Planar Dimer Tilings

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LIRMM (Montpellier, France)

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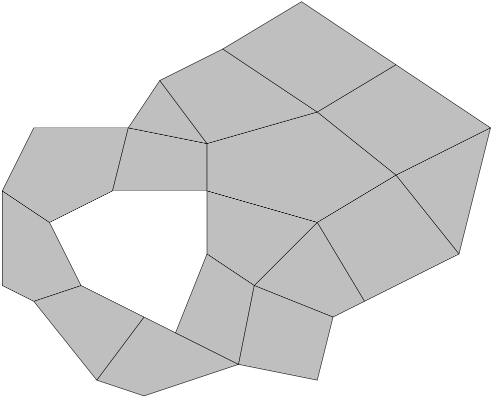
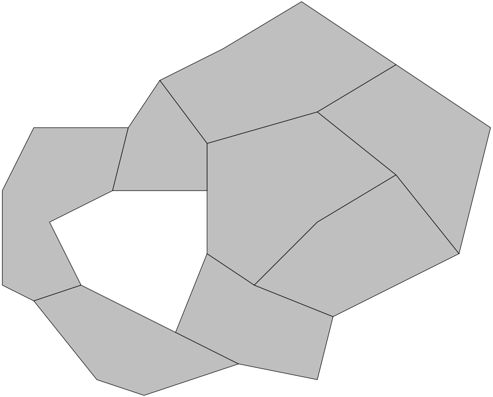
Planar dimer tiling



Domino and lozenge tilings espacially studied (statistical physics).

Planar dimer tiling



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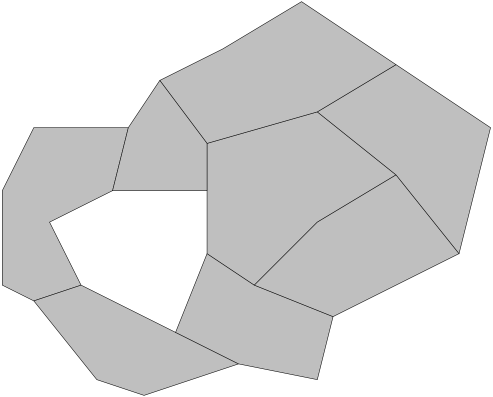

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Outline

1. The Thurston's algorithm (lozenge tilings)
   - Weights, heights and flips
   - Thin out a simply connected domain
2. The general case
   - Binary counters and heights
   - Relaxation: real counters
3. Structure of the set of tilings
   - Generalized flips
   - A distributive lattice

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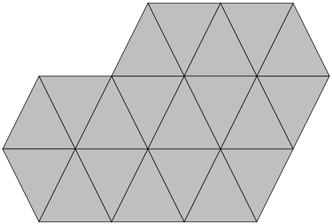
Triangular cells ↔ directed graph:



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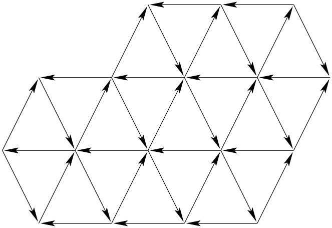
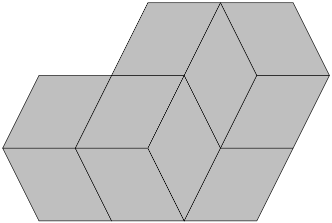
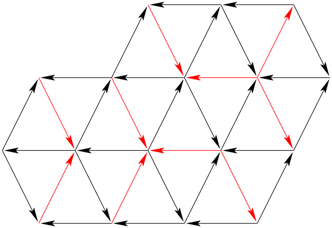


Dimer (lozenge) tiling $\leftrightarrow$ weighted edges (black: 1, red: -2):

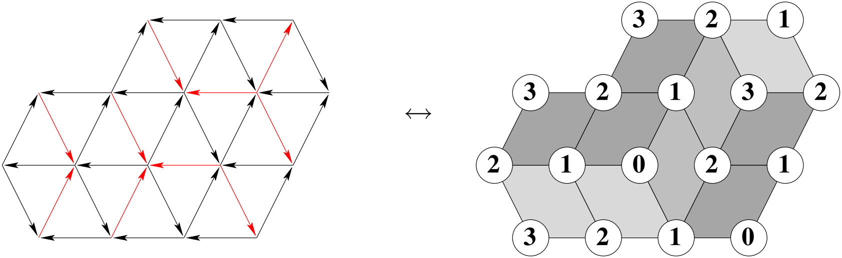


$\leftrightarrow$



Note: directed closed paths have weight 0.

