A Characterization of Flip-accessibility for Rhombus Tilings of the Whole Plane

O. Bodini[1], T. Fernique[2], and E. Rémila[3]

[1] LIP6 UMR 7606 (CNRS - Université Paris 6),
4 place Jussieu 75005 Paris - France,
Olivier.Bodini@ufr-info-p6.jussieu.fr
[2] LIRMM UMR 5506 (CNRS - Université Montpellier 2),
161 rue Ada 34392 Montpellier Cedex 5 - France,
Thomas.Fernique@lirmm.fr
[3] LIP UMR 5668 (CNRS - INRIA - ENS Lyon - Université Lyon 1),
46 allée d'Italie 69364 Lyon Cedex 7 - France,
Eric.Remila@ens-lyon.fr

Abstract. It is known that any two rhombus tilings of a polygon are flip-accessible, *i.e.* linked by a finite sequence of local transformations called flips. This paper consider flip-accessibility for rhombus tilings of the *whole plane*, asking whether any two of them are linked by a *possibly infinite* sequence of flips. The answer turning out to depend on tilings, a *characterization* of flip-accessibility is provided. This yields, for example, that any tiling by Penrose tiles is flip-accessible from a Penrose tiling.

Introduction

A *rhombus tiling* of $D \subset \mathbb{R}^2$ is a set of rhombus-shaped compact sets, namely *rhombus tiles*, whose interiors are disjoint, which meet edge-to-edge and whose union is D. Fig. 1 depicts celebrated rhombus tilings of $D = \mathbb{R}^2$ (see also [6]).

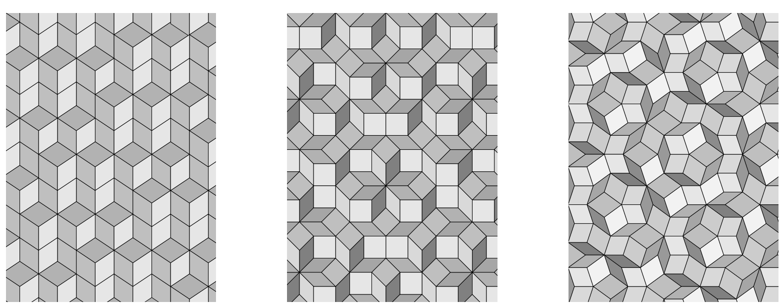

Fig. 1. Rauzy-dual, Ammann-Beenker and Penrose rhombus tilings (from left to right).

Then, the *flip* is a well-known local transformation over rhombus tilings which just exchanges three rhombus tiles sharing a vertex (see *e.g.* [1, 2, 5, 9, 11, 15], and also Fig. 2). Flips rise the question of *flip-accessibility*: can a given rhombus tiling be transformed into another one by performing a sequence of flips?

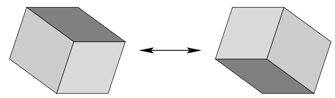

Fig. 2. A flip is an exchange of three rhombus tiles sharing a vertex.

A motivation for studying flip-accessibility for rhombus tilings comes from statistical physics. Indeed, rhombus tilings appeared to be a suitable model for the structure of recently discovered quasicrystalline alloys (see [14]). Moreover, elementary transformations of real quasicrystal, called *phasons*, seem being efficiently modeled by flips (see [10]). This led to study flip dynamics, thus the preliminary question of flip-accessibility.

In the case of rhombus tilings of a polygon, it is proven in [9] that any two rhombus tilings are linked by a finite sequence of flips. In other words, rhombus tilings of a polygon are all mutually flip-accessible. Many results concerning flip dynamics, in particular random sampling, have been obtained (see *e.g.* [5, 11]). The case of rhombus tilings of the whole plane is more complicated. First, note that it is natural to consider flip-accessibility in terms of *possibly infinite* sequences of flips. Then, even with this definition, tilings turn out to be not always flip-accessible. Thus, answering the question of flip-accessibility amounts to *characterize* flip-accessibility between pairs of tilings.

The paper is organized as follows. In Section 1, we more formally define rhombus tilings of the whole plane and the corresponding notion of flip-accessibility. We also show that rhombus tilings are naturally associated with a useful higher-dimensional notion, namely *stepped surfaces*. Section 2 then states the main result of this paper, that is, a characterization of flip-accessibility in terms of *shadows* (Theorem 1). As a corollary, we show that there is a large class of rhombus tilings, namely the *canonical projection tilings*, from which any other rhombus tiling over the same set of rhombus tiles is flip-accessible. The last section is devoted to the proof of this characterization. In particular, we rely on the de Bruijn lines of [3] to introduce *de Bruijn cones*, a tool which could be used for achieving efficient algorithms in the finite case.

1 General settings

Let us first define rhombus tilings of the whole plane. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ be $d \geq 3$ non-colinear unit vectors of $\mathbb{R}^2$. *Rhombus tiles* are the $\binom{d}{2}$ compact sets of nonempty interior defined for $1 \leq i < j \leq d$ by:

$$T_{ij} = \{\lambda \boldsymbol{v}_i + \mu \boldsymbol{v}_j,\ 0 \leq \lambda,\ \mu \leq 1\}.$$

Then, for $\boldsymbol{x} \in \oplus_i \mathbb{Z}\boldsymbol{v}_i$, we denote by $\boldsymbol{x} + T_{ij}$ the rhombus tile obtained by translating T_{ij} by $\boldsymbol{x}$. Note that there is no loss of generality by considering rhombus tiles translated in $\oplus_i \mathbb{Z}\boldsymbol{v}_i$ (instead of the whole $\mathbb{R}^2$) because we are here interested in flip-accessibility; this restriction will be useful in Prop. 1, below. Let us now define rhombus tilings of the whole plane:

Definition 1. *A* $d \to 2$ rhombus tiling *is a set* $\mathcal{T}$ *of translated rhombus tiles of disjoint interiors, meeting edge-to-edge*[4] *and whose union is the whole plane* $\mathbb{R}^2$*.*

For example, Fig. 1 depicts $d \to 2$ rhombus tilings for, respectively, $d = 3, 4, 5$.

