Local Rules for Planar Computable Tilings

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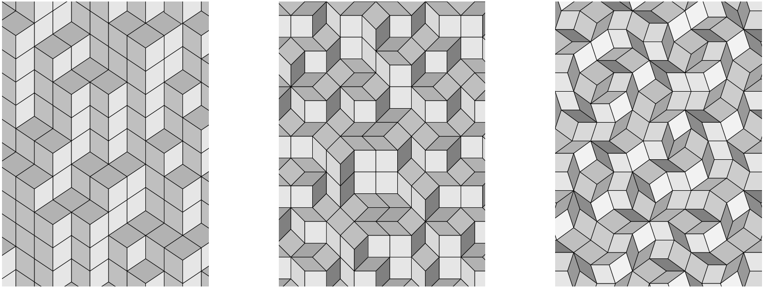
Bastia, September 20th, 2012

1. The Problem
2. The Tool
3. The Proof

Canonical $n \to d$ tilings



n pairwise non-colinear vectors of $\mathbb{R}^d \rightsquigarrow \binom{n}{d}$ tiles $\rightsquigarrow$ tiling of $\mathbb{R}^d$.

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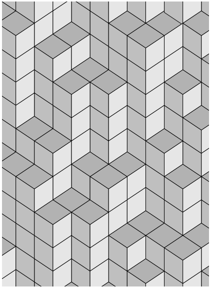
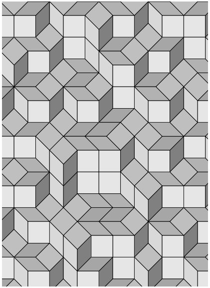
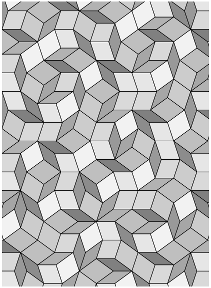
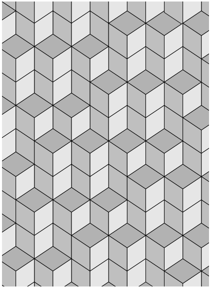


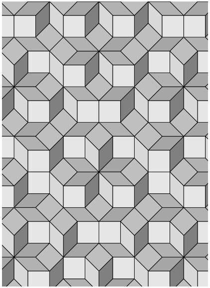


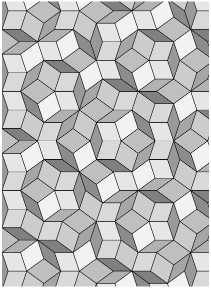


Lift: homeomorphism which maps tiles on d-faces of unit n-cubes.

Planarity

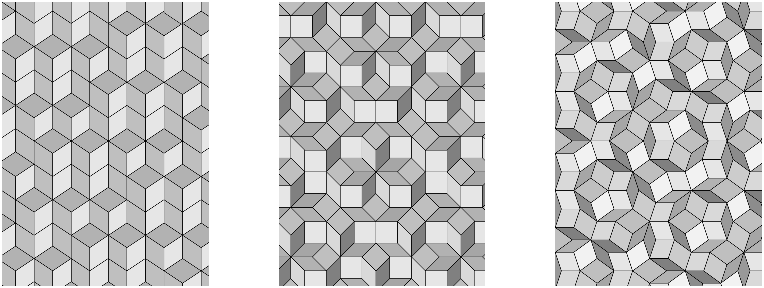






Planar: lift in $E + [0, t]^n$, where E is the slope and t the thickness.

Planarity



Perfect: planar with the minimal thickness $t = 1$.

Local rules

Definition

A slope E admits *local rules* if there is a finite set of patterns s.t. any canonical tiling without these patterns is planar with slope E.

Local rules are said to be

- *strong* if the tilings satisfying them are perfect;
- *natural* if the perfect tilings satisfy them;
- *weak* otherwise (the thickness is thus just bounded).

Local rules can also be *decorated*, with a tile playing different roles.

The Computability barrier

Computable number: within precision ε by a Turing machine.

Proposition

If a slope admits local rules (of any type), then it is computable.

Proof sketch:

- let t be the thickness (assumed to be known if rules are weak);
- let ε be the wanted precision;
- form a pattern covering a ball of radius $r \geq t/\varepsilon$;
- take d free vectors of length r in this pattern;
- they span a space at distance less than t/r from the slope.

