that this technique alone is not going to fare well with more complex shapes: in order to count up to w, it needs w tile types. Generating this many tile types is not really more economical than hand-crafting a $w \times h$ rectangle, and in fact a self-assembling system starting from a $w \times h$ rectangle as its seed and doing nothing does just as well as this last system.

Let us now define $R_{h,w}$ as the set of all rectangle with lower-left corner at $(0, 0)$, height at least h and width at least w. A self-assembling system $\mathcal{S}$ assembles R if the set of domains of its final productions is R. In informal terms, such a tile system may have an arbitrary glues and tiles as long as the shapes of its productions are all (large enough) rectangles. Since it is mandated that $\mathcal{S}$ assembles rectangle of all sizes larger than $w \times h$ with a constant number of tiles, it must do so efficiently, and no fixed inert seed will do.

A question of temperature The temperature parameter, which determines when patterns are stable and when tile additions are allowed is crucial in getting control of how the assembly proceeds. Let us first review what happens at temperature 1. Whenever a meta-tile has one side with a glue of positive strength that matches a border of a production, a transition is possible by adding that tile. This makes it very hard to assemble at once only rectangles, but also arbitrarily large rectangles. In fact, donig so is an open problem.

On the other hand, let $\mathcal{S}$ be a self-assembling system with temperature 2, let $r \in \mathbb{Z}^2$ be such that all tiles in the seed of $\mathcal{S}$ are at positions (x, y) with $x < r$; in any sequence of productions, the first production with a tile $t@(r, y)$ in row r must be such that $W(t)$ is a glue with strength 2. Likewise for other directions. This property is key to how the self-assembling system $\mathcal{S}_{\text{Rect}}$ of figure 7.4 works.

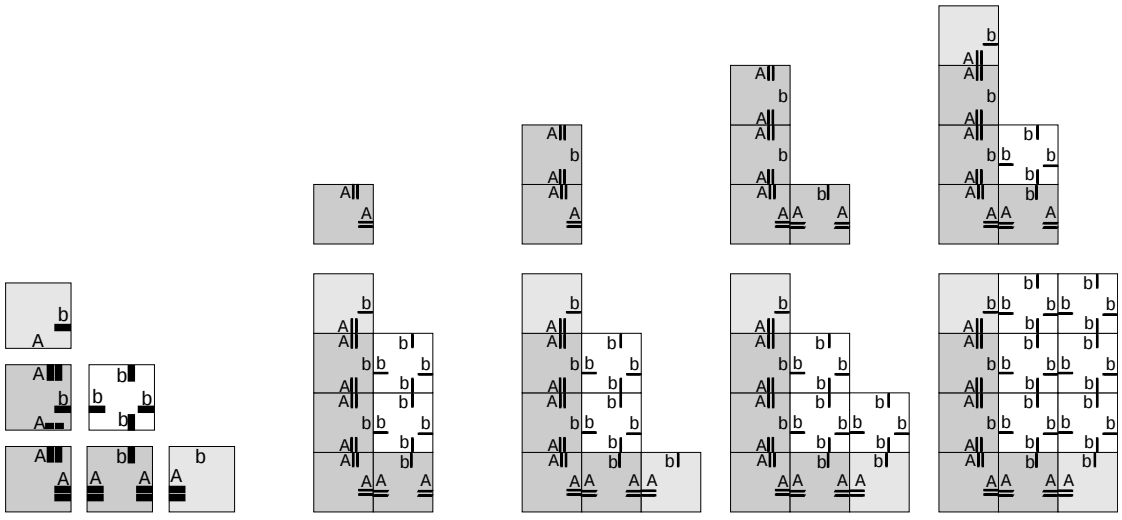


Figure 7.4: A temperature 2 self-assembling system that assembles arbitrary-size rectangles (left) and an example of its growth (right)

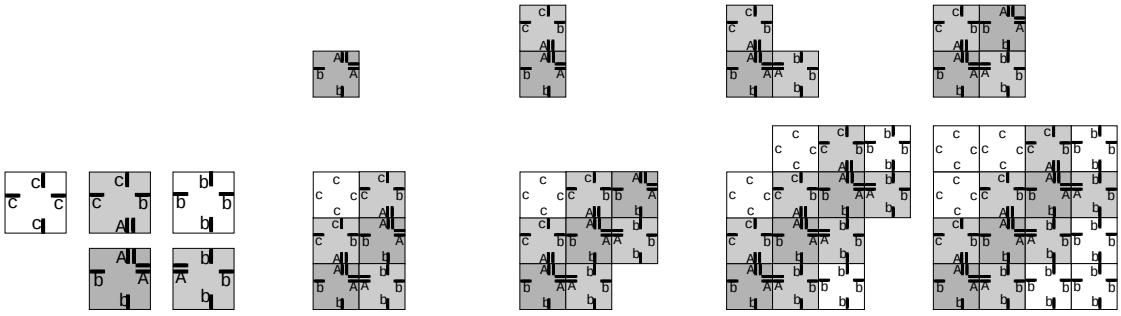


Figure 7.5: A temperature 2 self-assembling system that assembles arbitrary-size squares.