Let us now define *flip-accessibility* for $d \to 2$ rhombus tilings. Introduced in [15] for finite domino or lozenge tilings, flips are similarly defined for rhombus tilings (see Fig. 3).

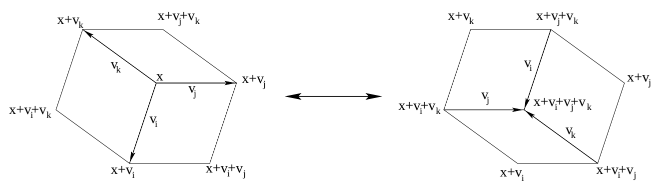

Fig. 3. A flip is a local exchange of three rhombus tiles sharing a vertex.

Clearly, performing a flip on a rhombus tiling yields a (new) rhombus tiling. This also holds for a finite sequence of flips, but we need to be more precise in the case of an *infinite* sequence of flips. Let us define the distance $d(\mathcal{T}, \mathcal{T}')$ between two tilings $\mathcal{T}$ and $\mathcal{T}'$ by:

$$d(\mathcal{T}, \mathcal{T}') = \inf\{2^{-r} \mid \mathcal{T}_{|B(\mathbf{0},r)} = \mathcal{T}'_{|B(\mathbf{0},r)}\},$$

where $\mathcal{T}_{|B(\mathbf{0},r)}$ denotes the set of rhombus tiles in $\mathcal{T}$ which belong to the 2-dimensional ball of center $\mathbf{0}$ and radius r. This allows us to indiscriminately consider finite or infinite sequences of flips for defining flip-accessibility:

Definition 2. *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be two rhombus tilings of the whole plane. If there is a sequence* $(\mathcal{T}_n)_{n \geq 0}$ *of rhombus tilings such that* $\mathcal{T}_0 = \mathcal{T}$*,* $\mathcal{T}_{n+1}$ *is obtained by performing a flip on* $\mathcal{T}_n$ *and* $d(\mathcal{T}_n, \mathcal{T}')$ *tends towards* 0*, then one says that* $\mathcal{T}'$ *is* flip-accessible *from* $\mathcal{T}$*, and one writes:*

$$\mathcal{T} \xrightarrow{flips} \mathcal{T}'$$

[4] that is, two intersecting tiles share either a point $\boldsymbol{x}$ or an edge $\{\boldsymbol{x} + \lambda \boldsymbol{v}_i,\ 0 \leq \lambda \leq 1\}$

Last, let us show how rhombus tilings and flips can be seen from a higher-dimensional viewpoint. This will be very useful in the following sections.

Let $(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d)$ be the canonical basis of $\mathbb{R}^d$. For $1\leq i<j\leq d$ and $\boldsymbol{x}\in\mathbb{Z}^d$, the *unit face* of *type* t_{ij} *located* at $\boldsymbol{x}$ is the subset of $\mathbb{R}^d$ defined by:

$$(\boldsymbol{x},t_{ij})=\{\boldsymbol{x}+\lambda\boldsymbol{e}_i+\mu\boldsymbol{e}_j,\ 0\leq\lambda,\ \mu\leq 1\}.$$

Let then $\Psi:\mathbb{R}^d\to\mathbb{R}^2$ be the linear map defined by:

$$\Psi(x_1,\ldots,x_d)=\sum_{i=1}^{d}x_i\boldsymbol{v}_i.$$

We are now in a position to introduce so-called *stepped surfaces*:

Definition 3. *A* $d\to 2$ stepped surface *is a set* $\mathcal{S}$ *of unit faces of* $\mathbb{R}^d$ *such that* Ψ *is a homeomorphism from the union of these unit faces onto* $\mathbb{R}^2$*.*

A stepped surface is thus a sort of fairly rugged subset of $\mathbb{R}^d$ homeomorphic to a plane. Rhombus tilings and stepped surfaces turn out to be naturally connected:

Proposition 1. *If* $\mathcal{S}$ *is a* $d\to 2$ *stepped surface, then* $\Psi(\mathcal{S})$ *is a* $d\to 2$ *rhombus tiling. Conversely, if* $\mathcal{T}$ *is a* $d\to 2$ *rhombus tiling, then there is a* $d\to 2$ *stepped surface* $\mathcal{S}$ *such that* $\Psi(\mathcal{S})=\mathcal{T}$*, and* $\mathcal{S}$ *is unique up to a translation in* $\ker(\Psi)\cap\mathbb{Z}^d$*.*

Proof. Let $\mathcal{S}$ be a stepped surface. First, Ψ clearly maps unit faces onto rhombus tiles whose vertices belong to $\oplus_i\mathbb{Z}\boldsymbol{v}_i$. Then, note that unit faces are of disjoint interiors and meet edge-to-edge: this still holds by applying the homeomorphism Ψ. Last, Ψ is onto $\mathbb{R}^2$. This shows that $\Psi(\mathcal{S})$ is a rhombus tiling of $\mathbb{R}^2$.
Conversely, let $\mathcal{T}$ be a rhombus tiling of $\mathbb{R}^2$. Let $\boldsymbol{x_0}$ be a vertex of $\mathcal{T}$. Since $\boldsymbol{x}_0\in\oplus_i\mathbb{Z}\boldsymbol{v}_i$ (by definition), there is some $\boldsymbol{y}_0\in\mathbb{Z}^d$ such that $\Psi(\boldsymbol{y}_0)=\boldsymbol{x}_0$, and $\boldsymbol{y}_0$ is unique up to a translation in $\ker(\Psi)\cap\mathbb{Z}^d$. One then define a function h from the vertices of $\mathcal{T}$ to $\mathbb{Z}^d$ as follows:

$$h(\boldsymbol{x}_0)=\boldsymbol{y}_0\quad\text{and}\quad\boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{v}_i\Rightarrow h(\boldsymbol{x}')=h(\boldsymbol{x})+\boldsymbol{e}_i.$$

