Characterizations of Flip-accessibility for Domino Tilings of the Whole Plane

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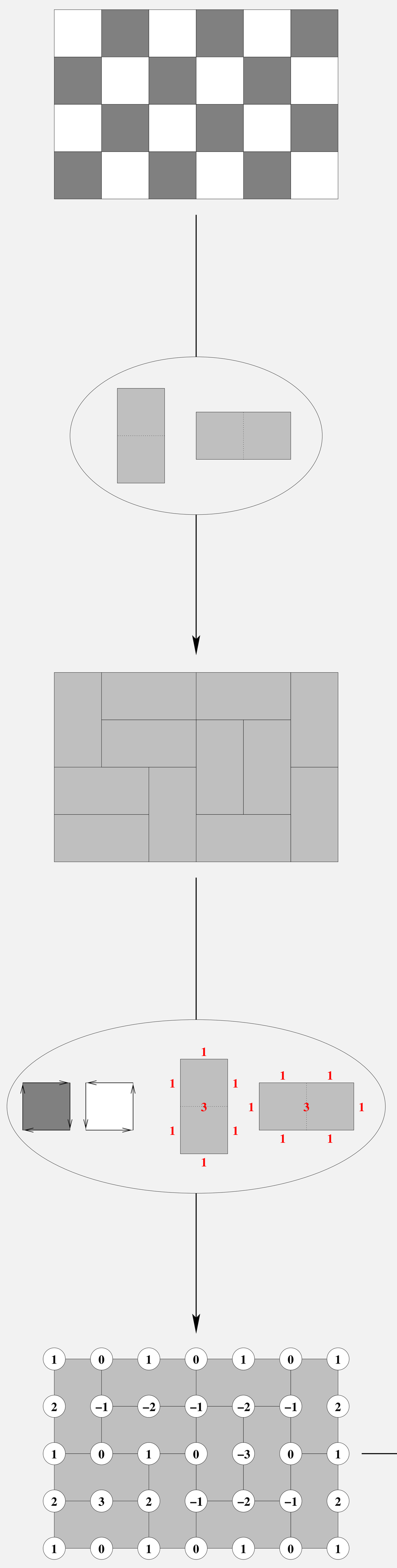
From tilings to surfaces



We here define *domino tilings* and provide a 3-Dim. viewpoint (steps 1–6 are illustrated, left)

1. We consider the whole plane as an infinite checkerboard made of black and white unit squares of $\mathbb{Z}^2$, called cells;

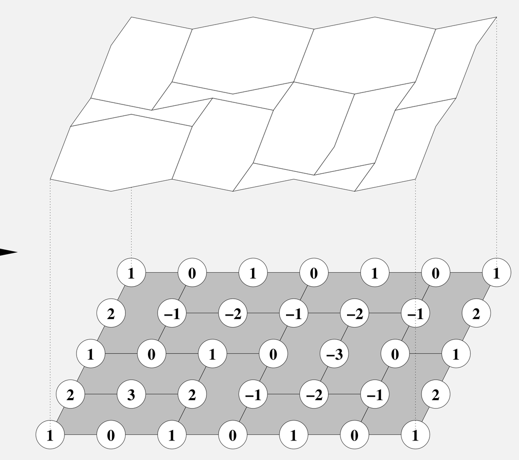
2. a domino is the union of two cells sharing an edge, either horizontally or vertically (shared edges are depicted dashed on the picture, left, in the first ellipse);

3. a domino tiling is a set of dominoes covering without overlap all the cells of the checkerboard;

4. we set a clockwise (resp. counterclockwise) orientation for black (resp. white) cells and we assign weight 1 (resp. 3) to boundary edges (resp. shared edges) of dominoes (see picture, left, in the second ellipse);

