# Compact packings with three discs

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Density: limsup of the proportion of $B(0, r)$ covered.
Question: densest packings?

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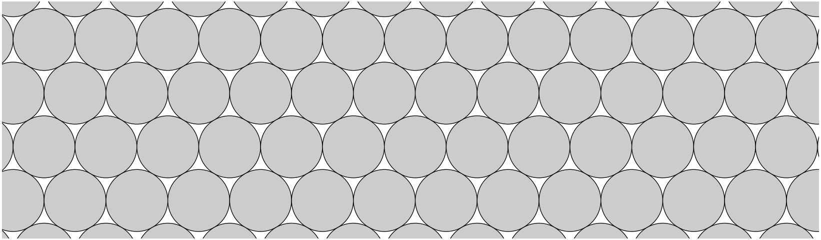
### Theorem (Thue, 1910)

*The densest packing in* $\mathbb{R}^2$ *is the* hexagonal compact *packing.*



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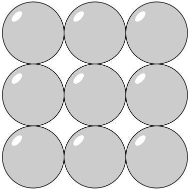
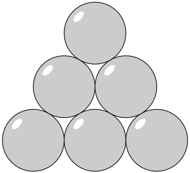
*The densest packings in* $\mathbb{R}^3$ *are the* close-packings*.*





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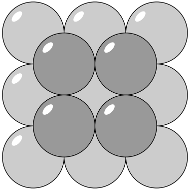

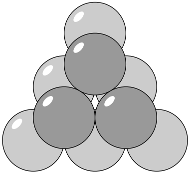

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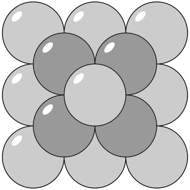

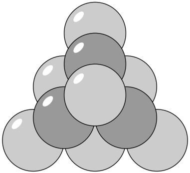

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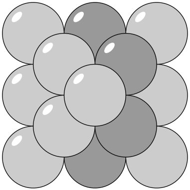

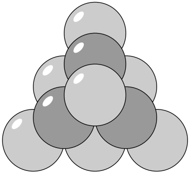

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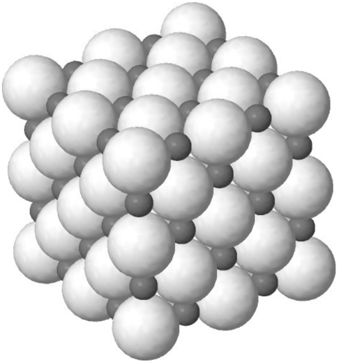
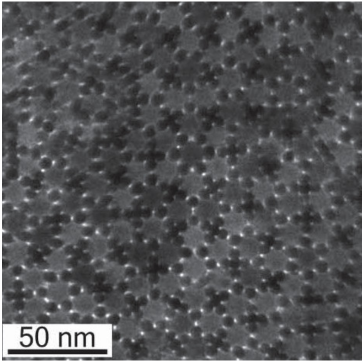
Theorem (Vyazovska *et al.*, 2017)

*The densest packings are known in* $\mathbb{R}^8$ *and* $\mathbb{R}^{24}$.

# Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes.
Natural problem in materials science!





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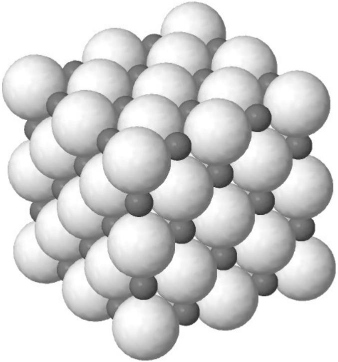
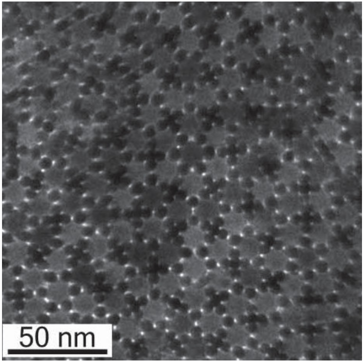
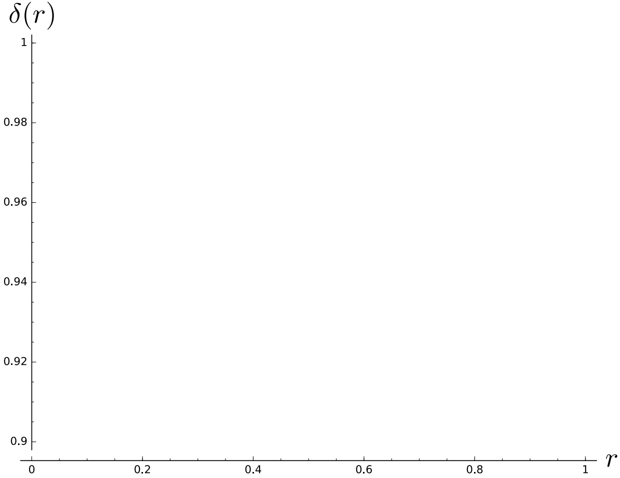