Showing that the final productions of $\mathcal{S}_{\text{Rect}}$ are rectangle shaped is done by inductively proving that in each of its production, the rightmost tile is on the $y = 0$ row, and the uppermost tile is on the $x = 0$ row.

7.3 Assembling squares

Let now S_k be the set of all squares with lower-left corner at $(0, 0)$ and edge length at least k. Compared with $R_{h,w}$, assembling S_k poses the problem of synchronizing the width and height to keep them equal. The system $\mathcal{S}_{\text{Square}}$ on figure 7.5 does that by deciding the length of the *diagonal*, and sending that information back to the corners.

Tile complexity $\mathcal{S}_{\text{Square}}$ has 5 different tiles. This figure is its *tile complexity*. It is the most common measure of complexity for tile-assembly systems. There is also *glue complexity*, which is the number of distinct glues. Tile complexity and glue complexity are important because they tell how involved the process of engineering the tiles before putting them into the test tube is going to be.

7.4 Controlling the expected size

The previous systems yield rectangles or squares, but with no way to control the size of their final productions. One way to control the size of these assembly is to control how likely it is for the stopping tiles to be added at each production where they are attachable.

Definition 7.5 (Concentration function, rate) *Let $\mathcal{S}$ be a self-assembling system. A* concentration function *is a function $c : T(\mathcal{S}) \to \mathbb{R}^+$.*

Given $\mathcal{S}$ with a concentration function, and $t = p \to p'$ a transition of $\mathcal{S}$, the rate *of t is the concentration of the meta-tile that is added by t.*

Given a self-assembling system $\mathcal{S}$ and a concentration function s, we can define a Markov Process as follows: the states of this Markov Process are the productions of $\mathcal{S}$,

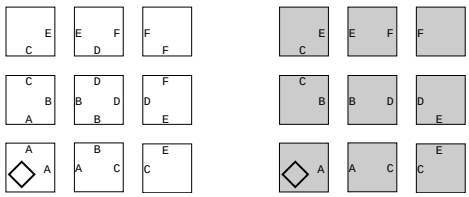


Figure 7.6: Two systems for assembling a 3×3 square at temperature 1. The rightmost one is faster as it allows more parallelism.

its transitions are the transitions of $\mathcal{S}$. Given a production p, let $succ(p)$ be the set of all transitions starting at p. For $t \in succ(p)$, the probability of t is $\frac{P(t|p)=rate(t)}{\Sigma_{t' \in succ(p)} rate(t')}$. In such a Markov Process, the concentrations can be tuned in the previous constructions to make the expected dimensions of the final productions be any arbitrary value. In [51], Kao and Schweller give more advanced techniques to get close to the desired size with high probability.

7.5 Timing self-assembly

7.5.1 The kinetic model

Another measurement of performance of a self-assembly system is the time it takes to assemble one of its final productions. Of course, taking each transition to be a unit of time, this measure already has a name... the surface of the final production. This is of no interest.

Meanwhile, in the test tube, a pattern has grown to be about a few hundreds particles in diameter. Particles in the solution can stick to the pattern, but this has to be *local* interactions. Thus, attachments at opposite ends of the pattern do not interfere: rather, they occur in parallel. In other words, the time it takes for a particle to attach does not depend on what attachments are possible in other parts of the pattern.

A Continuous Time Markov Process allows to model precisely this evolution. Again, the states are the productions of $\mathcal{S}$, the Markov transitions are the transitions of $\mathcal{S}$. A notion of *continous* time is added: every time a transition t is made, the time τ it takes is drawn randomly according to the law $P(\tau < x) = e^{-x/r}$, where r is the rate of t.

This kind of process can also incorporate a slight rate of erroneous attachments, and detachments of tiles that have been attached. In [104] and [96], this is used to make self-assembling system robust to errors.

With this Continuous Time Markov Process, the expected time taken to reach a given final production yields a non-trivial measurement. Note $t^M(\mathcal{S}, c, p)$ for this expected time. In particular, computing $t^M(\mathcal{S}, c, p)$ for the two systems of figure 7.6 both of which assemble a 3×3 square shows that the zig-zagging process is slower when the sum of the concentrations is fixed.

7.5.2 Combinatorial Time

Computing the expected time taken to reach a final production is arduous in most non-trivial cases. In some cases, there is a simpler combinatorial measure of time which gives the same results.

Let $\mathcal{S}$ be self-assembling system, p a production with support U. Let T be a sequence of transitions leading to p. A position $z \in U$ *depends* on one of its neighbors $z' \in U$ according to T if a tile is attached at z after one has been attached at z'. The dependency order associated $<_T$ with T is the transitive closure of the "depends" relation.