Weighted edges (black: 1, red: -2) $\leftrightarrow$ heights of vertices:



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Proposition

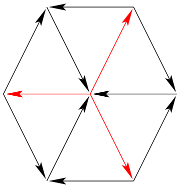
A vertex of maximum height locally enforces weights:

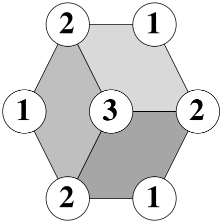
- incoming edges have positive weights (black edges)
- outcoming edges have negative weights (red edges)

proposition

A tileable domain admits a tiling whose vertices of max. height are on the boundary.

proof: flip:





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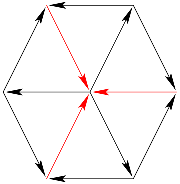

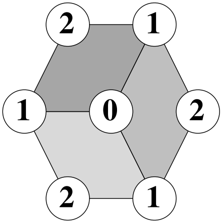

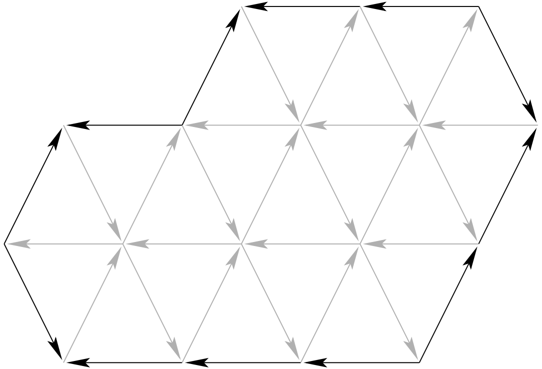

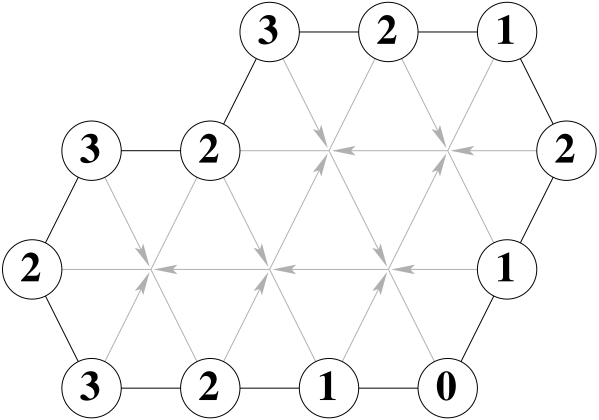
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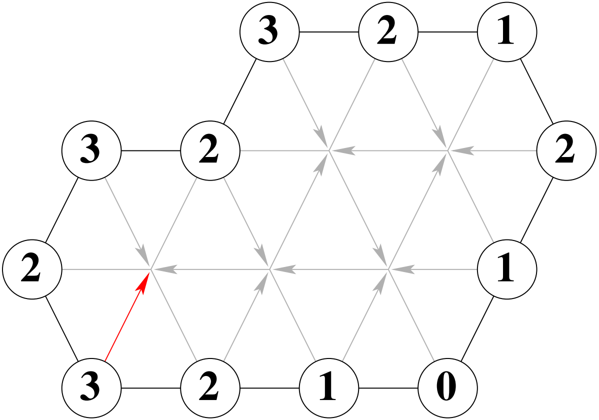
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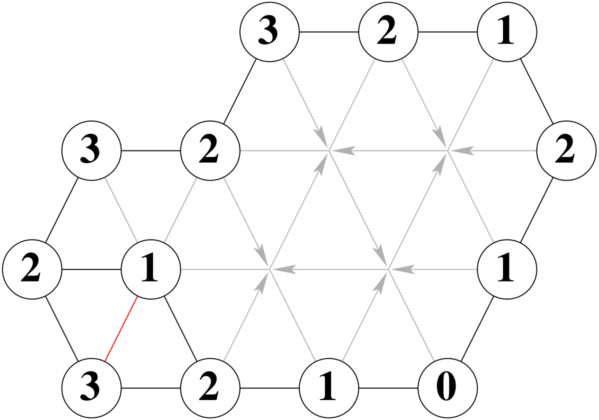
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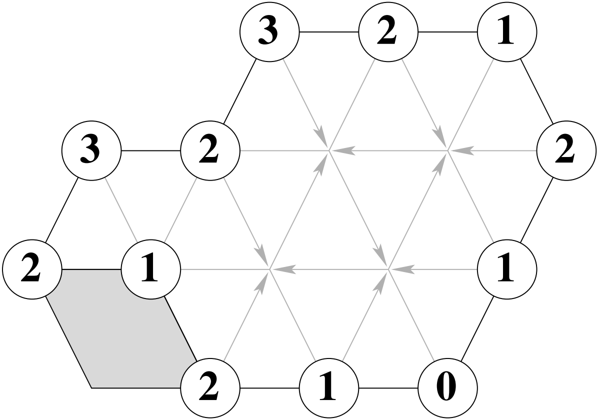
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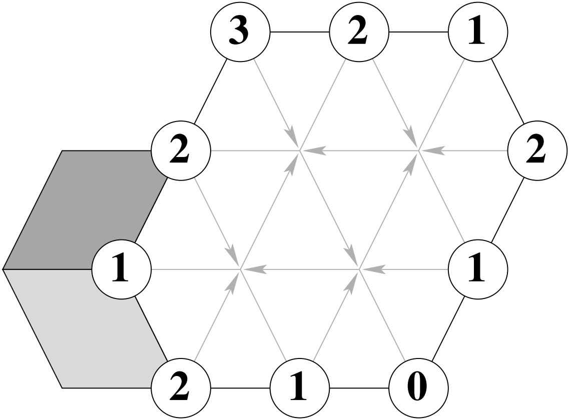
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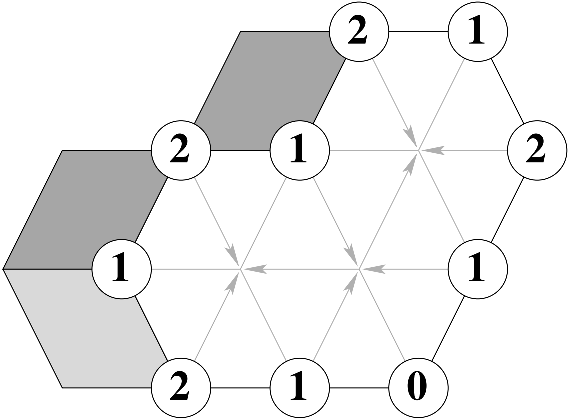