Theorem (Heppes-Kennedy, 2004–2006)

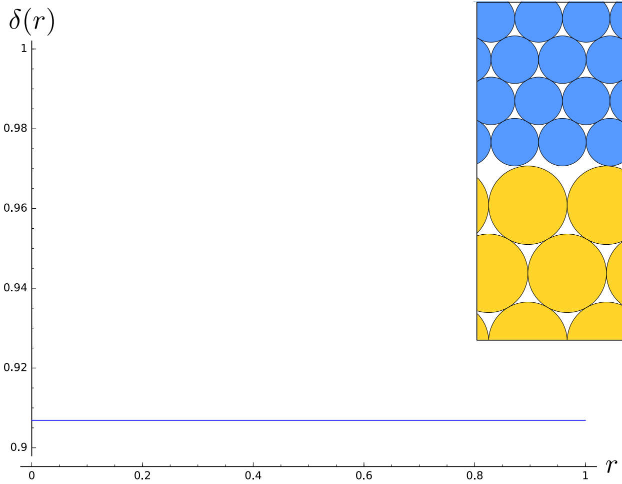
*The densest packings with* *<u>two discs</u>* *are known for* *<u>seven</u>* *ratios.*

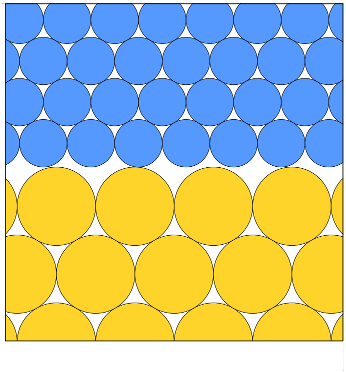
# Two discs




The maximal density is a function $\delta(r)$ of the ratio $r \in [0, 1]$.

# Two discs



The hexagonal compact packing yields a uniform lower bound.

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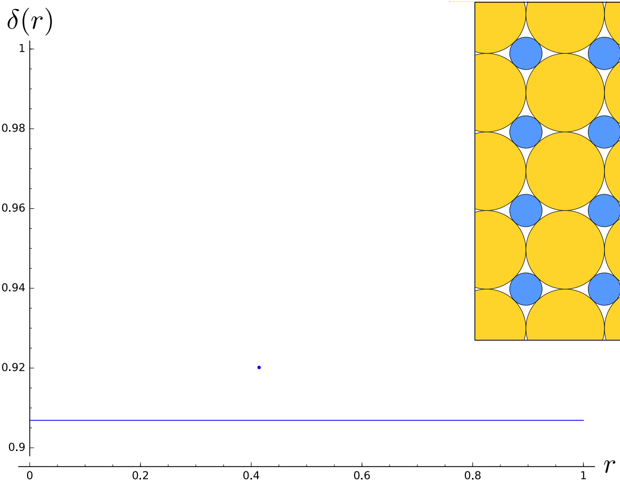


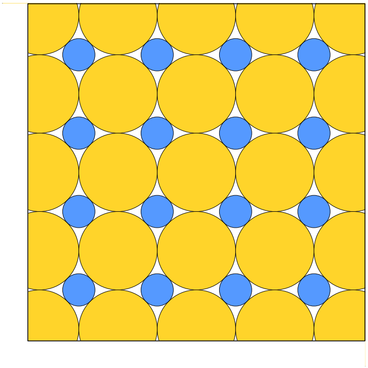

Any given packing yields a lower bound for a specific $r$.