A production p of a self-assembling system $\mathcal{S}$ is *ordered* if every sequence of transitions leading to p has the same dependency order. In this case, $<_p$ will be the order associated with any sequence of transitions leading to p. $\mathcal{S}$ itself is ordered if each of its productions is ordered.

Theorem 7.1 *Let $\mathcal{S}$ be a self-assembling system, and p an ordered production of $\mathcal{S}$. Let c be the concentration function defined by $\forall t \in T(\mathcal{S}), c(t) = 1$, and let $d(\mathcal{S}, p)$ be the length of a maximal decreasing sequence in $<_p$. Then $t^M(\mathcal{S}, c, p) = \Omega(d(\mathcal{S}, p))$.*

7.6 Computing with self-assembly

In order to get more complex shapes than just squares and rectangles, the ability to do computation is necessary. This section reviews how to do computation in self-assembling systems by implementing a counter.

In [89], Winfree et al. give a method to assemble a square or rectangle of precise dimensions through the use of a counter. In contrast with the previous examples, this is an example of a *deterministic* system, that is one with a unique final production.

7.6.1 Simple Computation

The self-assembling system of figure 7.7 assembles an $n \times 2^n$ rectangle, and its ith row codes the integer i in binary. The seed configuration is a length n bar, represented at the bottom. On the right, one can see the computation in progress, and its asynchronous nature: some rows have started to be computed before the previous one has finished.

Each row initiates the construction of the next one through its lowest-weight 0. From this position, the row grows towards least-significant bits by flipping all bits to 0, and towards most-significant bits by flipping all bits to 1 until the first 0. The position of the next least-significant 0 is alternatively the least-significant bit or the first 0 to the left of the first tile in the row.

7.6.2 Universal Computation

This counter construction can be generalized to simulate an arbitrary Turing machine (in other words, to run an arbitrary program written in a common programming

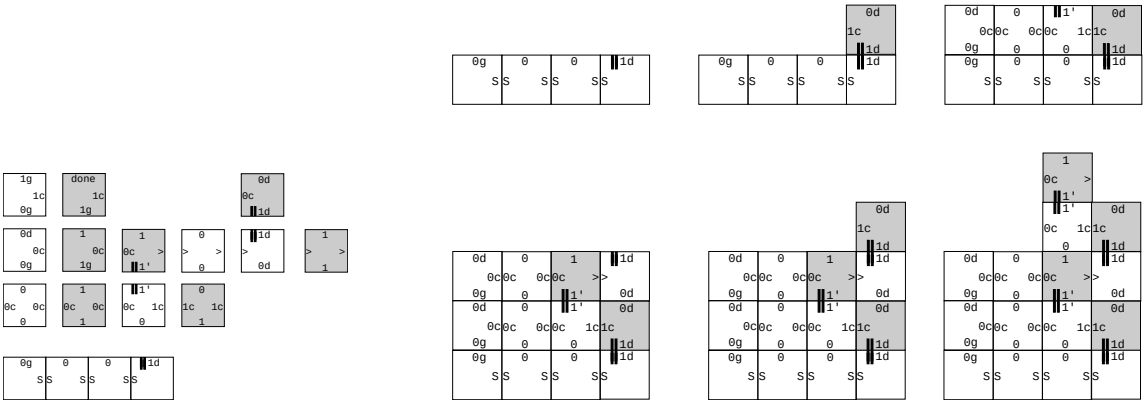


Figure 7.7: A counter at temperature 2. The colour of each meta-tile indicates the bit value it represents, and the evolution of its assembly

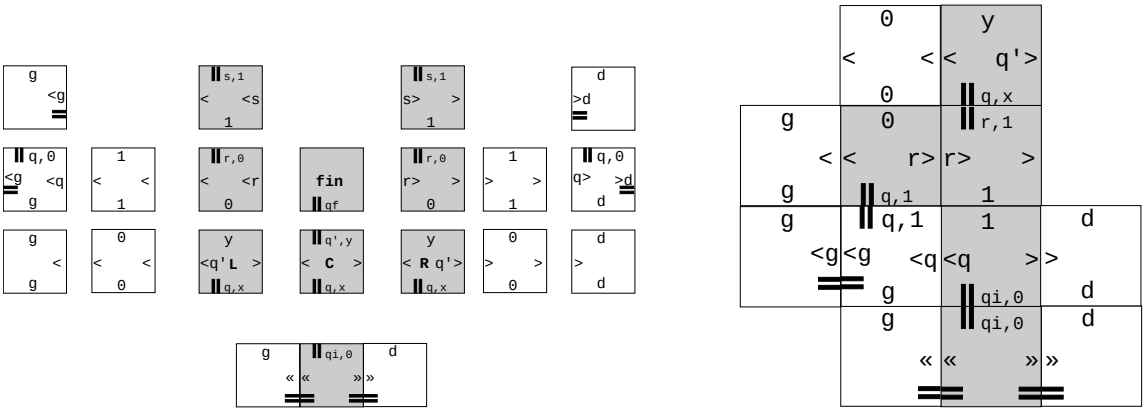


Figure 7.8: The self-assembling system (with temperature 2) for simulating a Turing Machine, and an assembly in progress.

