Substitutions on Multidimensional Sequences

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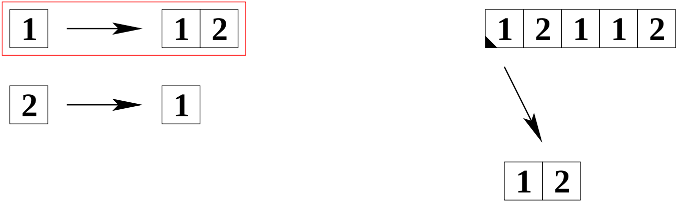
WORDS'05
September 13-17, 2005

Classic substitutions on words



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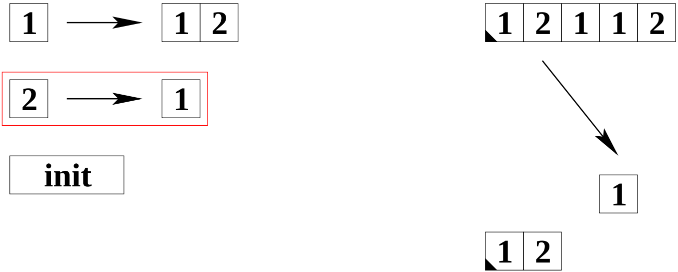

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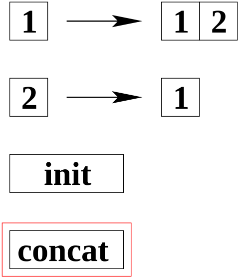

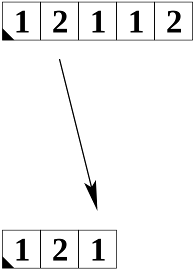

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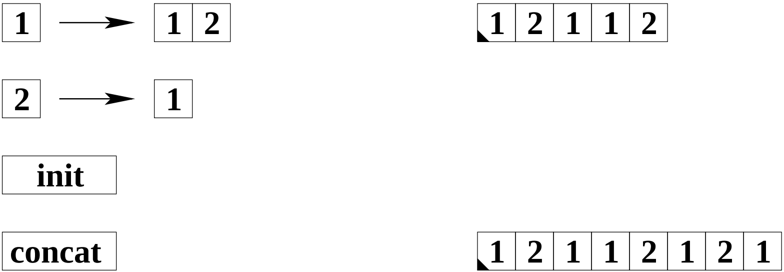
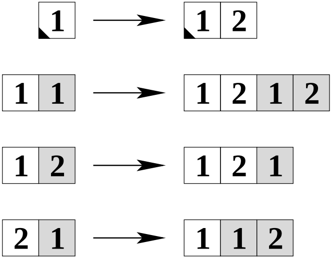


How to avoid the concatenation



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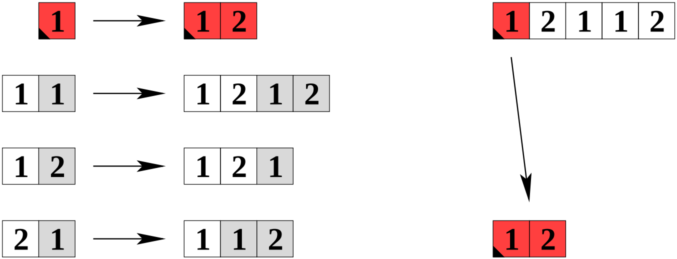

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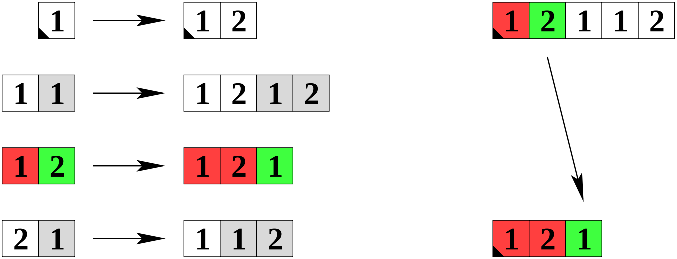

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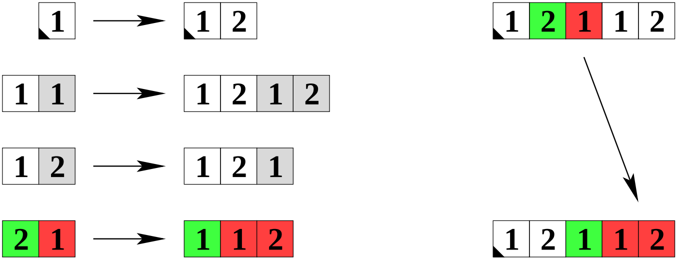

