Aperiodic Tilings

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This project is devoted to an interesting area of mathematics called tilings. We consider some finite set of tiles (figures on the plane) and try to cover the plane by placing our tiles next to each other such that there are no empty spaces between the tiles. The main question is to understand how the local constraints (due to the way tiles can fit next to each other) enforce global properties of tilings. In particular, an interesting global property is non-periodicity, and a tile set which can form only non-periodic tilings is said to be *aperiodic*.

There are a lot of problems separated into several cycles. Some problems are simple, some other more complicated, and some are even open questions: it would be great if someone make an advancement for these problems.

A One dimension

Let us consider the one dimensional case. *Tiles* can be seen as *letter* over a finite alphabet, and *tilings* as *two-side infinite words*. We call *pattern* of a word any of its finite subwords. Letter are allowed to stay next to each other or not, and a common way to specify such constraints is to give a finite set of forbidden patterns. For example, if aa and bb are forbidden, then the only infinite word that can be formed is periodic with period (ab).

A.1 Let us consider the set S of infinite words in the alphabet $\{a, b\}$ which contain runs of b's of length at most three. Does there exist a finite set of forbidden words determining the S?

A.2 Count the minimal number of forbidden words to determine the following infinite words (given by its periods): (ab), (aab), $(aabaabab)$, $(aabaababaabaababaabab)$.

A.3 Consider the set of infinite words over the alphabet $\{a, b\}$ where the patterns (subword) ba^nb are forbidden, for any $n > 1$. Can you find a finite set of forbidden patterns which defines the same set of words?

A.4 What about the previous question if, in addition, you are now allowed to color letters (using finitely many different colors), that is, for example, to make a difference between a blue a and a green a?

A.5 Let S be the set of infinite words whose finite runs of a's are all of even length only. Is it possible to determine the S with some finite number of forbidden words? And if we can color letters using finitely many different colors?

A.6 Same questions if the length of finite runs of a's is asked to be odd.

A.7 Let $u_0 = a$, $u_1 = ab$, $u_{n+2} = u_n u_{n+1}$. Count the minimal number of necessary forbidden words to enforce infinite periodical words with period u_n.

B Two dimensions

A *2D infinite word* over a given alphabet is obtained by writing a letter of the alphabet in each cell of the 2D grid, and a *pattern* is a restriction of such a word to a finite region.

B.1 Find a small set of forbidden patterns which defines 2D infinite words where the a's and b's alternate as the black and white squares on a checkerboard.

B.2 Consider the set of 2D infinite words over the alphabet $\{a, b\}$ such that any finite connected (by side) component of a's has even size. Can you find a finite set of forbidden patterns which defines this set? And if colors are allowed?

B.3 Same question if the size of the connected component of a's is asked to be odd.

B.4 Show that if a finite set of forbidden words allow to form patterns which cover arbitrarily large disk, then one can form a 2D word without any such forbidden words.

Given finitely many polygons called *tiles*, a *tiling* is a covering of the plane by isometric copies of these tiles that can intersect only on vertices or along whole edges. Tiles can in addition be colored using finitely many different colors. 2D infinite words are thus tilings: tiles are square with a color corresponding to the letter a or b. Convince yourself that colors and forbidden patterns are equivalent to notching of tiles or decorations that must match where tiles intersect.

B.5 Consider the hexagonal tiles whose side can have a bump or a dent (see, for example, Fig. 1): for which tiles can we form a tiling using only one of these tiles?

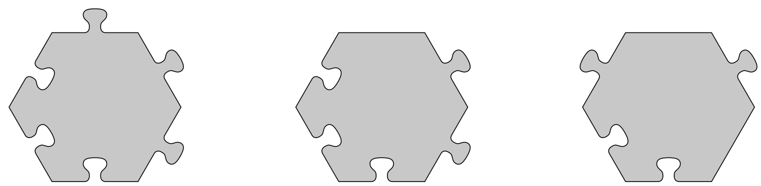

Figure 1: Some notched regular hexagons.

C Periodicity and Quasiperiodicity

Let us return to one dimensional situation. We shall here focus on the question of periodicity, and introduce a much larger notion, *quasiperiodicity*.

C.1 Consider a periodic infinite word. Prove that it can be determined by a finite set of forbidden words.

C.2 Show that any finite set of forbidden words which allows at least one infinite word also allows a periodic word (even if finitely many colors are allowed).

Hence, periodicity is unavoidable in dimension one. But how far can we go?

C.3 Assume that one can use only n forbidden words. Let p be the smallest period of periodical words. How large p could be?

C.4 Now we can use any number of forbidden words, but each contains at most n letters. How large p could be?

An infinite word is said to be *quasiperiodic* if, for any pattern w that appears somewhere in this word, there is a number r such that w appears at distance at most r from any letter of the word.

C.5 Show that any periodic word is quasiperiodic.

C.6 Find a non-periodic but quasiperiodic word, and a non-quasiperiodic word.

Consider now the two-dimensional case. Periodicity and quasiperiodicity can be easily generalized (do it). As we will later see, periodicity is however no more unavoidable! But quasiperiodicity does:

C.7 Show that any finite set of forbidden patterns which allows at least one tiling also allows a quasiperiodic tiling.

D Robinson tilings

The first aperiodic tile set has been discovered in 1964 by Robert Berger and contains 20426 tiles. The much simpler tile set on Fig. 2 have been discovered in 1971 by Rafael Robinson. Bumps or dents (which can have a symmetry or not) enforce the way tiles can be assembled, while the segments drawn on tiles form an interesting picture.

D.1 Show that the tiles on Fig. 2 can form a tiling of the plane.

D.2 Show that no tiling by the tiles on Fig. 2 is periodic.