language). This gives the following result [32]:

Theorem 7.2 *There is no algorithm for deciding if a given self-assembling system has a finite final production.*

The claim holds as long as it is possible to simulate a Turing machine in self-assembly. The construction is presented on figures 7.8. Symbols q, q', r, s are states, and x and y alphabet symbols. A version of tile L is present for each Turing Machine rule with the head going left, R when the heads goes right, and C when it stays put. The seed is in the lower part of the figure, with qi the initial state. Figure 7.8 shows that the computation is asynchronous: in the depicted configuration, the head must wait for the next symbol (d for the right end of the tape) before adavancing.

Chapter 8

Flip dynamics

8.1 Tiling space induced by a domain

We work in the square lattice, or in the triangular lattice of $\mathbb{Z}^2$. We fix a *domain* D of the lattice, i.e. a finite union of closed cells which, moreover, is simply connected (i.e. connected with no hole).

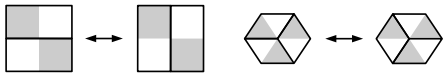


Figure 8.1: Flips in domino tilings (left) and in lozenge tilings (right).

We are interested in tilings of the domain D whose tiles are formed with two neighbors cells (i.e. dominoes in the square lattice, lozenges in the triangular lattice).

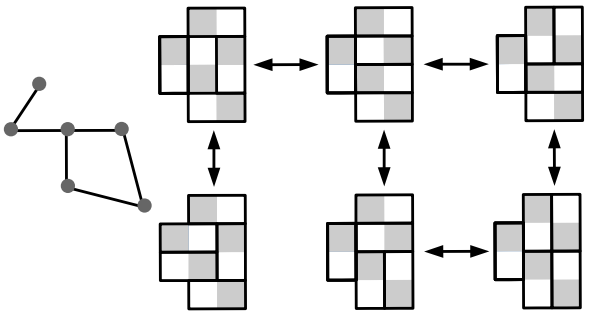


Figure 8.2: An example of tiling space (at left, in a compressed way with no details about tilings).

A *flip* in a tiling (see Figure 8.1) is a local rearrangement of tiles which allows to construct another tiling. Domino flips involve two tiles, and lozenge tilings involve three tiles. In the following, we will be focused in domino tilings, the study of lozenge tiling being very similar. The *tiling space* induced by a domain D (see Figure 8.2) is the undirected graph whose vertex set is the set of tilings of the domain, and two

tilings are neighbors if they only differ by a single flip. The main goal of this lecture is to investigate the structure of tilings spaces.

8.2 Height functions

8.2.1 Construction

Let V_D be the set of lattice vertices which are in the domain D. A tiling of D can be encoded by a function $h_T : V_D \to \mathbb{Z}$ as follows (see Figure 8.3):

1. Fix an origin vertex v_0 on the boundary D, for which $h_T(v_0) = 0$,

2. Direct all edges of the lattice, in such a way that black cells are surrounded clockwise, and white cells are surrounded counterclockwise,

3. For each pair v, v' of vertices of V_D such that

   - there exists and edge of the lattice, directed from v to v' according to the direction given above.
   - the segment $[v, v']$ is not the central axis of a tile of T,

   we have : $h_T(v') = h_T(v) + 1$.

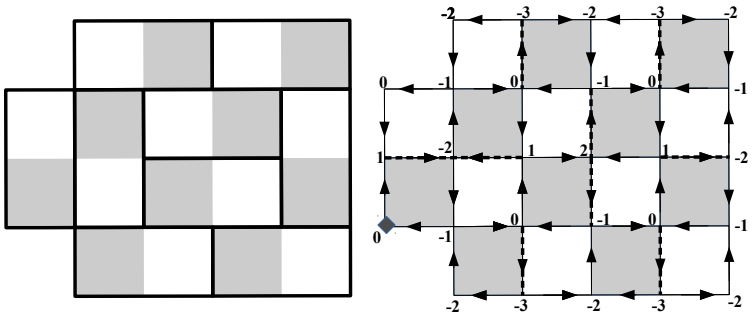


Figure 8.3: Encoding of a tiling with a height function

Since D is simply connected, this construction is consistent.