Subshifts

Definition

A nD subshift over an alphabet $\mathcal{A}$ is a translation invariant closed subset of $\mathcal{A}^{\mathbb{Z}^n}$, where $d(u,v) = 2^{-sup_{k\geq 0} u_{|[-k,k]^n} = v_{|[-k,k]^n}}$.

A subshift is said to be

- *effective* if its forbidden patterns are recursively enumerables;
- of *finite type* if defined by finitely many *forbidden* patterns;
- *sofic* if it is a letter-to-letter image of a subshift of finite type.

Projective subaction

Theorem (Aubrun-Sablik'10, Durand-Romashchenko-Shen'10)

If $X \subset \mathcal{A}^{\mathbb{Z}}$ is effective, then $\{y \in \mathcal{A}^{\mathbb{Z}^2},\ \forall j,\ y_j = y_0 \in X\}$ is sofic.

Proof sketch:

- Take the Robinson tiles;
- Add a layer which allows to vertically repeat any line in $\mathcal{A}^{\mathbb{Z}}$;
- Enumerate the forbidden patterns in the Robinson boards;
- Check that no forbidden pattern appears on the repeated line.

Quasi-Sturmian words

Sturmian word $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ:

$$s_{\rho,\alpha}(n) = 0 \Leftrightarrow (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

Distance over words in $\{0,1\}^{\mathbb{Z}}$:

$$d(u,v) := \sup_{p \leq q} \left| |u(p)\cdots u(q)|_0 - |v(p)\cdots v(q)|_0 \right|.$$

Proposition (Morse-Hedlund, 1940)

Two Sturmian words with the same slope are at distance at most 1.

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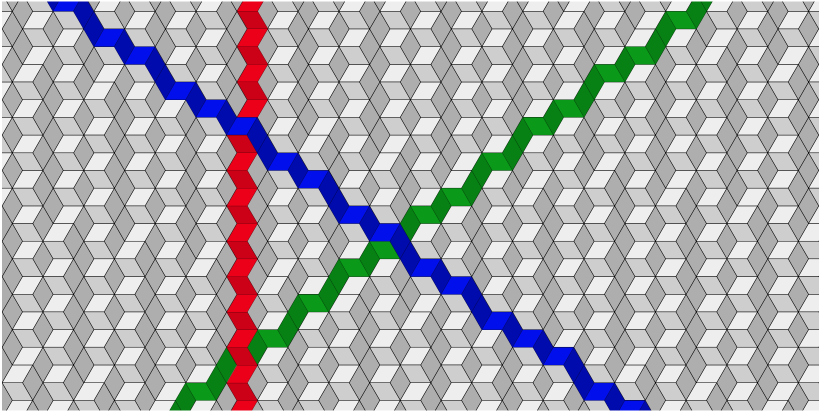
Definition

Quasi-Sturmian word: at distance at most 1 from a Sturmian word.

Stripes of perfect $3 \to 2$ tilings



Perfect $3 \to 2$ tiling: intertwined stripes encoding Sturmian words.

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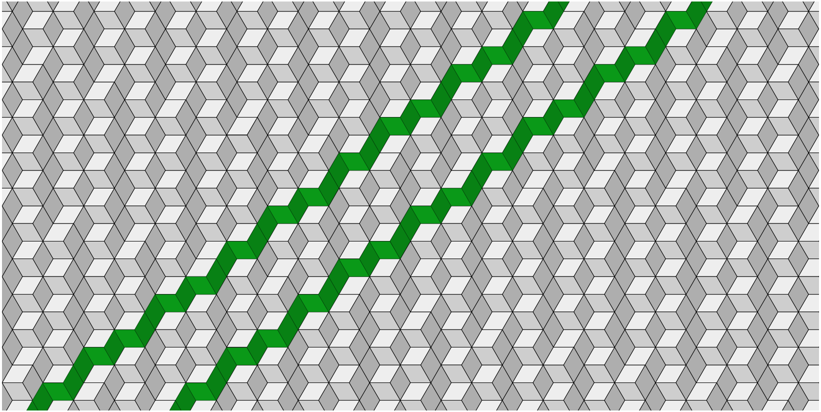


Parallel stripes encode quasi-Sturmian words with the same slope.

A sofic subshift

Proposition

The Sturmian words of comput. slope α form an effective subshift.

Proof sketch: compute patterns up to have $n+1$ factors of size n.