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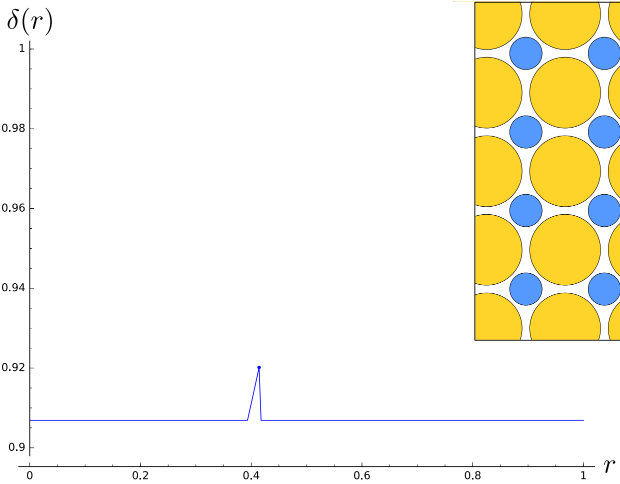
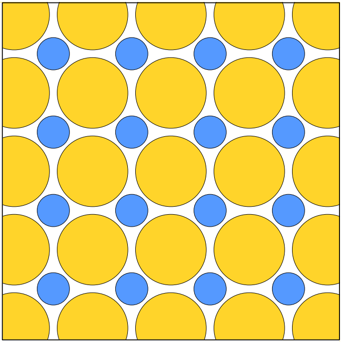
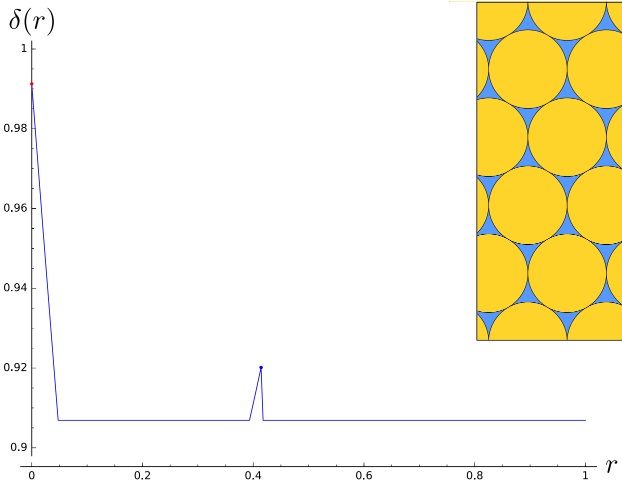
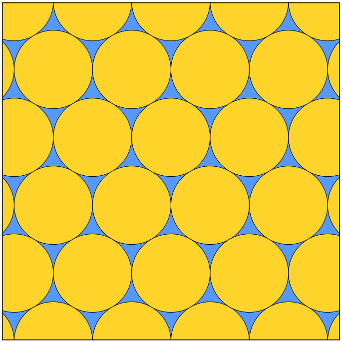


It actually yields a lower bound in a neighborhood of $r$.

# Two discs



$$\lim_{r \to 0} \delta(r) = \delta(1) + (1 - \delta(1))\delta(1) \simeq 0.99133.$$

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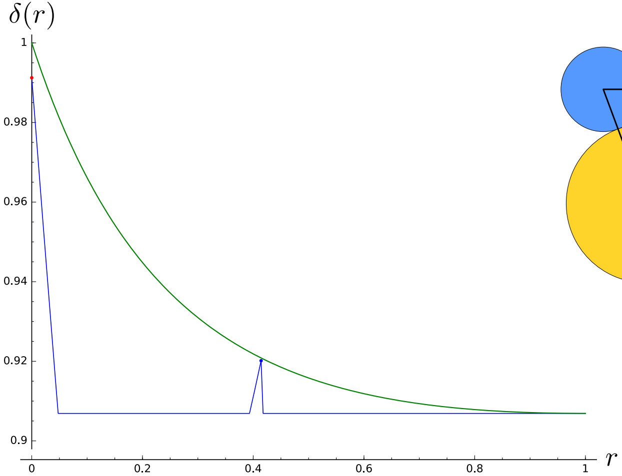
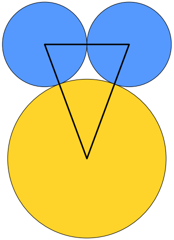
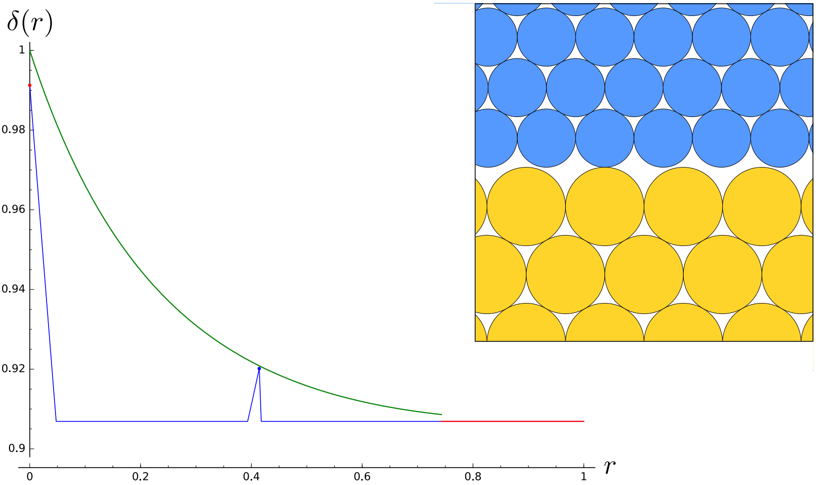