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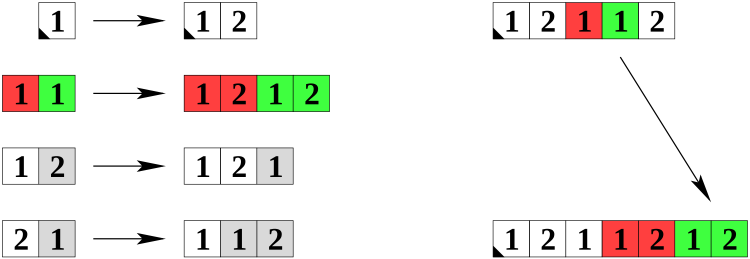

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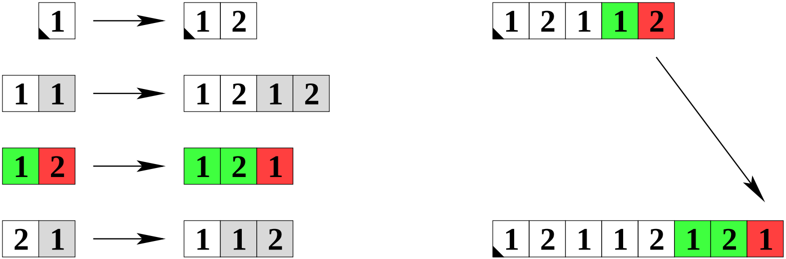

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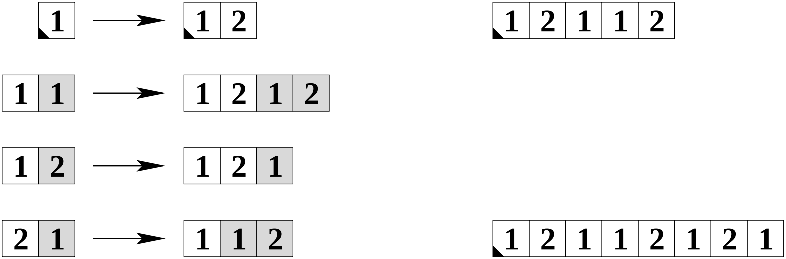
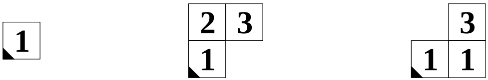


Basic notions 1/2

Pointed letter: $L = (x, \ell) \in \mathbb{Z}^n \times \mathcal{A}$, $x = Supp(L)$. $\rightsquigarrow$ Set $\mathcal{L}$.

Pointed pattern: pointed letters with distinct supports. $\rightsquigarrow$ Set $\mathcal{P}$.



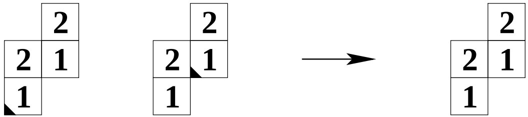
Consistence of $P, Q \in \mathcal{P}$: $((x, \ell) \in P \wedge (x, \ell') \in Q) \Rightarrow \ell = \ell'$.

Basic notions 2/2

$\mathbb{Z}^n$ acts on $\mathcal{L}$ and $\mathcal{P}$ by translation: $y + (x, \ell) = (y + x, \ell)$.

$\rightsquigarrow$ *Non-pointed letters* and *patterns*: $\overline{\mathcal{L}} = \mathcal{L}/\mathbb{Z}^n$ and $\overline{\mathcal{P}} = \mathcal{P}/\mathbb{Z}^n$.

P and Q *congruent*: $\overline{P} = \overline{Q}$, *i.e.* $P = Q + y$.

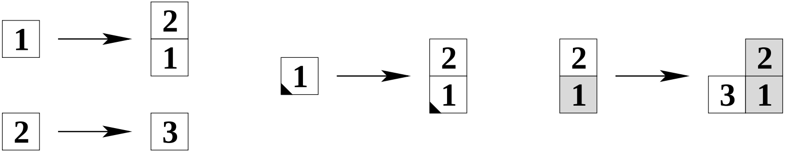


Definition

A *non-pointed substitution* is a map $\overline{\sigma} : \overline{\mathcal{L}} \to \overline{\mathcal{P}}$.

Definition (Local rules for $\overline{\sigma}$)

- *Initial rule* λ^*: def. on $I(\lambda^*) = \{L\}$ s.t. $\overline{\lambda^*(L)} = \overline{\sigma}(\overline{L})$;
- *extension rule* λ: def. on $E(\lambda) = \{L_1, L_2\}$ s.t. $\overline{\lambda(L_i)} = \overline{\sigma}(\overline{L_i})$.