Actually, h is nothing but a *height function*, and is thus consistent (see *e.g.* [4]). Here, note that $\Psi(h(\boldsymbol{x}))=\boldsymbol{x}$ for any vertex $\boldsymbol{x}$ of $\mathcal{T}$, and let us define the following set of unit faces:

$$\mathcal{S}=\{(h(\boldsymbol{x}),t_{ij})\mid\boldsymbol{x}+T_{ij}\in\mathcal{T}\}.$$

It follows from the construction of $\mathcal{S}$ that the restriction of Ψ to the union of unit faces of $\mathcal{S}$, denoted by $\Psi_{|\mathcal{S}}$, is a bijection onto $\mathbb{R}^2$. It is continuous as Ψ does, and its inverse is also continuous since $\Psi_{|\mathcal{S}}$ is closed. Thus, Ψ is a homeomorphism from $\mathcal{S}$ onto $\mathbb{R}^2$, that is, $\mathcal{S}$ is a stepped surface. Last, $\mathcal{S}$ is unique up to the initial choice of $\boldsymbol{y}_0$, that is, up to a translation in $\ker(\Psi)\cap\mathbb{Z}^d$. □

In other words, stepped surfaces are nothing but rhombus tilings seen from a higher-dimensional viewpoint. Actually, this is just a generalization of ideas introduced in [15] for finite domino or lozenge tilings. Note also that the case $d = 3$ corresponds to the notion introduced in [8], where the 3-dimensional viewpoint is very natural (see, for example, the leftmost tiling of Fig. 1).

The notion of flip is then defined over stepped surfaces so that if a stepped surface $\mathcal{S}'$ is obtained by performing a flip on a stepped surface $\mathcal{S}$, then the rhombus tiling $\Psi(\mathcal{S}')$ is obtained by performing a flip on the rhombus tiling $\Psi(\mathcal{S})$ (it suffices to replace $\boldsymbol{v}_i$ by $\boldsymbol{e}_i$ on Fig. 3). If, moreover, one says that two stepped surfaces $\mathcal{S}$ and $\mathcal{S}'$ are at distance less than 2^{-r} if they share the same set of unit faces within the *d-dimensional* ball $B(\mathbf{0}, r)$, then this leads to a notion of flip-accessibility for stepped surfaces which satisfies:

Proposition 2. *For two stepped surfaces $\mathcal{S}$ and $\mathcal{S}'$, one has:*

$$\Psi(\mathcal{S}) \overset{flips}{\longrightarrow} \Psi(\mathcal{S}') \;\Leftrightarrow\; \exists \boldsymbol{a} \in \ker(\Psi) \cap \mathbb{Z}^d \;\; s.t.\; \mathcal{S} \overset{flips}{\longrightarrow} \boldsymbol{a} + \mathcal{S}',$$

where $\boldsymbol{a} + \mathcal{S}'$ denotes the stepped surface obtained by translating $\mathcal{S}'$ by $\boldsymbol{a}$.

Fig. 4 illustrates the notion of flip-accessibility. Note that, contrarily to the case of rhombus tilings of a polygon, flip-accessibility does not always holds, and is moreover even not symmetric.

2 Characterization by shadows

The aim of this section is to provide a characterization of flip-accessibility for stepped surfaces (which can be then restated in terms of rhombus tilings according to Prop. 1 and 2). Let us first define the following maps, for $1 \leq i < j \leq d$:

$$\pi_{ij} \;:\; \begin{array}{rcl} \mathbb{R}^d & \rightarrow & \mathbb{R}^2 \\ (z_1, \ldots, z_d) & \mapsto & (z_i, z_j) \end{array}$$

In particular, π_{ij} maps the unit face $(\boldsymbol{x}, t_{kl})$ onto a unit square if $i = k$ and $j = l$, onto a unit segment if $i = k$ or $j = l$ and onto a point otherwise. We then use these maps to define the *shadows* of a stepped surface (see *e.g.* Fig. 4):

Definition 4. *The* shadows *of a $d \to 2$ stepped surface $\mathcal{S}$ are the $\binom{d}{2}$ subsets of $\mathbb{R}^2$ defined, for $1 \leq i < j \leq d$, by:*

$$\pi_{ij}(\mathcal{S}) = \bigcup_{(\boldsymbol{x},t) \in \mathcal{S}} \pi_{ij}(\boldsymbol{x}, t).$$

A simple but fundamental property of shadows is that they are invariant by performing a flip (this can be easily checked on Fig. 3). This also holds for finite sequences of flips, but we have only a weaker property for infinite sequences:

Proposition 3. *If a stepped surface $\mathcal{S}'$ is flip-accessible from a stepped surface $\mathcal{S}$, then the shadows of $\mathcal{S}'$ are included in the shadows of $\mathcal{S}$:*

$$\mathcal{S} \overset{flips}{\longrightarrow} \mathcal{S}' \;\Rightarrow\; \forall i,\; \forall j,\; \pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S}).$$