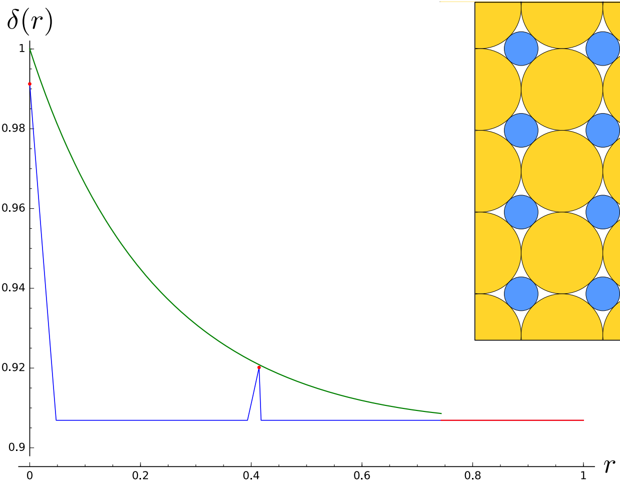
The density in the r1r triangle is an upper bound (Florian, 1960).

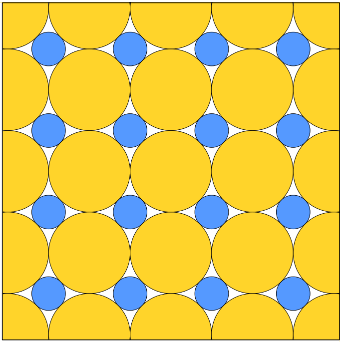
# Two discs




For $r \geq 0.74$, two discs do not pack better than one (Blind, 1969).

# The seven "magic" ratios



0.41, root of $X^2 - 2$.

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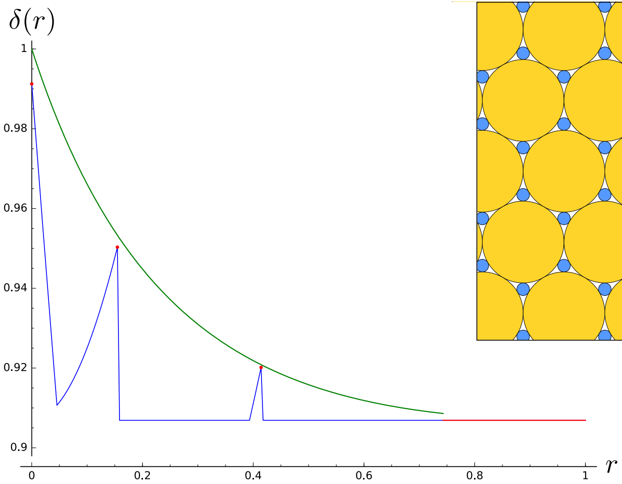


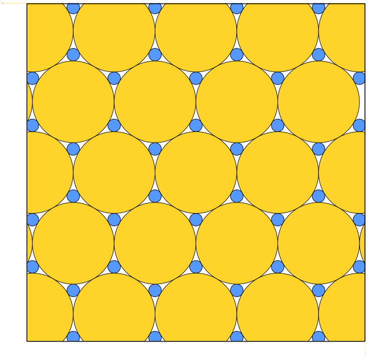

0.15, root of $3X^2 + 6X - 1$.