Proposition 8.1 *For any tiling T of D, there exists a unique function h_T satisfying the condition above.*

8.2.2 Basic properties of height functions

For each tiling T and each directed edge, from v to v', such that $[v, v']$ is the central axis of a tile of T, we have : $h_T(v') = h_T(v) - 3$. Therefore, h_T is really an encoding, i.e. the knowledge of h_T allows to construct the whole tiling T. Conversely, we have the following characterization:

Proposition 8.2 *Let h be a function $v_D \to \mathbb{Z}$ that:*

- $h(v_0) = 0$,
- *for each pair v, v' of vertices of V_D such that there exists an edge from v to v', either $h(v') = h(v) + 1$ or $h(v') = h(v) - 3$,*
- *when, moreover, the segment $[v, v']$ is contained in the boundary of D, then $h(v') = h(v) + 1$.*

Then there exists a tiling T such that $h = h_T(v)$.

All tiling functions have similarities which are pointed below. Let T and T' be two tilings of D.

- For each vertex v on the boundary of D, we have $h_T(v) = h_{T'}(v)$,
- For each vertex v, we have $h_T(v) \equiv h_{T'}(v) \mod [4]$.

8.3 Order on tilings

We say that $T \leq T'$ if for each vertex v of V_D, $h_T(v) \leq h_{T'}(v)$. Assume that T and T' are neighbors in the tiling space and let v be the center of the flip support linking T and T'. We have:

- either $h_{T'}(v) = h_T(v) - 4$, or $h_{T'}(v) = h_T(v) + 4$,
- $h_T(v') = h_{T'}(v')$ when $v' \neq v$.

When $h_T(v) + 4 = h_{T'}(v)$, we say that we have an *upward* flip from T to T'. Note that, in this case, we have: $T \leq T'$.

Theorem 8.1 *Let T and T' be two tilings of D.*

- *$T \leq T'$ if and only if one can pass from T to T' by a sequence of upward flips*
- *Let $\min(h_T, h_{T'})$ and $\max(h_T, h_{T'})$ be the functions defined by, for each vertex v of V_D, $\min(h_T, h_{T'})(v) = \min(h_T(v), h_{T'}(v))$ and $\max(h_T, h_{T'})(v) = \max(h_{T'}(v), h_{T'}(v))$. There exists a tiling, denoted by $\min(T, T')$, such that $h_{\min(T,T')} = \min(h_T, h_{T'})$, and a tiling, denoted by $\max(T, T')$, such that $h_{\max(T,T')} = \max(h_{T'}, h_{T'})$.*

Corollary 8.1 *The tiling space is connected.*

The order relation on tilings defined above confers to the set of tiling of D a structure of *distributive lattice.*

8.4 Applications

8.4.1 Flip distance

Definition 8.1 *Let T and T' be a pair of tilings of D. The flip distance $\Delta(T, T')$ is defined as the distance between T and T' in the tiling space, i.e. the minimal number of necessary flips to transform T into T'.*

Proposition 8.3 *For any pair T and T' of tilings of D, we have the equality:*

$$\Delta(T, T') = \frac{1}{4} \sum_{v \in D_V} |h_T(v) - h_{T'}(v)|$$

Proof strategy:

1. Prove the equality in the particular case when $T \leq T'$,
2. In the general case, use $\min(T, T')$ as intermediate tiling, for being reduced to the previous item.

8.4.2 Tiling algorithm

Problem: given as input a domain D, how can we construct a tiling of D, (or see that there is no tiling) ?

A good idea: from Theorem 8.1, there exists a tiling $T_{\min}$ of D such that, for each tiling T of D, we have $T_{\min} \leq T$. We will try to construct $T_{\min}$.

Lemma 8.1 *Let $h_{\min}$ be the height function encoding $T_{\min}$. For any vertex v of V_D which is not of the boundary of D, there exists a neighbor v' of V such that $h_{\min}(v') = h_{\min}(v) + 1$.*

Let $M = \max\{h_{\min}(v), v \in V_D\}$. From this lemma, it follows that

- if $h_{\min}(v) = M$, then v is on the boundary of D,
- if $h_{\min}(v) = c - 1$, with $c \leq M$, then there exists a neighbor v' of v, such that $h_{\min}(v') = c$.

This allows to construct $h_{\min}$ from one vertex to its neighbors, by the algorithm below. First, in the initialization, we determine

$$V_M = \{v \in V_D, h_{\min}(v) = M\}.$$

Indeed, Following clockwise the boundary, one can determine $h_{\min}(v)$ for each v on the boundary. Thus V_M can be determined, from the Lemma. Afterward, passages the main loop allow to successively construct sets

$$V_c = \{v \in V_D, h_{\min}(v) = c\}$$

with c decreasing at each passage through the loop, starting with vertices v such that $h_{\min}(v) = M$, and progressively decreasing the value $h_{\min}$. Each passage through the main loop is decomposed into two steps. In the first step, the algorithm checks that the new values defined for $h_{\min}$ are compatible with the previous (and larger) ones, using the second item of Proposition 8.2 . In the second step, the algorithm extends $h_{\min}$ to a larger subset of V_D.