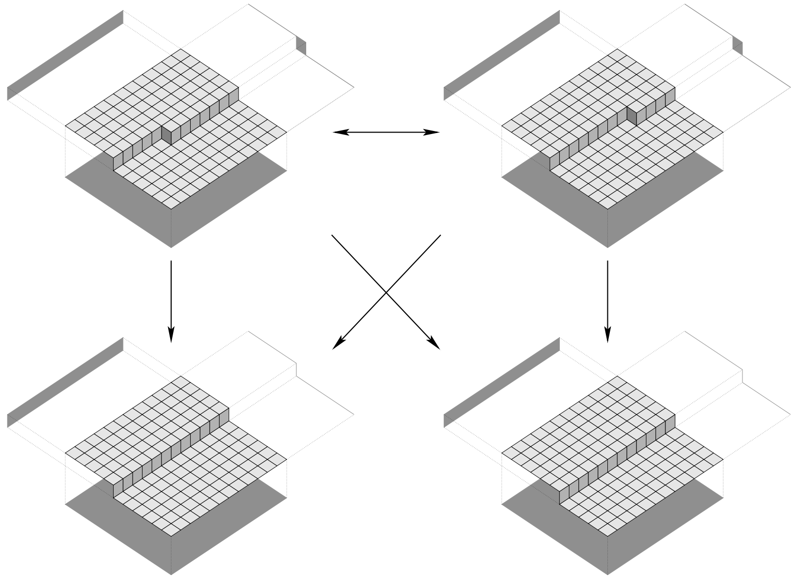

Fig. 4. Four patches of $3 \to 2$ stepped surfaces and their shadows (see Def. 4, below). Flip-accessibility is represented by arrows: the top two stepped surfaces are mutually flip-accessible (by a finite sequence of flips), and the bottom two stepped surfaces are flip-accessible from them (by an infinite sequence of flips rejecting the "corner" to infinity in one of the two possible directions). The bottom two stepped surfaces are sort of *dead ends*: no flip can be performed on them. It is worth noticing that a stepped surface is flip-accessible from another one if and only if the shadows of the latter are included in the shadows of the former (this illustrates Th. 1, below).

Proof. Let $\mathcal{S}_n$ be a sequence of stepped surfaces, obtained by performing flips on $\mathcal{S}$, which tends towards $\mathcal{S}'$. Let $z \in \pi_{ij}(\mathcal{S}')$: z belongs to the projection of a face $(\boldsymbol{x}, t) \in \mathcal{S}'$. Let $r \in \mathbb{R}$ such that $(\boldsymbol{x}, t) \subset B(0, r)$ and $N \in \mathbb{N}$ such that $d(\mathcal{S}_N, \mathcal{S}') \leq 2^{-r}$. In particular, $(\boldsymbol{x}, t) \in \mathcal{S}_N$. Since $\mathcal{S}_N$ is obtained from $\mathcal{S}$ by performing a finite number of flips, both have the same shadows. Thus, $z \in \pi_{ij}(\boldsymbol{x}, t) \subset \pi_{ij}(\mathcal{S}_N)$ yields $z \in \pi_{ij}(\mathcal{S})$. This proves $\pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S})$. □

In the previous proposition, inclusions of shadows can be strict (see, for example, Fig. 4). Actually, the main result of this paper is that the converse of this proposition also holds:

Theorem 1. *A stepped surface $\mathcal{S}'$ is flip-accessible from a stepped surface $\mathcal{S}$ iff the shadows of $\mathcal{S}'$ are included in the shadows of $\mathcal{S}$:*

$$\mathcal{S} \xrightarrow{flips} \mathcal{S}' \;\Leftrightarrow\; \forall i,\ \forall j,\ \pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S}).$$

Th. 1 is proven in the following section. Before this, let us provide an interesting corollary. We need the following definition:

Definition 5. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *be two vectors of* $\mathbb{R}^d$ *with non-zero entries. The* $d \to 2$ stepped plane $\mathcal{P}_{\boldsymbol{u},\boldsymbol{v}}$ *is defined as the set of all unit faces which lie (entirely) in the following "slice" of* $\mathbb{R}^d$*:*

$$\mathbb{R}\boldsymbol{u} + \mathbb{R}\boldsymbol{v} + [0,1]^d.$$

Roughly speaking, the stepped plane $\mathcal{P}_{\boldsymbol{u},\boldsymbol{v}}$ is an approximation by unit faces of the real plane $\mathbb{R}\boldsymbol{u}+\mathbb{R}\boldsymbol{v}$ (this corresponds to a viewpoint developed in discrete geometry, see *e.g.* [12]). Actually, stepped planes are nothing but the stepped surfaces which are associated by Prop. 1 with so-called *canonical projection tilings*. These are rhombus tilings obtained by the *cut and project* method (see [7, 13]). For example, the Rauzy-dual, Ammann-Beenker and Penrose tilings depicted on Fig. 1 are canonical projection tilings associated with $d \to 2$ stepped planes for, respectively, $d = 3, 4, 5$ (see [6]).

Now, let us note that $\pi_{ij}(\mathbb{R}\boldsymbol{u}+\mathbb{R}\boldsymbol{v}) = \mathbb{R}^2$. This easily yields that $\pi_{ij}(\mathcal{P}_{\boldsymbol{u},\boldsymbol{v}}) = \mathbb{R}^2$. In particular, the shadows of the stepped plane $\mathcal{P}_{\boldsymbol{u},\boldsymbol{v}}$ contain the shadows of any other stepped surface. We thus obtain as an immediate corollary of Th. 1:

Corollary 1. *Any stepped surface is flip-accessible from a stepped plane.*

In terms of rhombus tilings, this means that any rhombus tiling is flip-accessible from a canonical projection tiling over the same set of rhombus tiles.