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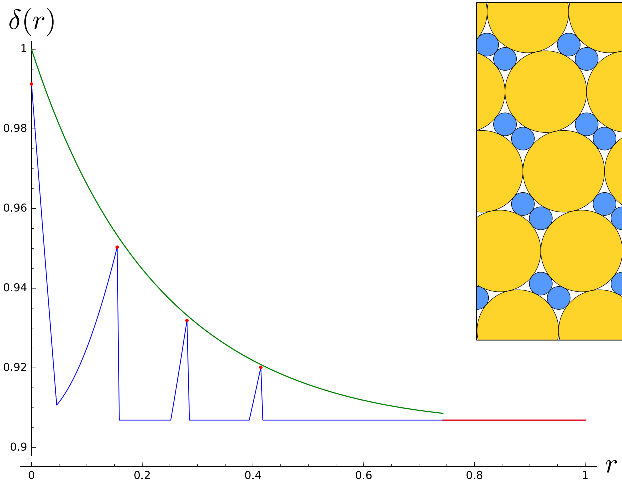


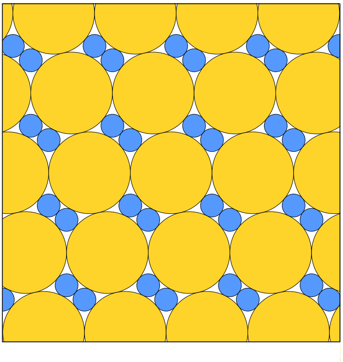

0.28, root of $2X^2 + 3X - 1$.

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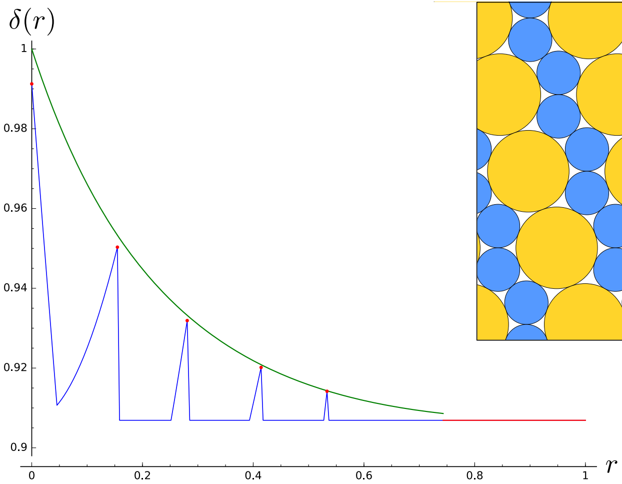


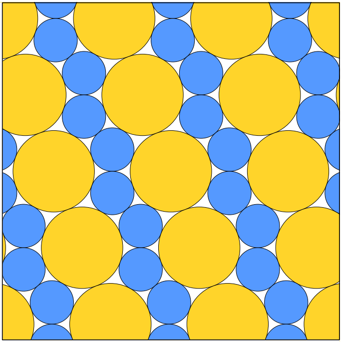

0.53, root of $8X^3 + 3X^2 - 2X - 1$.

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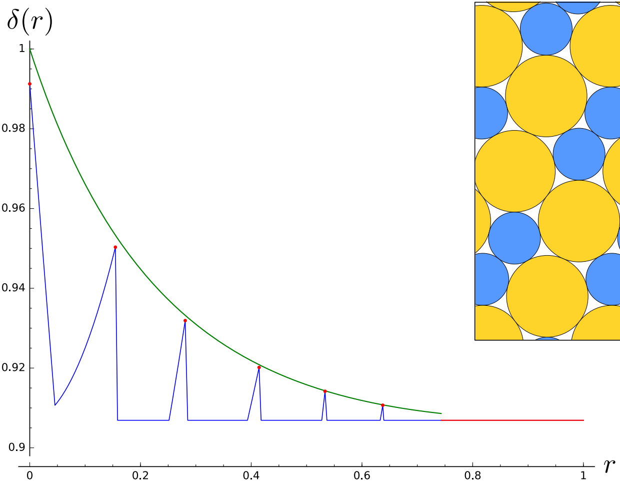


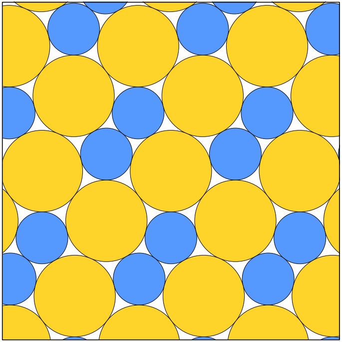

0.64, root of $X^4 - 10X^2 - 8X + 9$.