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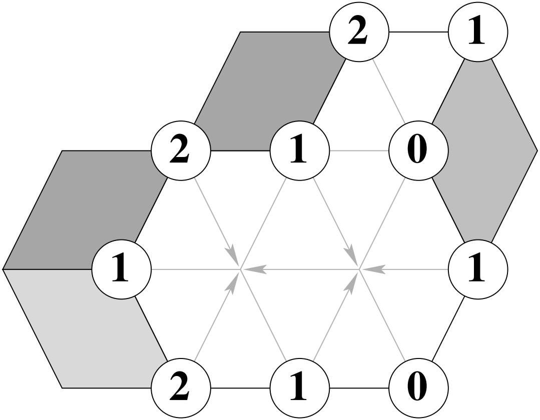
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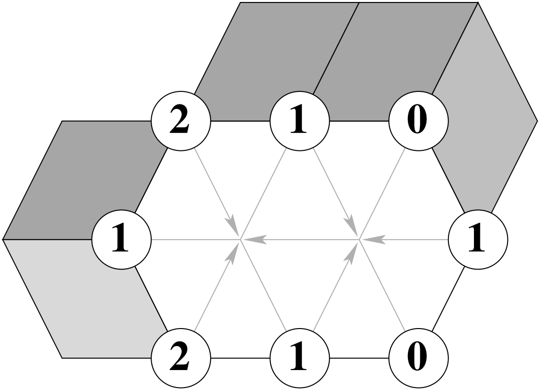
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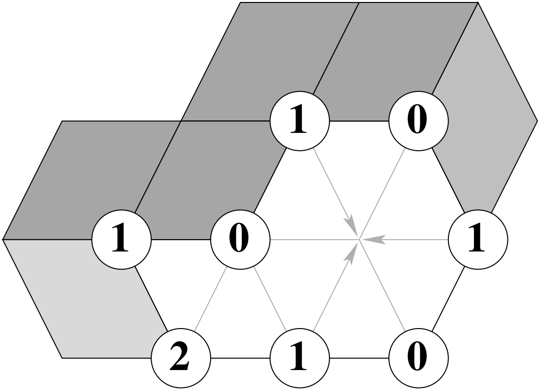
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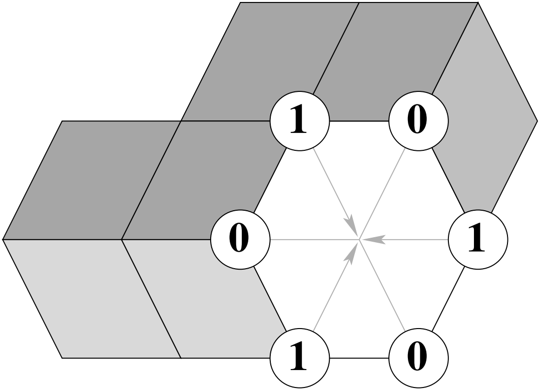
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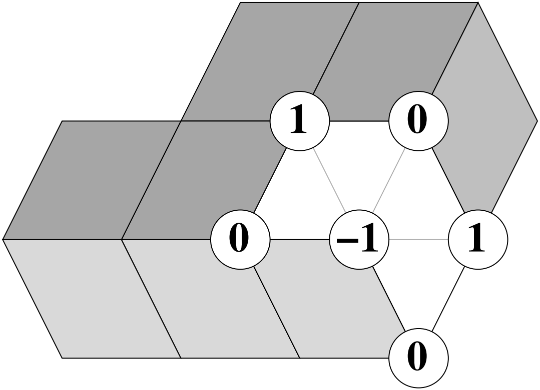
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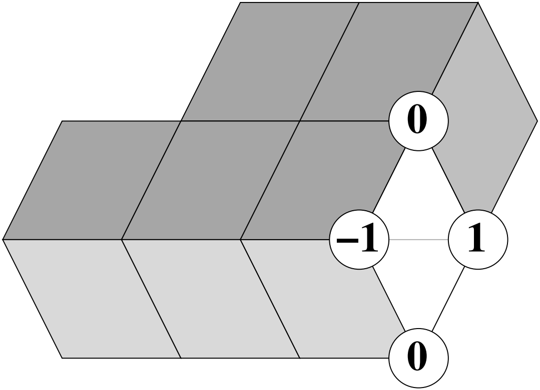
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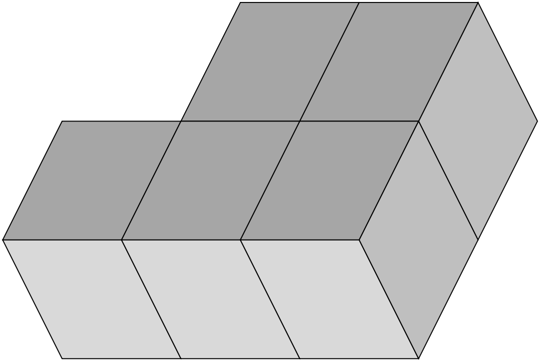