3 Proof of the characterization

This section provides a proof of the characterization stated in Theorem 1. The necessary condition is proven by Prop. 3. Let thus $\mathcal{S}$ and $\mathcal{S}'$ be two stepped surfaces such that the shadows of $\mathcal{S}'$ are included in the shadows of $\mathcal{S}$, and let us prove that $\mathcal{S}'$ is flip-accessible from $\mathcal{S}$.

Since the proof is not so short, it is worth giving a brief outline. The general idea is to transform $\mathcal{S}$ into $\mathcal{S}'$ by moving one by one unit faces. More precisely, for $(\boldsymbol{x}', t_{ij}) \in \mathcal{S}'$, inclusion of shadows ensure that there is a unit face $(\boldsymbol{x}, t_{ij}) \in \mathcal{S}$ such that $\pi_{ij}(\boldsymbol{x}', t_{ij}) = \pi_{ij}(\boldsymbol{x}, t_{ij})$. We would like to move $(\boldsymbol{x}, t_{ij})$ to $(\boldsymbol{x}', t_{ij})$. We proceed as follows. While there is k such that $x_k < x'_k$, we choose such a k and we define a set $F^*_k(\boldsymbol{x}, t_{ij})$ such that, by performing a finite number flips over this set, we can translate $(\boldsymbol{x}, t_{ij})$ by $\boldsymbol{e}_k$ (Lem. 1, 2 and 3). Similarly, we can translate $(\boldsymbol{x}, t_{ij})$ by $-\boldsymbol{e}_k$ for k such that $x_k > x'_k$. Hence, we can move $(\boldsymbol{x}, t_{ij}) \in \mathcal{S}$ to $(\boldsymbol{x}', t_{ij}) \in \mathcal{S}'$ by performing a finite number of flips. The last step will be to show that we can, in this way, obtain unit faces of $\mathcal{S}'$ over growing balls centered in $\mathbf{0}$ (Lem. 4), that is, that $\mathcal{S}'$ is flip-accessible from $\mathcal{S}$ (see Def. 2).

Let us now start the proof. We first define a useful tool:

Definition 6. *Let $\mathcal{S}$ be a stepped surface, $k \in \mathbb{Z}$ and $1 \leq i \leq d$. If not empty, the following set of unit faces defines the k-th* de Bruijn section of type i *of $\mathcal{S}$:*

$$S_{i,k} = \{((x_1, \ldots, x_d), t_{ij}) \in \mathcal{S} \mid x_i = k\}.$$

It is easily seen that $S_{i,k}$ is an infinite stripe of unit faces two by two adjacent along vectors $\boldsymbol{e}_i$. Then, removing $S_{i,k}$ naturally splits $\mathcal{S}$ into the two following connected sets of unit faces (see Fig. 5):

$$T^+_{i,k} = \{((x_1, \ldots, x_d), t) \in \mathcal{S} \mid x_i > k\} \quad \text{and} \quad T^-_{i,k} = \mathcal{S} \backslash (S_{i,k} \cup T^+_{i,k}).$$

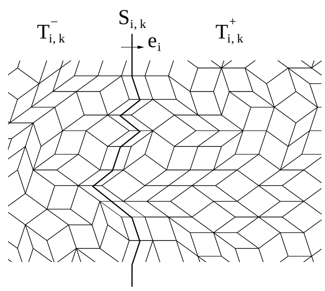

Fig. 5. A de Bruijn section $S_{i,k}$, here represented by a broken line crossing its unit faces, splits a stepped surface into two connected sets of unit faces, $T^-_{i,k}$ and $T^+_{i,k}$.

Actually, de Bruijn sections turn out to be the set of unit faces associated by Prop. 1 with the well-known de Bruijn lines introduced in [3]. In other words, $S_{i,k}$ is a de Bruijn section of $\mathcal{S}$ iff $\Psi(S_{i,k})$ is a de Bruijn line of the rhombus tiling $\Psi(\mathcal{S})$. In particular, two de Bruijn sections share at most one face, as well as de Bruijn lines. In such a case, they are said to *intersect*. Note that, if $(\boldsymbol{x}, t_{kl}) = S_{i,n} \cap S_{j,m}$, then $k = i$, $l = j$, $x_i = n$ and $x_j = m$. In particular, only sections of different types can intersect, although they can also not intersect.

We use de Bruijn sections to define so-called *de Bruijn triangles*:

Definition 7. *For $(\boldsymbol{x} = (x_1, \ldots, x_d), t_{ij}) \in \mathcal{S}$ and $1 \leq k \leq d$, $k \neq i$, $k \neq j$, the* de Bruijn triangle $F_k(\boldsymbol{x}, t_{ij})$ *is the set of unit faces of $\mathcal{S}$ defined by:*

$$F_k(\boldsymbol{x}, t_{ij}) = (S_{i,x_i} \cup T^{\varepsilon_i}_{i,x_i}) \cap (S_{j,x_j} \cup T^{\varepsilon_j}_{j,x_j}) \cap (S_{k,x_k} \cup T^-_{k,x_k}),$$

where ε_i and ε_j respectively denote the signs of entries of $\boldsymbol{v}_k$ in the basis $(\boldsymbol{v}_i, \boldsymbol{v}_j)$.