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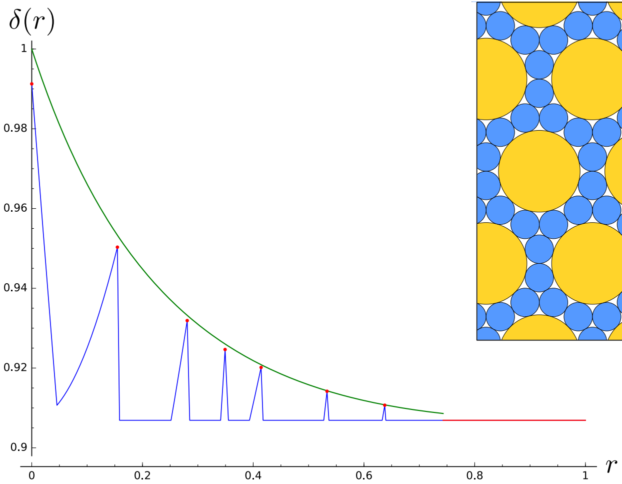


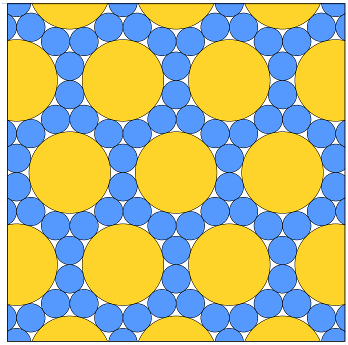

0.35, root of $X^4 - 28X^3 - 10X^2 + 4X + 1$.

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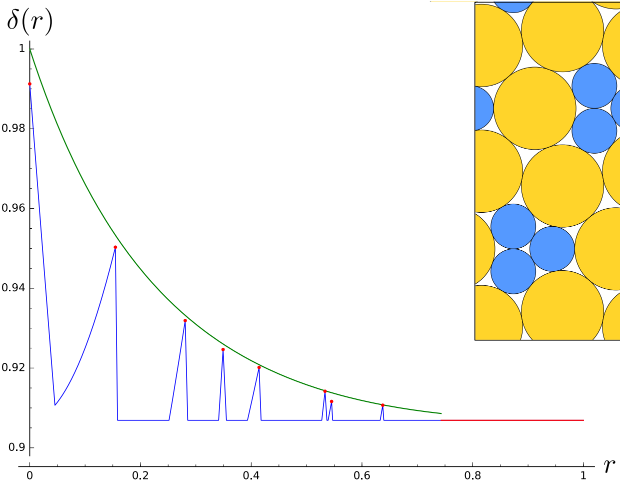


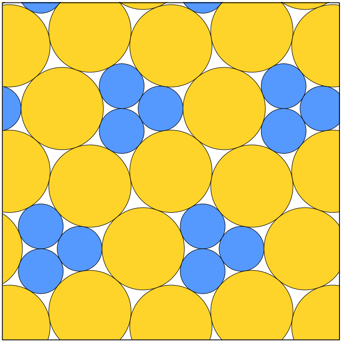

0.55, root of $X^8$-8$X^7$-44$X^6$-232$X^5$-482$X^4$-24$X^3$+388$X^2$-120$X$+9.

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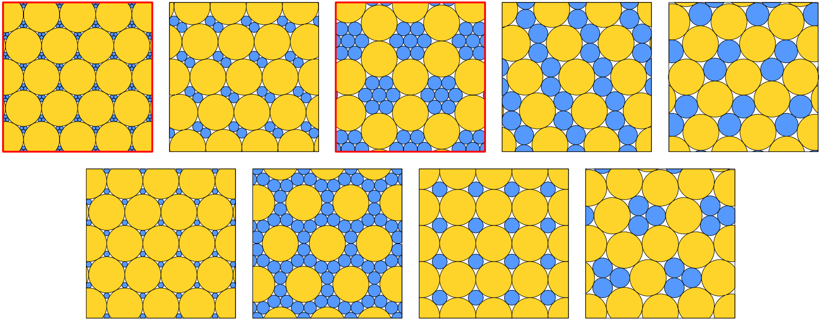
Compact packings are candidates to provably maximize the density.