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1. The Thurston's algorithm (lozenge tilings)
   - Weights, heights and flips
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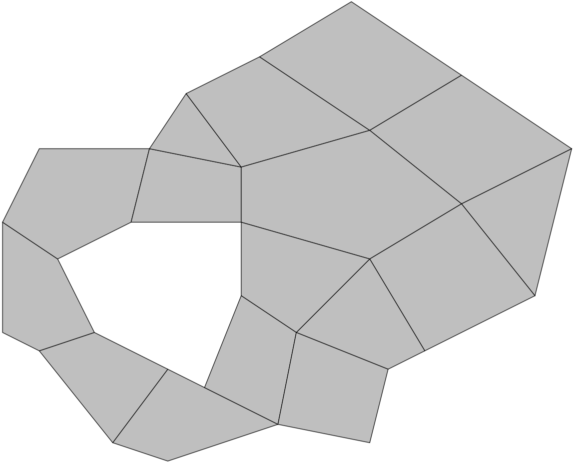
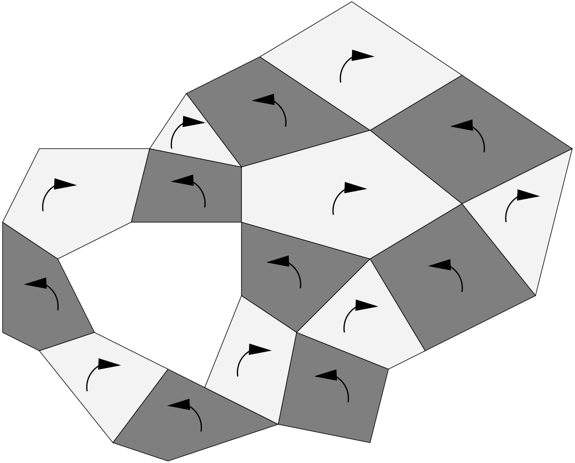
Intuitively:

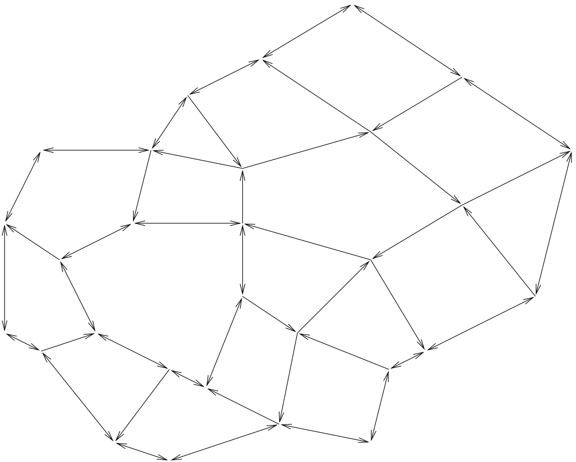
- lozenge tilings: constrained enough for deriving the whole from the boundary
- dimer tilings: holes/irregularities can "hide" information to the boundary



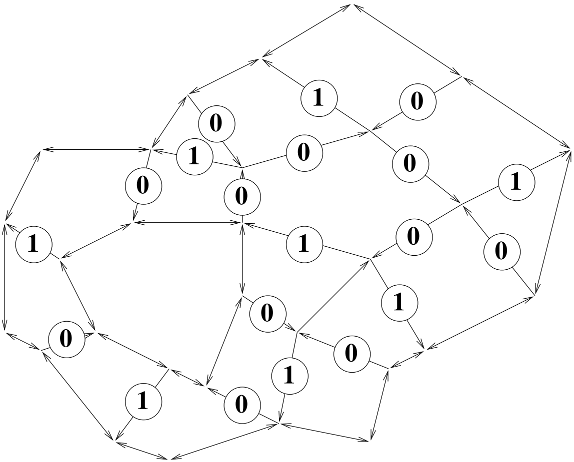
Consider a set of polygonal cells



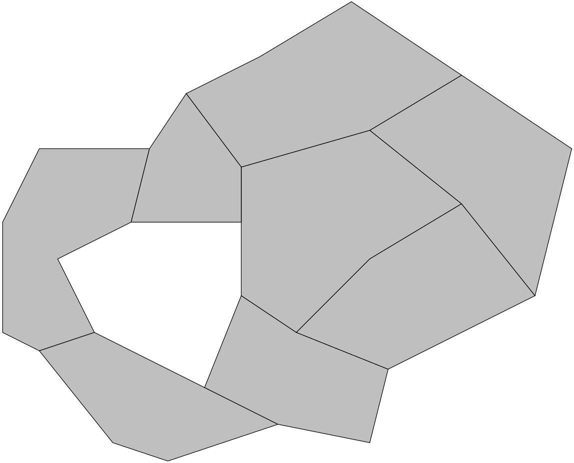
Suppose it is bipartite $\rightsquigarrow$ gears-like orientation of cells



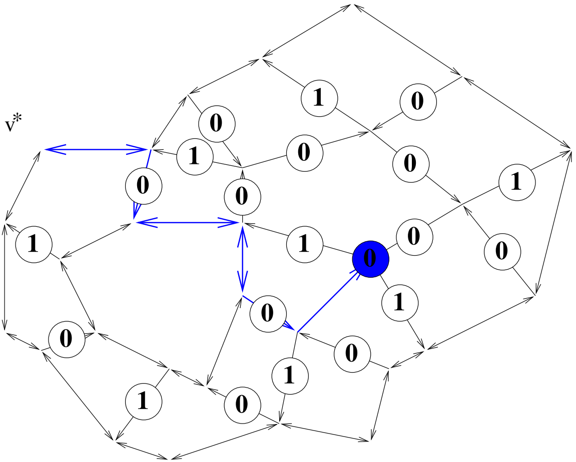
This allows to define a directed graph.



Binary counter: 0-1 weight function δ s.t. $\delta(\mathit{cell}) = 1$



Trival bijection with dimer tilings.



Height function: $h_\delta : v \mapsto \min\{\delta(p) \mid p : v^* \rightsquigarrow v\}$, for a fixed v^*.

But heights do not more *locally* enforce weights!

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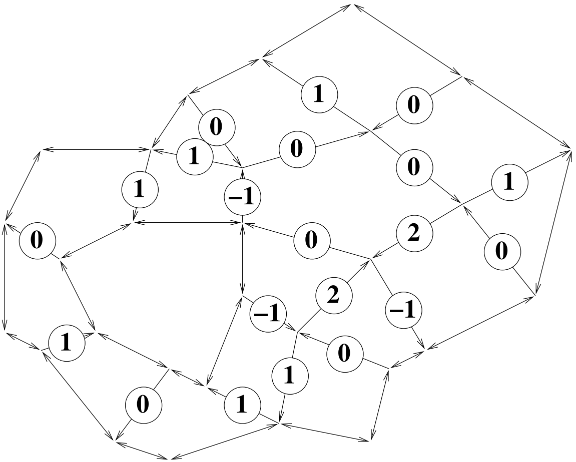

Counter: *real* weight function ψ s.t. $\psi(\mathit{cell}) = 1$.

Proposition

One can compute a counter in linear time (using a spanning tree).

Theorem

If ψ is a counter, then one defines a binary counter by:

$$\delta : (v, v') \mapsto \psi(v, v') - (h_\psi(v') - h_\psi(v)).$$

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Algorithm

1. construct a counter ψ in time $\mathcal{O}(n)$;
2. compute h_ψ in time $\mathcal{O}(n \ln^3 n)$ (SSSP for a planar graph);
3. derive a binary counter in time $\mathcal{O}(n)$.

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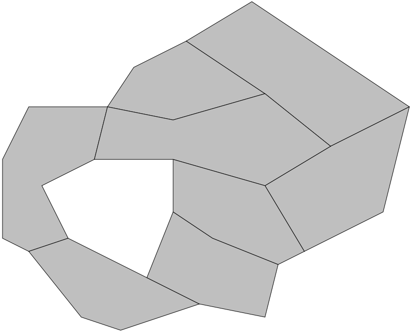
Lozenge tilings of simply connected domains are connected by flips.
Which notion of flip for general dimer tilings?