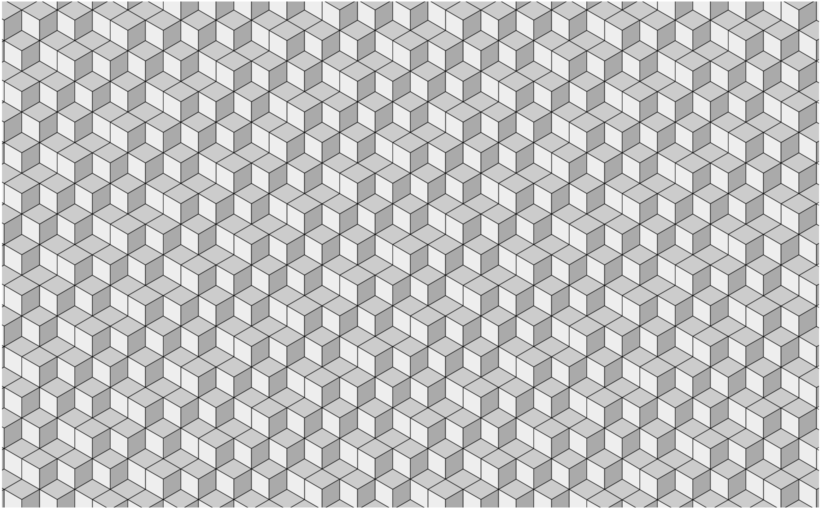
The following 2D subshift is thus sofic:

$$Z_\alpha = \{y \in \{0,1\}^{\mathbb{Z}^2},\ \forall j,\ y_j = y_0 = s_{\alpha,0}\},$$

and the *quasi-Sturmian subshift* also (use a "carry" layer):

$$Z'_\alpha = \{y \in \{0,1\}^{\mathbb{Z}^2},\ \forall j,\ d(y_j, s_{\alpha,0}) \leq 1\}.$$

From perfect 3 → 2 tilings to quasi-Sturmian subshifts



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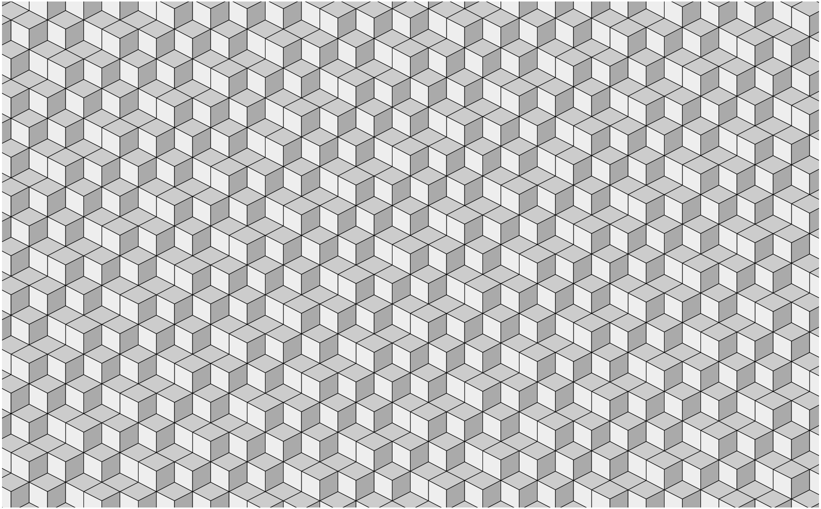

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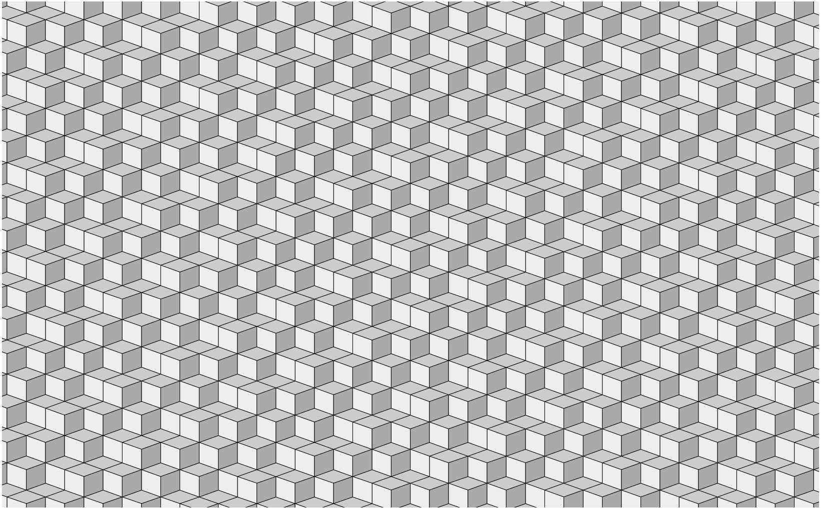