# Compact packings with two discs



Theorem (Kennedy, 2006)

*There are <u>nine</u> ratios allowing a compact packing with two discs.*

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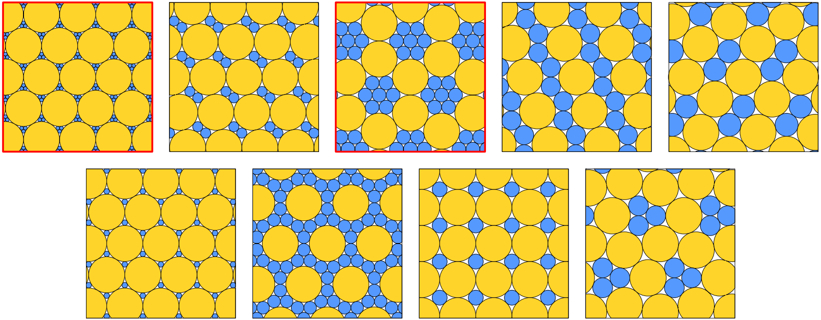

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Remark: two have (still?) not been proven to maximize the density.

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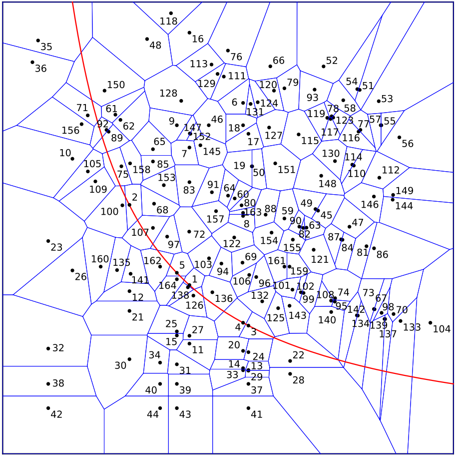


## Theorem (F.-Hashemi-Sizova)

*There are* 164 *ratios allowing a compact packing with three discs.*

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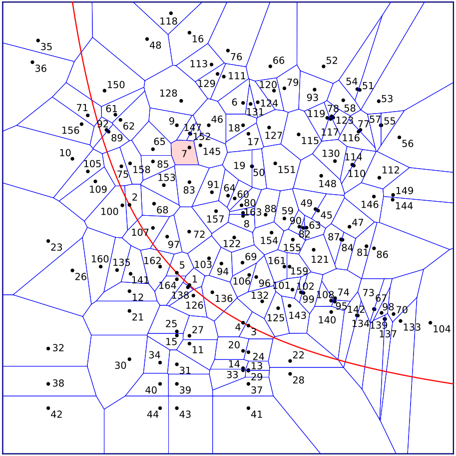

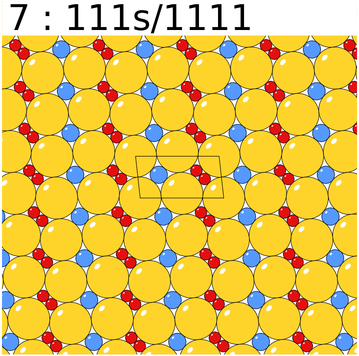


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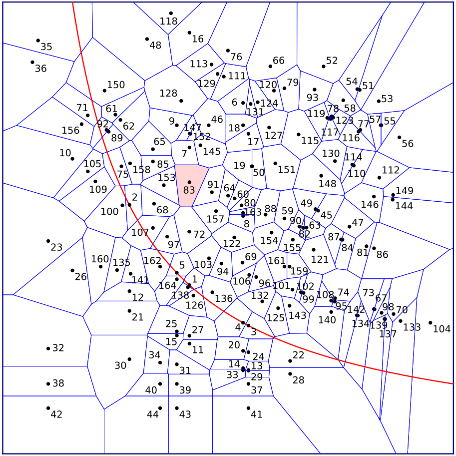