Roughly speaking, $F_k(\boldsymbol{x}, t_{ij})$ is the triangle defined by the three "lines" S_{i,x_i}, S_{j,x_j} and S_{k,x_k} (see Fig. 6, left). Note that it could be infinite, since the de Bruijn sections S_{i,x_i} or S_{j,x_j} do not necessarily intersect S_{k,x_k}. We will later avoid this case (Lem. 3). Intuitively, for translating $(\boldsymbol{x}, t_{ij})$ by $\boldsymbol{e}_k$, we first need to translate by $\boldsymbol{e}_k$ the unit faces in $F_k(\boldsymbol{x}, t_{ij})$. However, moving a unit face of $F_k(\boldsymbol{x}, t_{ij})$ requires, in turn, to move some others unit faces before. Therefore, we extend de Bruijn triangles by so-called de Bruijn cones (see also Fig. 6, right):

Definition 8. *With the convention $F_k(A \cup B) = F_k(A) \cup F_k(B)$, we define:*

$$F_k^0(\boldsymbol{x}, t_{ij}) = (\boldsymbol{x}, t_{ij}) \quad \textit{and} \quad F_k^{n+1}(\boldsymbol{x}, t_{ij}) = F_k(F_k^n(\boldsymbol{x}, t_{ij})).$$

Then, the de Bruijn cone $F_k^*(\boldsymbol{x}, t_{ij})$ *is defined by:*

$$F_k^*(\boldsymbol{x}, t_{ij}) = \bigcup_{n \geq 0} F_k^n(\boldsymbol{x}, t_{ij}).$$

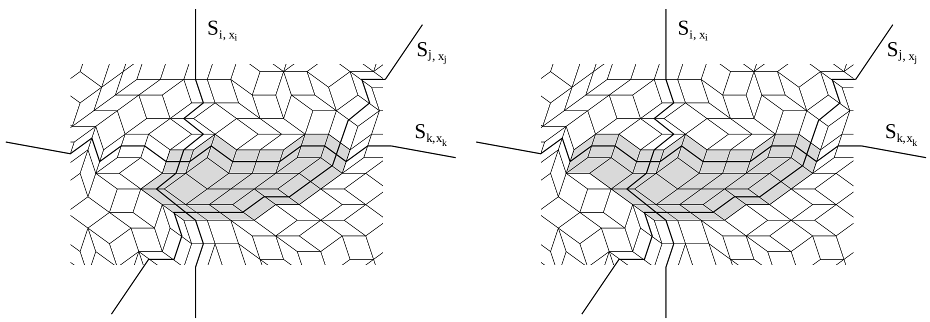

Fig. 6. A de Bruijn triangle $F_k(\boldsymbol{x}, t_{ij})$ (the shaded unit faces, left) and its closure, the de Bruijn cone $F_k^*(\boldsymbol{x}, t_{ij})$ (right). Recall that one has always $(\boldsymbol{x}, t_{ij}) = S_{i,x_i} \cap S_{j,x_j}$.

Let us now show that $(\boldsymbol{x}, t_{ij})$ can be translated by performing flips over $F_k^*(\boldsymbol{x}, t_{ij})$:

Lemma 1. *If $F_k^*(\boldsymbol{x}, t_{ij})$ is finite, then one can translate $(\boldsymbol{x}, t_{ij})$ by $\boldsymbol{e}_k$ by performing card$(F_k^*(\boldsymbol{x}, t_{ij}) \backslash S_{k,x_k})$ flips over $F_k^*(\boldsymbol{x}, t_{ij})$.*

Proof. Def. 8 yields, for any unit faces $(\boldsymbol{y}, t)$ and $(\boldsymbol{y}', t')$:

$$(\boldsymbol{y}, t) \in F_k^*(\boldsymbol{y}', t') \;\Rightarrow\; F_k^*(\boldsymbol{y}, t) \subset F_k^*(\boldsymbol{y}', t').$$

This naturally leads to define the following partial order over $F_k^*(\boldsymbol{x}, t_{ij})$:

$$\forall (\boldsymbol{y}, t), (\boldsymbol{y}', t') \in F_k^*(\boldsymbol{x}, t_{ij}), \quad (\boldsymbol{y}, t) \preceq (\boldsymbol{y}', t') \;\Leftrightarrow\; F_k^*(\boldsymbol{y}, t) \subset F_k^*(\boldsymbol{y}', t').$$

Let us now consider a unit face $(\boldsymbol{y}, t) \in F_k^*(\boldsymbol{x}, t_{ij}) \backslash S_{k,x_k}$ which is minimal for this order. It is not hard to check that $F_k^*(\boldsymbol{y}, t)$ is a set of three unit faces on which a flip can be performed (see, for example, Fig. 6, right). By performing this flip, $(\boldsymbol{y}, t)$ is translated by $\boldsymbol{e}_k$, so that the obtained face does no more belongs to $F_k^*(\boldsymbol{x}, t_{ij})$, which thus decreased (Fig. 7, left). This can be inductively repeated, up to translate by $\boldsymbol{e}_k$ the unit face which was originally maximal in $F_k^*(\boldsymbol{x}, t_{ij})$, that is, $(\boldsymbol{x}, t_{ij})$ itself (Fig. 7, right). Since there is one flip performed for each translated unit face, there is a total of card$(F_k^*(\boldsymbol{x}, t_{ij}) \backslash S_{k,x_k})$ flips performed. $\square$

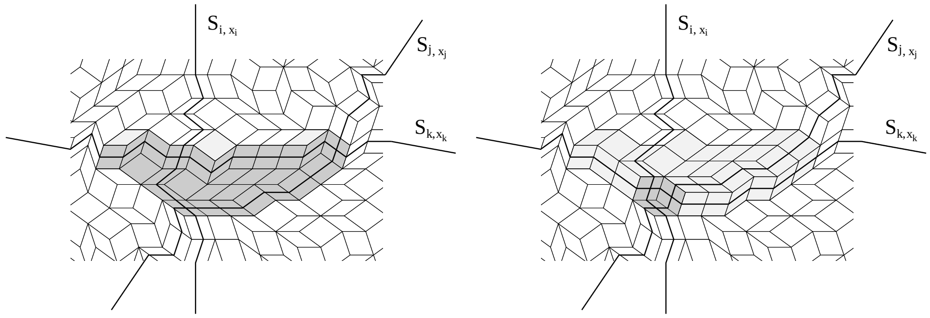

Fig. 7. Three flips have been performed on the minimal elements of the de Bruijn cone of Fig. 6 (left). This can be repeated, reducing the de Bruijn cone up to only three unit faces (right), on which performing a flip will translate the unit face $(\boldsymbol{x}, t_{ij})$ by $\boldsymbol{e}_k$.