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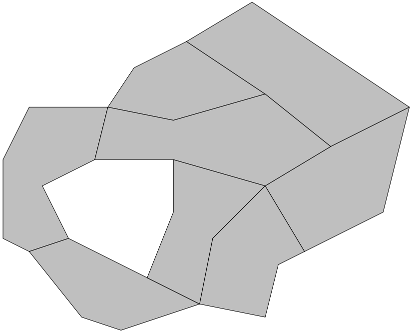

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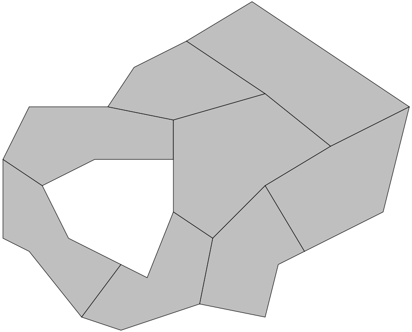
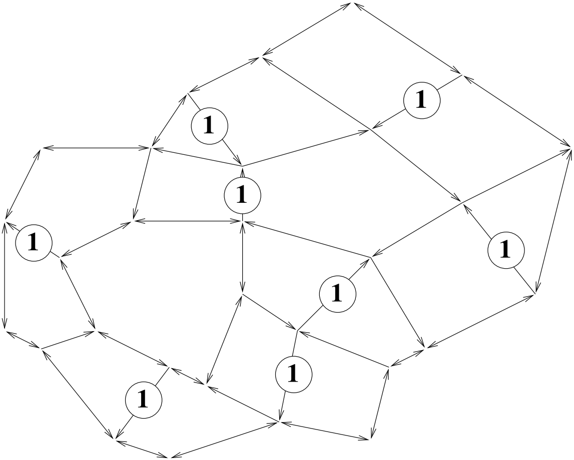


In terms of binary counter:



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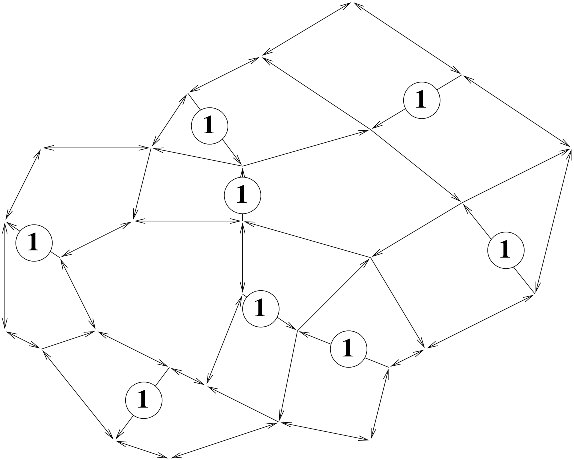

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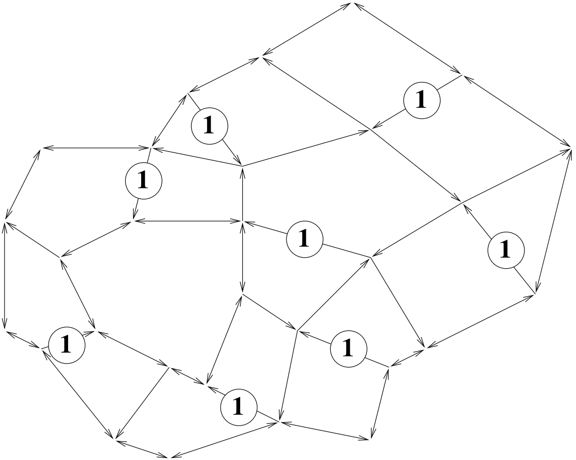

Definition (Flip)

Let A be vertices strongly connected by edges of weight 0 and s.t.

- all its incoming edges have weight 0;
- all its outcoming edges have weight 1.

Then, a *flip* on A exchanges these weights ($0 \leftrightarrow 1$).

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