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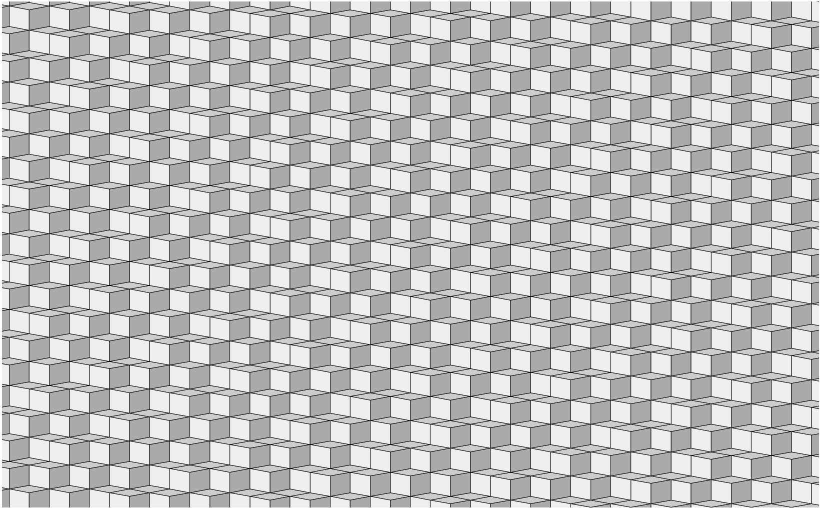

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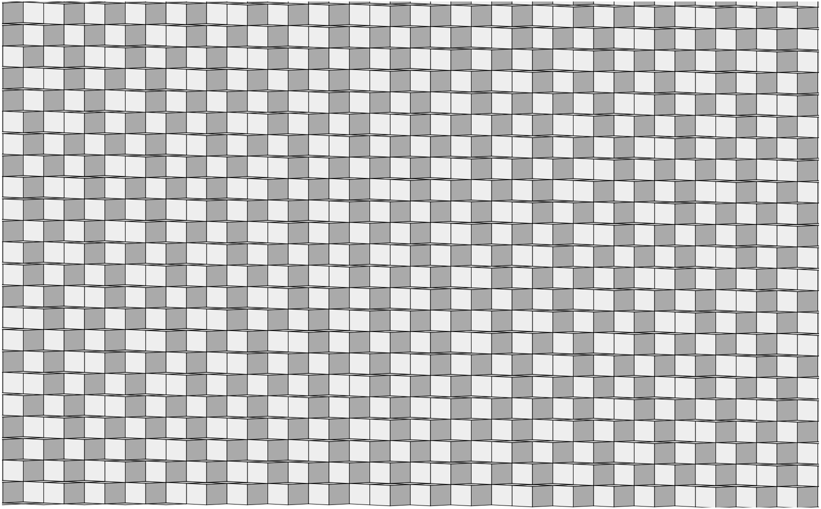

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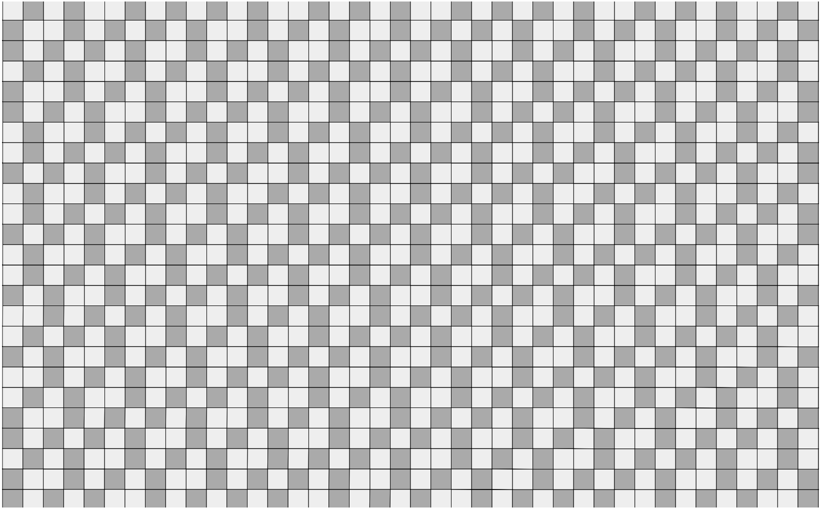