Although the definition of de Bruijn cones by transitive closure suffices to prove the previous lemma, the following stronger property actually holds:

Lemma 2. *One has* $F_k^*(\boldsymbol{x}, t_{ij}) = F_k^2(\boldsymbol{x}, t_{ij})$.

Proof. Let $(\boldsymbol{y}, t) \in F_k^2(\boldsymbol{x}, t_{ij})$. If $F_k(\boldsymbol{y}, t)$ is not included in $F_k^2(\boldsymbol{x}, t_{ij})$, then a case study (relying on the fact that two de Bruijn sections intersect at most once) shows that one of the two de Bruijn sections containing $(\boldsymbol{y}, t)$, say $S_{k',y_{k'}}$, necessarily intersects $F_k(\boldsymbol{x}, t_{ij})$. Let thus $(\boldsymbol{y}', t') \in S_{k',y_{k'}} \cap F_k(\boldsymbol{x}, t_{ij})$. One has $F_k(\boldsymbol{y}, t) \subset F_k(\boldsymbol{y}', t')$, and $(\boldsymbol{y}', t') \in F_k(\boldsymbol{x}, t_{ij})$ yields $F_k(\boldsymbol{y}', t') \subset F_k^2(\boldsymbol{x}, t_{ij})$. Hence, $F_k(\boldsymbol{y}, t) \subset F_k^2(\boldsymbol{x}, t_{ij})$. Since this holds for any $(\boldsymbol{y}, t) \in F_k^2(\boldsymbol{x}, t_{ij})$, this proves $F_k^3(\boldsymbol{x}, t_{ij}) \subset F_k^2(\boldsymbol{x}, t_{ij})$. The result follows. □

We are now in a position to prove that one can choose k_0 such that $F_{k_0}^*(\boldsymbol{x}, t_{ij})$ is finite and $(\boldsymbol{x}, t_{ij})$ should be translated by $\boldsymbol{e}_{k_0}$ (the condition $k_0 \in D$ below). Lem. 1 then yields that $(\boldsymbol{x}, t_{ij})$ can be effectively translated by $\boldsymbol{e}_{k_0}$.

Lemma 3. *Let* $(\boldsymbol{x}', t_{ij}) \in \mathcal{S}'$ *and* $(\boldsymbol{x}, t_{ij}) \in \mathcal{S}$ *such that* $\pi_{ij}(\boldsymbol{x}', t_{ij}) = \pi_{ij}(\boldsymbol{x}, t_{ij})$. *If* $D = \{k \mid x'_k > x_k\} \neq \emptyset$, *then there is* $k_0 \in D$ *such that* $F_{k_0}^*(\boldsymbol{x}, t_{ij})$ *is finite.*

Proof. We first prove that $F_k(\boldsymbol{x}, t_{ij})$ is finite for any $k \in D$, and then that there is $k_0 \in D$ such that $F_{k_0}^*(\boldsymbol{x}, t_{ij}) = F_{k_0}^2(\boldsymbol{x}, t_{ij})$ is finite.
Let $k \in D$. Note that $F_k(\boldsymbol{x}, t_{ij})$ is finite iff both S_{i,x_i} and S_{j,x_j} intersect S_{k,x_k}. Suppose that S_{i,x_i} does not intersect S_{k,x_k}. Thus, $S_{i,x_i} \subset T_{k,x_k}^-$. Then, since the shadows of $\mathcal{S}'$ are included in the shadows of $\mathcal{S}$, there is $(\boldsymbol{z}, t) \in \mathcal{S}$ such that $\pi_{ik}(\boldsymbol{x}') \in \pi_{ik}(\boldsymbol{z}, t)$. This yields $z_i = x'_i = x_i$ and $z_k = x'_k > x_k$. In particular, $\boldsymbol{z} \in S_{i,x_i} \cap T_{k,x_k}^+$. Since this contradicts $S_{i,x_i} \subset T_{k,x_k}^-$, we deduce that S_{i,x_i} intersects S_{k,x_k}. Similarly, S_{j,x_j} intersects S_{k,x_k}. The first result is proven.
Let us now choose $k_0 \in D$ being minimal in D for the following partial order:

$$n \preceq m \;\Leftrightarrow T_{m,x_m}^+ \subset T_{n,x_n}^+.$$

In other words, k_0 is chosen such that there is no section S_{k,x_k} separating $(\boldsymbol{x}, t_{ij})$ from $S_{k_0,x_{k_0}}$, that is, such that $(\boldsymbol{x}, t_{ij}) \in T^-_{k,x_k}$ and $S_{k_0,x_{k_0}} \subset T^+_{k,x_k}$. This yields that a unit face $(\boldsymbol{y}, t)$ of $F_{k_0}(\boldsymbol{x}, t_{ij})$ belongs to two de Bruijn sections which both intersect $S_{k_0,x_{k_0}}$. Thus, $F_k(\boldsymbol{y}, t)$ is finite. The second result follows. □

Note that the previous lemma only proves that *there is* $k_0 \in D$ such that one can (and should) translate $(\boldsymbol{x}, t_{ij})$ by $\boldsymbol{e}_{k_0}$. Actually, one can easily check that, for $d = 3$, *any* $k \in D$ is convenient, whereas this is no more true for $d > 3$. Without going into details, let us just say that it is strongly connected with the fact that the set of $d \to 2$ rhombus tilings of a polygon forms a *distributive* lattice for $d = 3$, whereas not for $d > 3$ (see [5, 11]).