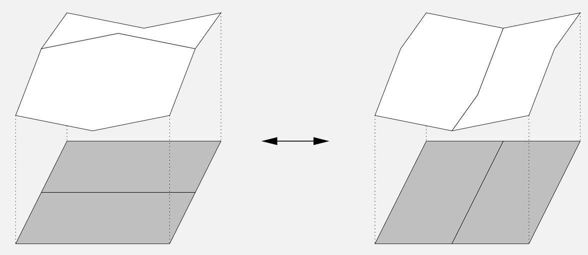
5. orientation of cells and weights over edges of dominoes allows to define a height function h over vertices of dominoes as follows (see picture, bottom-left):
   - we set $h(u_0) = 0$ for some arbitrary vertex u_0;
   - if (u, v) is an edge from u to v with weight $w \in \{1, 3\}$, then $h(v) = h(u) + w$;

6. last, heights of vertices naturally yield a three-dimensional viewpoint for domino tilings in terms of so-called stepped surfaces (last picture, below).



Flip-accessibility

A flip is a local modification of a domino tiling, with two vertical dominoes tiling a square being replaced by two horizontal dominoes tiling the same square (see below).

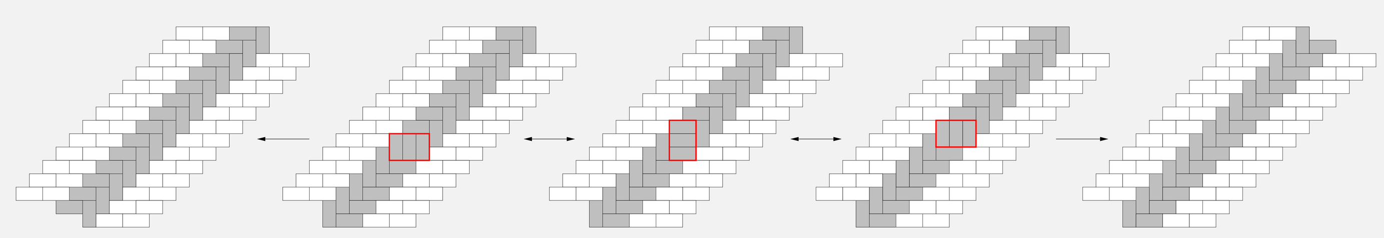


Note that only the height of the central vertex of the square changes: it increases or decreases by 4, according to the position of the square on the checkerboard.

The *distance* between two tilings is the infimum of 2^{-r}, for r such that they coincide within distance r from origin. This yields a notion of *limit* for sequences of tilings.

A tiling T' is said to be flip-accessible from a tiling T if there is a finite or infinite sequence $(T_n)_{n\geq 0}$ of tilings such that:
- $T_0 = T$;
- T_{n+1} is obtained by performing a flip on T_n;
- either $T_N = T'$ for some $N \geq 0$, or T_n tends towards T' when n goes to infinity.

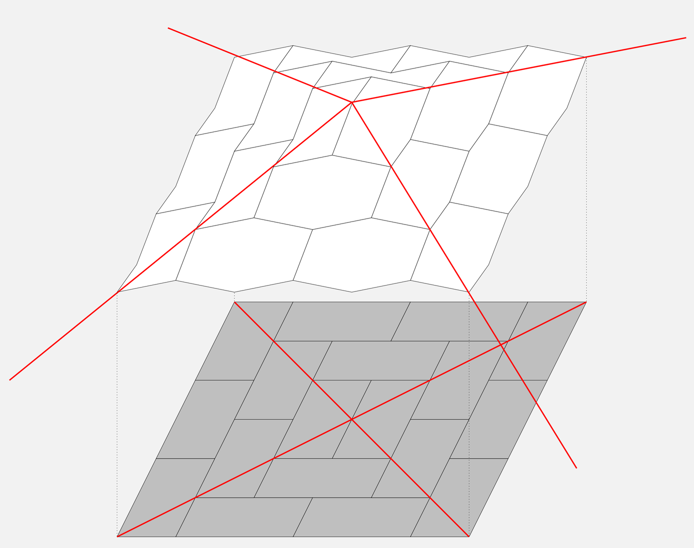
Below, domino tilings which differ on a thin infinite diagonal (grey dominoes) and agree everywhere else (white dominoes, arranged as brickwalls up to infinity).