Algorithm constructing $h_{\min}$

Input:
a domain D

Initialization: (construction of $h_{\min}$ on the boundary, and computation of M)
for each vertex v of the boundary of D, **do**
compute $h_{\min}(v)$
if a contradiction appears, **then**
STOP: there is no tiling
else, $c := \max\{h_{\min}(v), v \text{ on the boundary of } D\}$
end for

Main loop: (check of the compatibility of values of V_c, and computation of V_{c-1})
while there exists v such that $h_{\min}(v)$ is not checked, **do**
$c := c - 1$,
for each v for which $h_{\min}(v) = c$, **do**
for each v' such that $\{v, v'\}$ is an (undirected) edge, **do**

Checking process:
if $h_{\min}(v')$ is previously defined, **then**
if $h_{\min}(v')$ and $h_{\min}(v)$ are not compatible **then**
STOP: there is no tiling.

Extension process:
if $h_{\min}(v')$ is not yet defined **and**
$[v, v']$ is directed from v' to v **then**
$h_{\min}(v') := c - 1$
end for
Consider the vertex v as *checked*
end for

end while

8.5 Discrete Markov chains and tiling samplers

Markov chains play a important rule in tiling theory [27, 73, 102, 9]. Indeed they represent a efficient way to generate at random tilings and by this fact the possibility to visualize some properties of random tilings. By this approach, Arctic circles phenomena have been for the first time observed [49]. In this section, we introduce the concept Markov chains [68] and describe its applications to random tiling theory.

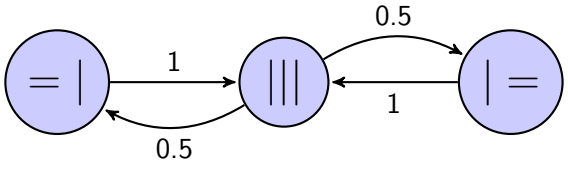
A *Discrete time Markov chain (DTMC)* is a sequence of random variables $X_1, X_2, ...$ with the following memorylessness property: $Pr(X_{n+1} = x|X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = Pr(X_{n+1} = x|X_n = x_n)$ The possible values of X_i form a countable set S called the *state space* of the chain. We can roughly say that the memorylessness property indicate that the evolution of the chain only depends on the current state where we are and not the way we access to it.

The move from a state to another is called a *transition*.

Moreover, we assume here that the Markov chains are *time-homogeneous*, i.e. $\forall n \geq 0, \forall (i,j) \in S^2, Pr(X_{n+1} = i|X_n = j) = Pr(X_1 = i|X_0 = j)$. In other words, the transitions from i to j does not independent on n. In this case, we can define the transition graph as the directed graphs on S where the edges (i,j) of the graph are labeled by the probabilities $Pr(X_1 = i|X_0 = j)$ and the transition matrix as the matrix $(p_{i,j})$ where $p_{i,j}$ is the probability of the transition from the state i to the state j: $p_{i,j} = Pr(X_1 = j|X_0 = i)$.

In particular, the transition matrix $P = (p_{i,j})$ is stochastic, i.e. the sum of the values on each line of P is 1: $\forall i \in S, \sum_{j \in S} p_{i,j} = 1$.

For instance, let us consider the set of the 3 tilings by domino of a 2×3 rectangle. We can define a DTMC as follows: the states are the tilings, and let T_1 and T_2 be two tilings just differing by one flip. $Pr(X_{n+1} = T_2|X_n = T_1) = \frac{1}{x}$ where x is the number of tilings from T_1 by one flip. During this course, we are going to modify this DTMC in order to obtain what we expect: A uniform sampler for this set of tilings.



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 8.4: The transition graph of 3×2 domino-tiling and its transition matrix.

Assume that we are initially on the tiling T_1, we can represent it by the vector $v = (1, 0, 0)$. At time 1, we are on the tiling T_2 with probability 1, this corresponds to $vP = (0, 1, 0)$. At the next step, we are in on T_1 or T_3 with probability $1/2$: this can be obtained by $vP^2 = (1/2, 0, 1/2)$. And so on...

Before continuing with this example, let us give some definitions.

8.5.1 Reducibility

A state j is *accessible* from a state i (denoted by $i \to j$) if starting in state i, we have a non-zero probability to reach the state j at some time. In other words, state j is accessible from state i if there exists an integer $n \geq 0$ such that $Pr(X_n = j | X_0 = i) > 0$.

A state i *communicate* with state j (denoted by $i \leftrightarrow j$) iff both $i \to j$ and $j \to i$. A set of states C is a *communicating class* if every pair of states in C communicates with each other, and no state in C is communicating with any state not in C.

A Markov chain is said to be *irreducible* if its state space is a single communicating class; in other words, if we can reach any state from any state, i.e. the transition graph is strongly connected.

So, our running example is an example of irreducible Markov chain.