So, following the outline given at the beginning of this section, we can now, by performing flips, translate $(\boldsymbol{x}, t_{ij})$ by some $\boldsymbol{e}_{k_0}$ such that $x'_{k_0} > x_{k_0}$. We can repeat this up to have $x'_k \leq x_k$ for any k. The way we can translate by $-\boldsymbol{e}_{k_0}$ a unit face $(\boldsymbol{x}, t_{ij})$ such that $x'_{k_0} < x_{k_0}$ is similar. So, we are able to move $(\boldsymbol{x}, t_{ij})$ to $(\boldsymbol{x}', t_{ij})$. The end of the proof relies on the following lemma:

Lemma 4. *Let $(\boldsymbol{x}', t_{ij}) \in \mathcal{S}'$ and $(\boldsymbol{x}, t_{ij}) \in \mathcal{S}$ such that $\pi_{ij}(\boldsymbol{x}', t_{ij}) = \pi_{ij}(\boldsymbol{x}, t_{ij})$. If $x'_k > x_k$, then $F^*_k(\boldsymbol{x}, t_{ij}) \cap \mathcal{S}' = \emptyset$.*

Proof. (sketch) Writing down a detailed proof is rather technical and obfuscating, but the underlying geometrical idea is quite easy. Indeed, $x'_k > x_k$ yields $(\boldsymbol{x}, t_{ij}) \in T^-_{k,x_k}$ and $(\boldsymbol{x}', t_{ij}) \in T'^+_{k,x_k}$, as depicted on Fig. 8. So, suppose that there is a unit face $(\boldsymbol{y}, t) \in F_k(\boldsymbol{x}, t_{ij}) \cap \mathcal{S}'$. Such a face thus should have the same position, in $\mathcal{S}$ and $\mathcal{S}'$, relatively to any de Bruijn section. For example, if $(\boldsymbol{y}, t)$ belongs to $T^+_{i,x_i} \cap T^+_{j,x_j} \cap T^-_{k,x_k}$ in $\mathcal{S}$ (as in the case of Fig. 8, left), then it should belongs to $T'^+_{i,x_i} \cap T'^+_{j,x_j} \cap T'^-_{k,x_k}$ in $\mathcal{S}'$. However, this last set turns out to be empty (see Fig. 8, right). Thus, $F_k(\boldsymbol{x}, t_{ij}) \cap \mathcal{S}' = \emptyset$. Suppose now that $(\boldsymbol{y}, t) \in F^2_k(\boldsymbol{x}, t_{ij}) \cap \mathcal{S}'$. There is $(\boldsymbol{z}, t_z) \in F_k(\boldsymbol{x}, t_{ij})$ such that $(\boldsymbol{y}, t) \in F_k(\boldsymbol{z}, t_z)$. We prove $F_k(\boldsymbol{z}, t_z) \cap \mathcal{S}' = \emptyset$ as above, with $(\boldsymbol{z}, t_z)$ instead of $(\boldsymbol{x}, t_{ij})$. □

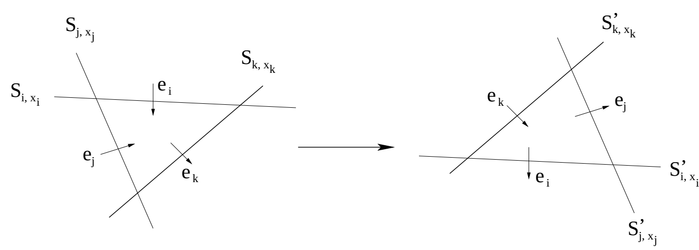

Fig. 8. If $(\boldsymbol{x}, t_{ij})$ must cross the section S_{k,x_k} to be transformed to $(\boldsymbol{x}', t_{ij})$, then any unit face inside the triangle $T^+_{i,x_i} \cap T^+_{j,x_j} \cap T^-_{k,x_k}$ must also cross one of the sections S_{i,x_i}, S_{j,x_j} or S_{k,x_k}, hence is moved.

This lemma ensures that, once a unit face of $\mathcal{S}'$ is obtained, it is no more moved. We thus can get unit faces of $\mathcal{S}'$ over growing balls, and Th. 1 follows. We end the paper by summing up the whole proof by the following pseudo-algorithm:

for r=0 **to** ∞
  while $\mathcal{S}_{B(\mathbf{0},r)} \neq \mathcal{S}'_{B(\mathbf{0},r)}$
    choose $(\boldsymbol{x}, t_{ij})$ in $\mathcal{S}_{B(\mathbf{0},r)} \backslash \mathcal{S}'_{B(\mathbf{0},r)}$
    $(\boldsymbol{x}', t_{ij}) \leftarrow S'_{i,x_i} \cap S'_{j,x_j}$ $\quad (\pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S}))$
    while $\boldsymbol{x} \neq \boldsymbol{x}'$
      choose k s.t. $x_k \neq x'_k$ and $F^*_k(\boldsymbol{x}, t_{ij})$ is finite $\quad$ (Lem. 3)
      $x_k \leftarrow x_k \pm 1$ by performing flips over $F^*_k(\boldsymbol{x}, t_{ij})$ $\quad$ (Lem. 1)
    endwhile
  endwhile $\quad$ (Lem. 4)
endfor

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