1. Local rules
   - Main idea
   - General definition
2. Substitutions on patterns
   - Λ-paths
   - Good sets of local rules
3. Derived local rules
   - Global rules
   - Generalized substitutions

Definition (Λ-path)

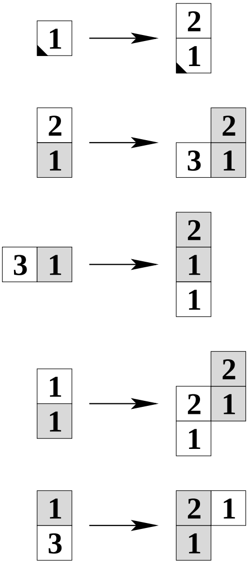
$R = (R_1, \ldots, R_k) \in \mathcal{L}^k$ is a Λ-path if there exist:

- an initial rule $\lambda^* \in \Lambda$ s.t. $I(\lambda^*) = R_1$;
- for $1 \leq i < k$, an extension rule $\lambda_i \in \Lambda$ and $x_i \in \mathbb{Z}^n$ s.t. $E(\lambda_i) = \{L_1^i, L_2^i\}$ and $(R_i, R_{i+1}) = (L_1^i, L_2^i) + x_i$.

Definition (Substitution on a Λ-path)

$(\overline{\sigma}, \Lambda, R)$ defined on R by induction:

- $(\overline{\sigma}, \Lambda, R)(R_1) = \lambda^*(R_1)$;
- $(\overline{\sigma}, \Lambda, R)(R_{i+1}) = \lambda_i(L_2^i) + \vec{v}(\lambda_i(L_1^i), (\overline{\sigma}, \Lambda, R)(R_i))$.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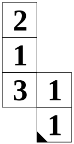
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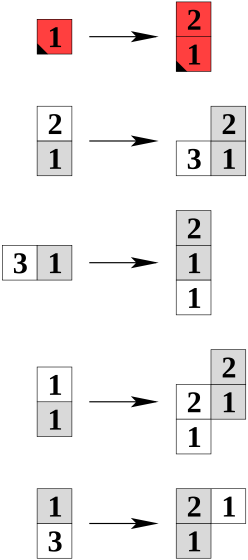
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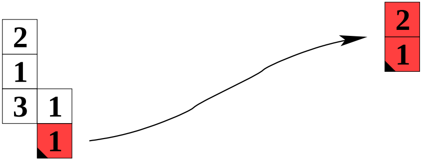
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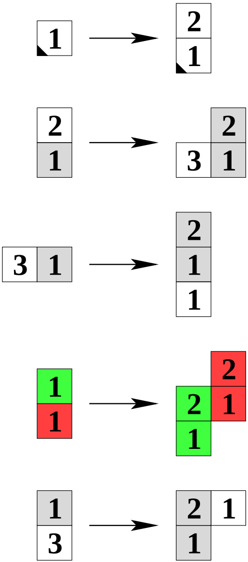