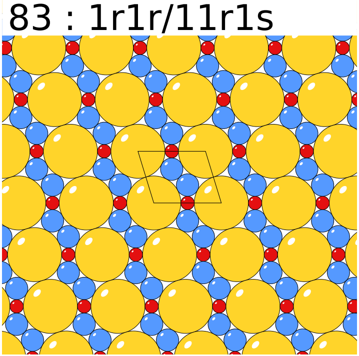


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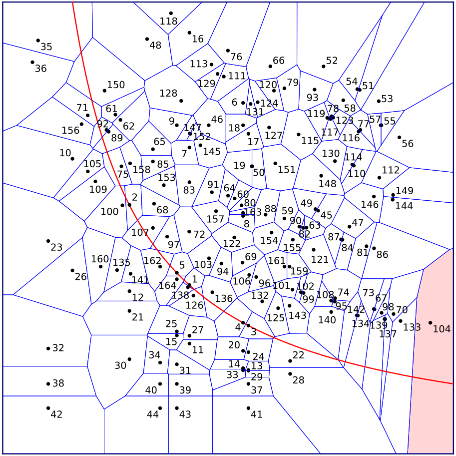


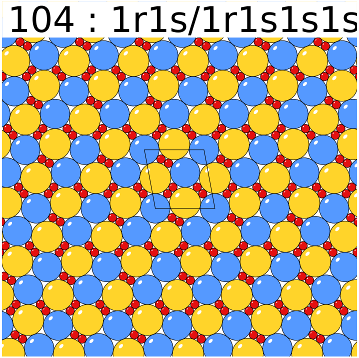


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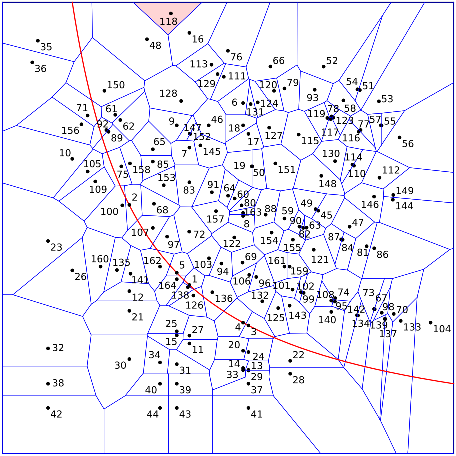

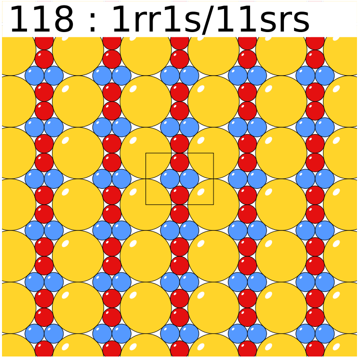


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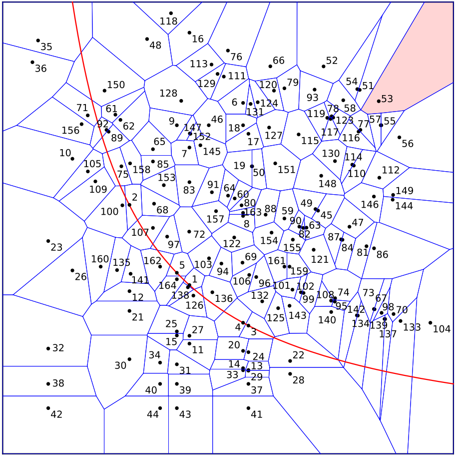

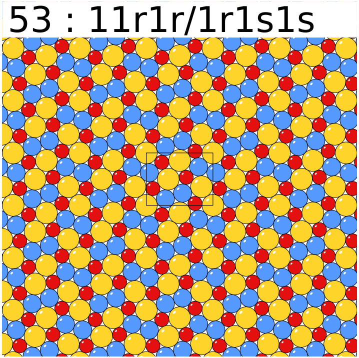


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# Proof sketch

Let $s < r < 1$ be the three sizes of discs.

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## Strategy

For each pair of $s$- and $r$-coronas, solve the polynomial system, then find all the possible coronas and finally find the packings.

# Poster teaser

During the poster sessions, you can get more details about. . .

the proof:

- how to associate a polynomial system with *s*- and *r*-coronas?
- why (and how) we used resultants and interval arithmetic?
- which alternative strategies do exist?
- is it that easy, given the coronas, to find a packing?

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the packings:

- admire a compact packing for each pair $(r, s)$.
- what diversity of packings allows each pair $(r, s)$?
- try yourself to pack it compact (javascript)!