8.5.2 Periodicity

A state i has *period* k if any return to state i must occur in multiples of k time steps. So, we can defined the period of a state as $k = \gcd\{n > 0; Pr(X_n = i | X_0 = i) > 0\}$ (where "gcd" is the greatest common divisor) provided that this set is not empty. Otherwise the period is not defined.
Note that even though a state has period k, it may not be possible to reach the state in k steps. For example, suppose it is possible to return to the state in {4,6} time steps; k would be 2, even though 2 does not belong to this list.
If $k = 1$, then the state is said to be *aperiodic*.
A Markov chain is *aperiodic* if every state is aperiodic.

Our running example is periodic of the period 2. Indeed, the transition graph is bipartite.

8.5.3 Transience

In this section, we consider that the space of states can be infinite (otherwise, the notions bring nothing new!). A state i is *transient* if, starting in state i, there is a non-zero probability that we will never return to i. In other word, let the random variable T_i be the first return time to state i: $T_i = \inf\{n > 1; X_n = i | X_0 = i\}$, state i is transient if $Pr(T_i < \infty) < 1$. State i is *recurrent* iff it is not transient.

Even if the hitting time is finite with probability 1, its expectation is not necessarily finite. The *mean recurrence time* at state i is the expected return time: $M_i = E[T_i]$. State i is *positive recurrent* if M_i is finite; otherwise, state i is *null recurrent*.

If a Markov chain is irreducible and if its space of states is finite, all states are positive recurrent.

8.5.4 Stationary and limiting distributions

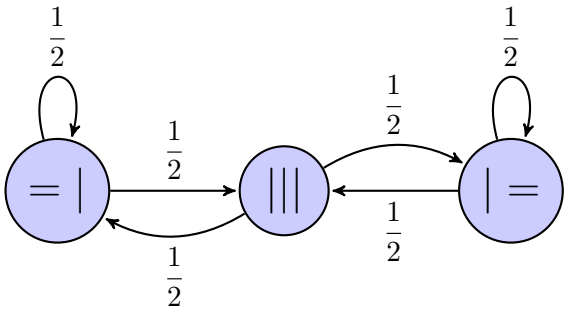
A vector π is called a *stationary distribution* if $\forall j \in S$ it satisfies: $0 \leq \pi_j \leq 1$, $\sum_{j\in S} \pi_j = 1$ and $\pi = \pi P$.

In general (finite or infinite state space), an irreducible chain has a stationary distribution if and only if all of its states are positive recurrent. In that case, π is unique and is related to the expected return time: $\pi_j = \dfrac{C}{M_j}$ where C is the normalizing constant. Further, if the positive recurrent chain is both irreducible and aperiodic, it is said to have a *limiting distribution*.

In finite state space, if the Markov chain is irreducible and aperiodic, then there is a unique stationary distribution π. Moreover, in this case P^k converges to a matrix in which each row is the stationary distribution π, that is, $\lim_{k\to\infty} P^k = \mathbf{1}\pi$

Our example has two problems. Firstly, it is not aperiodic. Secondly, its stationary distribution is not uniform. Indeed, the matrix transition P verifies $P^{2n} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$ and $P^{2n+1} = P$. So, the stationary distribution is $\pi = (2/3, 1/6, 2/3)$.

Let us modify the process as follow: choose a vertex on the $\mathbb{Z}^2$-grid inside the rectangle 3×2, if the tiling can be flip around this vertex, we do the flip, otherwise, we do nothing. We can describe this new process by its transition graph:



$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Figure 8.5: The transition graph of 3×2 domino-tiling and its transition matrix.

This new process is now irreducible and aperiodic. It is easy to show that the stationary distribution is the vector $\pi = (1/3, 1/3, 1/3)$. So, we get a way to generate at random our tilings.

Now, let us discuss this process. First of all, it can be easily generalised to the tiling by domino of every simply connected bounded region of the plan (and even more generally on cases of tiling by rhombus). In fact, we just need the condition of accessibility by flip, which is the case in various tiling problems.

In these cases, the process is defined as follows. Draw a vertex of the grid support (the $\mathbb{Z}^2$ grid, for instance, for the domino tiling), if we can make a flip on the tiling

around this point, then do it, otherwise, stay on the same tiling. Due to the flip-accessibility this process is irreducible and it is also aperiodic because of the non-zero probably loops. So, its stationary distribution exists and is unique. Moreover the stationary distribution is the uniform distribution.

It remains the question of the speed of convergence. In the case of a irreducible and aperiodic Markov chain with finite state space, the convergence is exponentially fast, i.e. there is $K > 0$ and $\alpha \in]0, 1[$ such that $\|\mu_n - \pi\| \leq K\alpha^n$.

8.6 Coupling from the past

This useful procedure has been introduced by Proop and Wilson [83]. Consider a finite state irreducible aperiodic Markov chain M with state space S and (unique) stationary distribution π. Assume that we have a probability distribution μ on the set of maps $f : S \to S$ with the property that for every fixed $s \in S$, its image $f(s)$ is distributed according to the transition probability of M from state s. An example of such a probability distribution is the one where $f(s)$ is independent from $f(s')$ whenever $s \neq s'$, but it is often worthwhile to consider other distributions. Now for $j \in \mathbb{Z}$ let f_j be independent samples from μ.