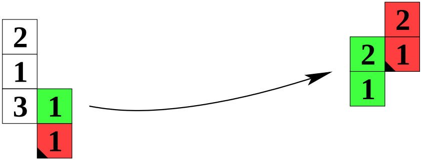

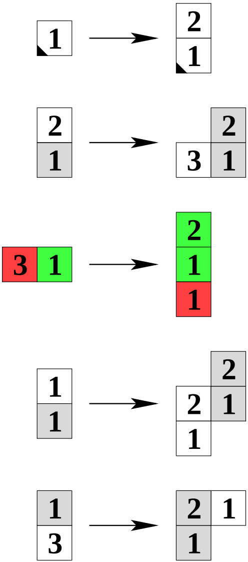

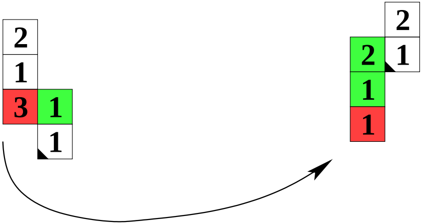

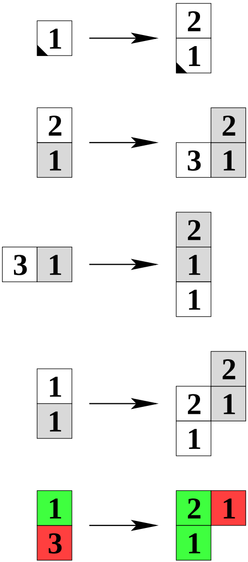
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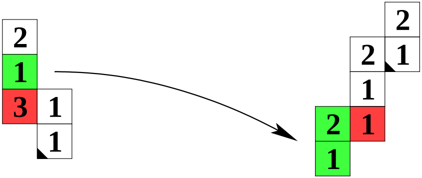
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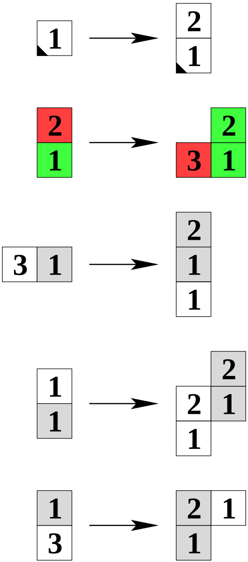

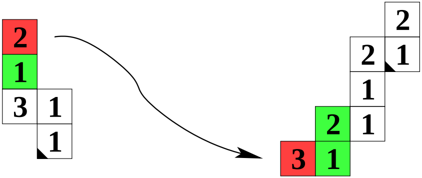

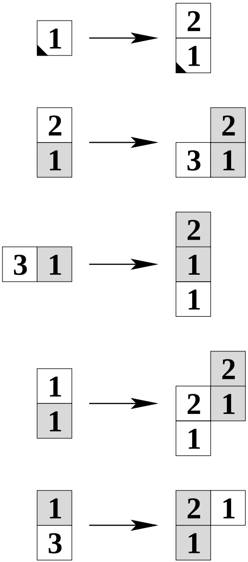

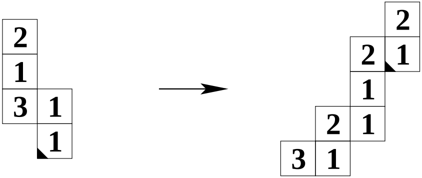

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Definition (Good set of local rules)

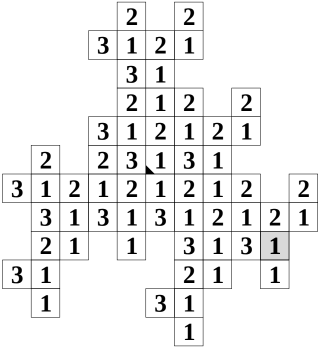
Λ is a *good* set of local rules on a pointed pattern U if:

- any pointed letter of U belongs to a Λ-path of U;
- the images of any two Λ-paths of U are consistent.

Definition (Substitution on a pattern)

If Λ is a good set of local rules on U, then one sets:

$$(\overline{\sigma},\Lambda)(U)=\bigcup\{(\overline{\sigma},\Lambda,R)(U)\mid R \text{ is a } \Lambda\text{-path of } U\}.$$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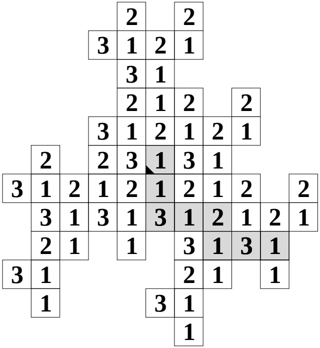
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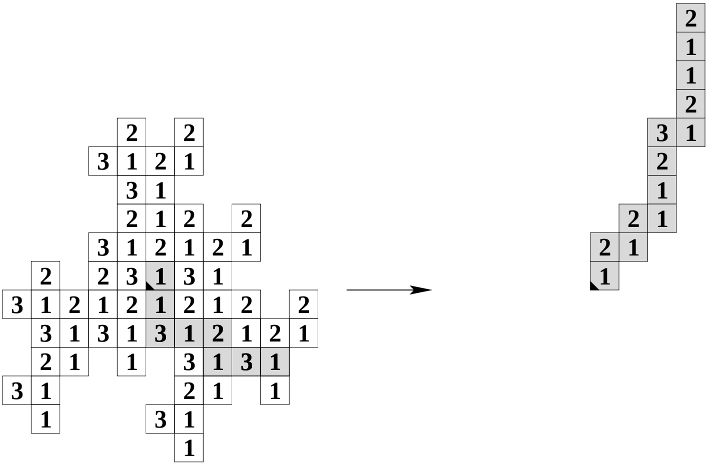