D.3 When segments drawn on tiles form a square which intersects a smaller square, then the former is called the *parent* of the latter. The *lineage* of a square is then be encoded by an infinite word over a four-letter alphabet, with each letter coding the corner which is intersected by the parent square. At which condition any two squares of a tiling have a common ancestor?

D.4 Show that there are uncountably many different tilings, even up to an isometry.

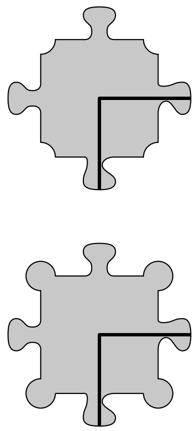

Figure 2: Six notched squares (note the two different type of bump/dent).

E Hierarchical tilings

We shall introduce *hierarchical tilings*, which provide the first general way to find aperiodic tile set, that is, finite tile sets forming only non-periodic tilings.

E.1 Show that the tiles on Fig. 3, discovered in 1974, do tile the plane

E.2 Same question for the tiles on Fig. 4, discovered in the early 90's.

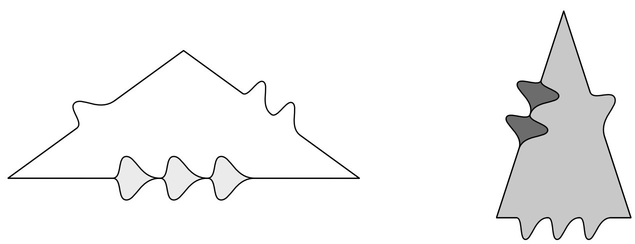

Figure 3: Notched isosceles triangles. Top angles are respectively 108° and 36°.

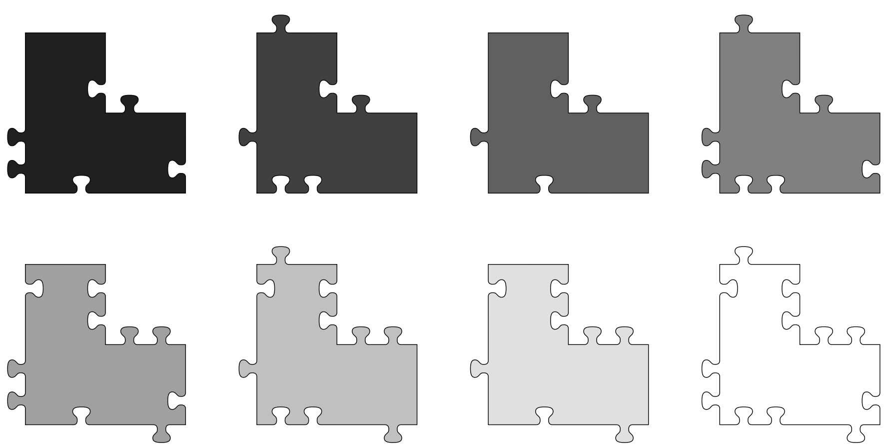

Figure 4: Notched L-shaped tiles.

E.3 If we cannot use colorings or notches on edges, can we nevertheless characterize the tilings of two previous questions characterized by finitely many of their patterns?

A *substitution* is a map which first inflates (by a common factor) the tiles of a given tile set and then subdivide them in non-overlaping tiles of this tile set (Fig. 5 This is thus a map on the tilings by this tile set. With a given substitution are then associated so-called *hierarchical tilings*: they are the tilings which admit a uniquely defined infinite sequence of preimages by this substitution.

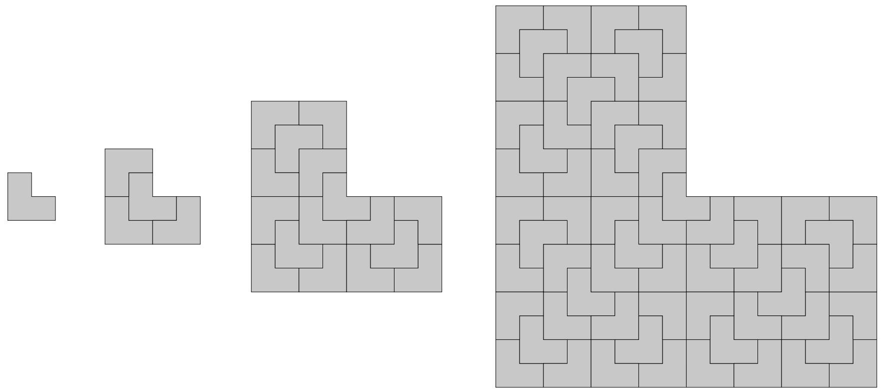

Figure 5: Example of substitution.

E.4 Show that hierarchical tilings are non periodic.

E.5 Are the hierarchical tilings associated with the substitutions on Fig. 6 characterized by finitely many patterns?

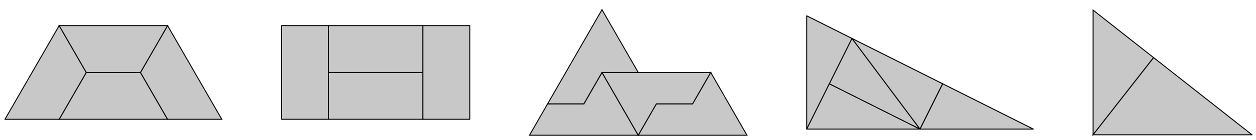

Figure 6: Substitutions (rightmost: tiles are homothetic but not isometric).

A general result discovered in 1998 states that, **given a substitution over a set of tiles, these tiles can be colored so that the hierarchical tilings associated with are characterized by finitely many patterns** (equivalently, tiles can be notched or decorated).

E.6 Can you find such decorations for the hierarchical tilings associated with the substitutions the previous question?