Suppose that x is chosen randomly according to π and is independent from the sequence f_j. (We do not worry for now where this x is coming from.) Then $f_{-1}(x)$ is also distributed according to π, because π is $M-$stationary and our assumption on the law of f. Define $F_j := f_{-1} \circ f_{-2} \circ \cdots \circ f_{-j}$. Then it follows by induction that $F_j(x)$ is also distributed according to π for every $j \in \mathbb{N}$. Now here is the main point. It may happen that for some $n \in \mathbb{N}$ the image of the map F_n is a single element of S. In other words, $F_n(x) = F_n(y)$ for each $y \in S$. This is what we call a *coalescence*. Therefore, we do not need to have access to x in order to compute $F_n(x)$. The algorithm then involves finding some $n \in \mathbb{N}$ such that $F_n(S)$ is a singleton, and outputing the element of that singleton.

Let us now show how to do that in the case of random sampling.

We previously saw that the domino tiling can be partially ordered which has a unique minimal element s_0 and a unique maximal element s_1; that is, every $s \in S$ satisfies $s_0 \leq s \leq s_1$. Also, suppose that μ may be chosen to be supported on the set of monotone maps $f : S \to S$. Then it is clear that $|F_n(S)| = 1$ if and only if $F_n(s_0) = F_n(s_1)$, since F_n is monotone. Thus, checking this becomes rather easy.

The process is the following, choose a vertex v on the $\mathbb{Z}^2$-grid inside the tiling region and a symbol in $\{\uparrow, \downarrow\}$. For every tiling such that it is possible to do a flip around v and which respect the symbol (a flip is $\downarrow$ is it decrease the height function, $\uparrow$ otherwise), do the flip.

The algorithm can proceed by choosing $n = n_0$ for some constant n_0, sampling the maps $f_{-1}, \ldots, f_{-n}$, and outputing $F_n(s_0)$ if $F_n(s_0) = F_n(s_1)$, that is to say if the coalescence has occurred. If $F_n(s_0) \neq F_n(s_1)$ the algorithm proceeds by doubling n and repeating as necessary until an output is obtained.

8.7 Random sampling of tilings and certification chains

In this section, we briefly illustrate on an example the notion of certification chains [46, 68].

Our problem consists in drawing uniformly at random a tiling of a rectangle $n \times m$ by 1×1 and 2×2 squares. We can present such a tiling as a $n \times m$ grid of 1 and 0 that respectively denote the occupation of the site or not by a 2×2 square. More precisely, we put a 1 in (i, j) if and only if $(i, j), (i + 1, j), (i, j + 1), (i + 1, j + 1)$ are covered by a unique 2×2 tile. By analogy with physics, we call the 2×2 squares, the *particles* and the 1×1 the *vacancy*. A natural Markov chain M for this problem consists to choose a site (i, j) at random and "0" or "1" equiprobably. Then

- if "1" is chosen, place a particle at the site (i.e. put "1" in (i, j)) if there are no overlapping with already existing particles (i.e. , there is no "1" in $(i+1, j)$, $(i, j+1)$, $(i + 1, j + 1)$)

- if "0" is chosen, put "0" on the site (i, j).

Following Hubert [46], we can draw a tiling according to the stationary distribution of M (which is the uniform distribution) by using the notion of certifications. We proceed as follows:

One can associate with each set of tilings a threevalued function on the sites of the grid, where the value "1" means that all tilings in the set are known to have a particle placed in this site, the value "0" means that all tilings in the set are known to have a vacancy at that site, and the value "?" means that it is possible that some of the tilings in the set have a particle there while others have a vacancy.

Initially we place a "?" at every site since the Markov chain could be in any tiling. In other words, we suppose that we have a tiling draw according to the stationary distribution but we know nothing about it!

We continue to work directly on this three-valued state-model in order to simulate our Markov chain M. The process selects a random site and proposes to place a particle there with probability 1/2 or proposes to place a vacancy there with probability 1/2. Any proposal to place a vacancy always succeeds for any tiling in the current set, so in this case a "0" is placed at the site. A proposal to place a particle at the site succeeds only if we are sure to have no overlapping with other particles. This appends if all neighbouring sites $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$ have a "0", and otherwise keeps the site unchanged.

After the update, the "0, 1, ?" configuration describes any possible tiling that the Markov chain M may be in after one operation. It is immediate that if the "0, 1, ?" Markov chain ever reaches a tiling in which there are no "?"'s, then the Markov chain M, using the same random proposals, maps all initial states into the same final state. This moment is called a *coalescence*.

Now, we can propose a process to draw a tiling at random. We cannot stop at the coalescence point, because we create a bias by this choice of stop. But, we can use *coupling from the past* principle [83].

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