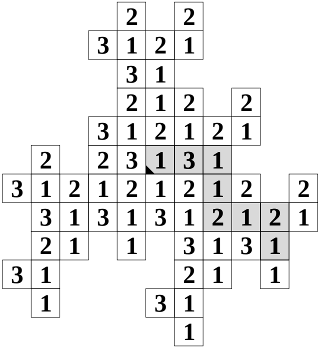
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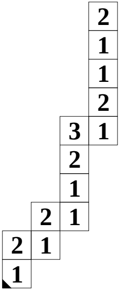
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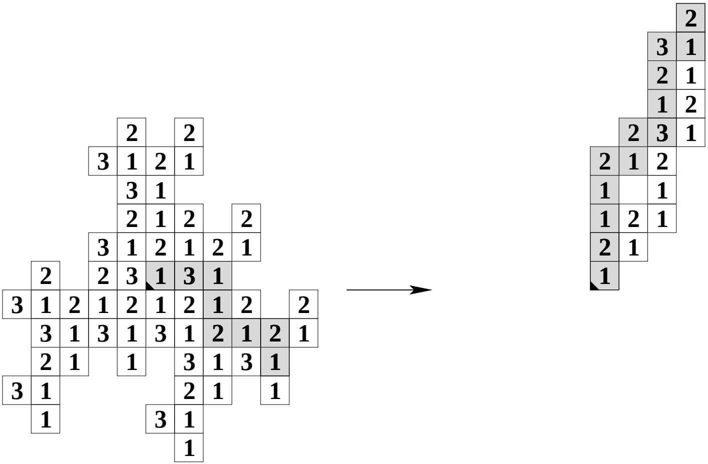
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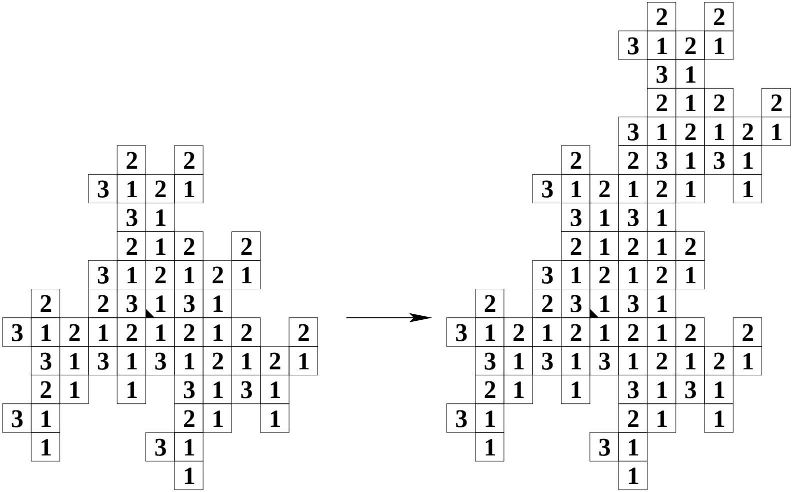

$\overline{\sigma}$ non-pointed substitution, $\mathcal{H} \subset \mathcal{P}$ and $\mathcal{L}(\mathcal{H}) = \{L \in U \mid U \in \mathcal{H}\}$.

Definition (Context-Free Global Rule)

A *CF global rule* on $\mathcal{H}$ for $\overline{\sigma}$ is a map $\Gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{P}$ such that:

- $\overline{\Gamma(L)} = \overline{\sigma}(\overline{L})$;
- $(\overline{U} = \overline{V}) \Rightarrow (\overline{\Gamma(U)} = \overline{\Gamma(V)})$;
- $Supp(L) \neq Supp(L') \Rightarrow Supp(\Gamma(L)) \cap Supp(\Gamma(L')) = \emptyset$;
- $\forall L \in \mathcal{L}(\mathcal{H})$, $\exists L'$ s.t. $L \in \Gamma(L')$.

Hyp: $\exists U_0 \in \mathcal{P}$ s.t. $\forall U \in \mathcal{H}$, U is 4-connected and $U \cap U_0 \neq \emptyset$.

Theorem

$\exists$ CF global rule $\Gamma \iff \exists$ finite good set Λ s.t. $(\overline{\sigma}, \Lambda) = \Gamma$ on $\mathcal{H}$.

Λ computable from Γ.

Sturmian hyperplane sequence: $(n-1)$-dimensional sequence over $\{1, \ldots, n\}$ which generalizes Sturmian words.

Generalized substitution: acts on such sequences.

Proposition

A generalized substitution is a CF global rule acting on the Sturmian hyperplane sequences.

$\rightsquigarrow$ Good sets of local rules

And now?

Defined: substitutions on multidimensional sequences.

◇ Fixed point sequences?

Classic substitutions: characterization by *return words* (Durand)
Local rules in $1D$: nothing more (Fogg)

◇ Fixed points among *some* sequences?

Sturmian words: *iff* quadratic slope (Crisp & Moran)
Sturmian hyperplane sequences: sufficient condition (F.)

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