The above tilings show how a 2×3 rectangle (a "bubble") can be moved upwards or downwars by performing flips. The limit tilings (leftmost and rightmost) do not contain any more this bubble: no flip can be performed.

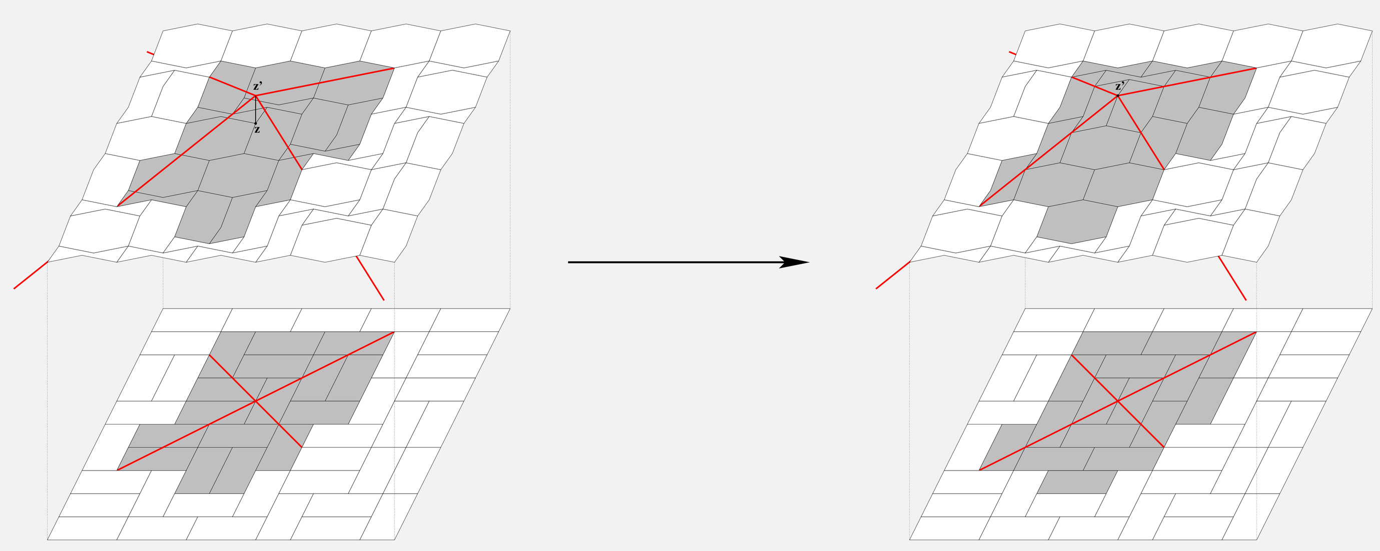
Characterizations

We introduce particular domino tilings: for $(\vec{v}, z) \in \mathbb{Z}^2 \times \mathbb{Z}$, the pyramid $\check{P}_{\vec{v},z}$ (resp. $\hat{P}_{\vec{v},z}$) has minimal (resp. maximal) height function among the domino tilings giving height z to the vertex $\vec{v}$.



Above, a pyramid $\hat{P}_{\vec{v},z}$ (both tiling and surface viewpoints). The red lines represent the edges of the pyramid: in the surface viewpoint, they have direction $(\pm 1, \pm 1, -2)$.

Consider a domino tiling T. Suppose that we want to increase the height of a vertex v from z to z'. By minimality of the height function of $\check{P}_{\vec{v},z'}$, we need to move, by performing flips, T "above" the pyramid $\check{P}_{\vec{v},z'}$.



One shows that this can be done by performing all the flips increasing heights of vertices between T and $\check{P}_{\vec{v},z'}$, that is, the vertices of the grey dominoes on the left picture, above. This leads to the tiling depicted on the right, where the vertex v has height z'. This is possible iff the zone between T and $\check{P}_{\vec{v},z'}$ is bounded: this provides our first characterization. Equivalent characterizations can be stated in terms of shadows or stepped lines.