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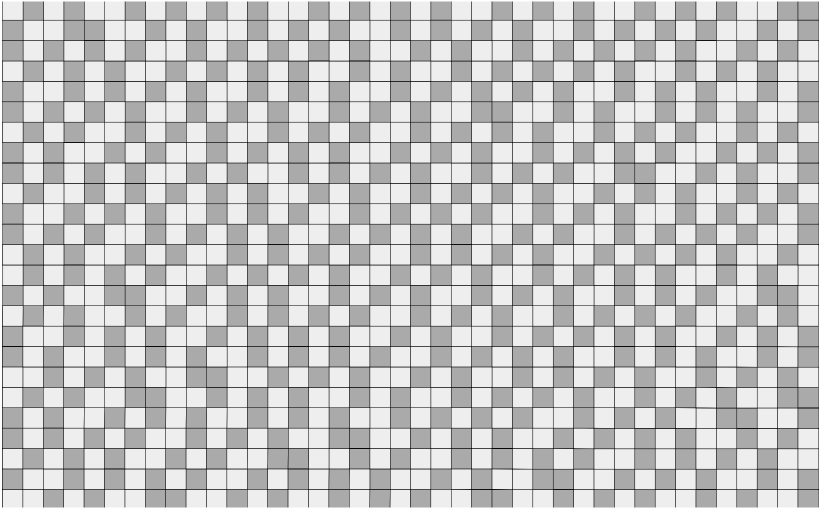

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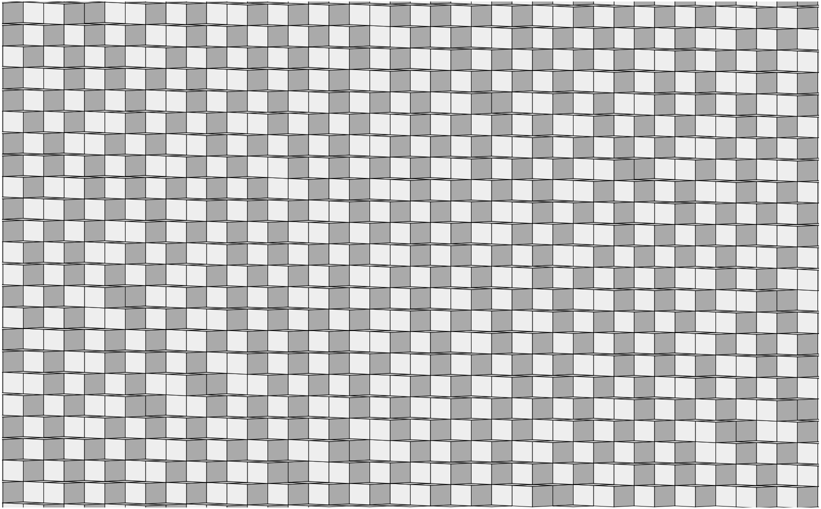

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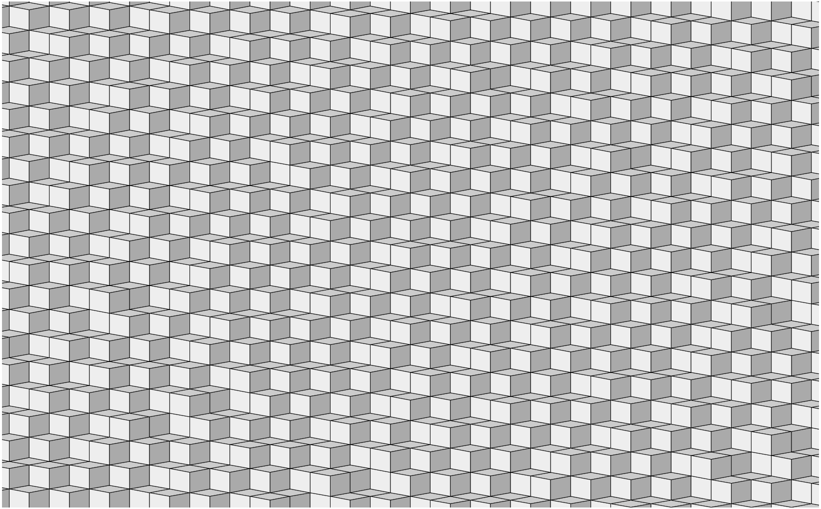

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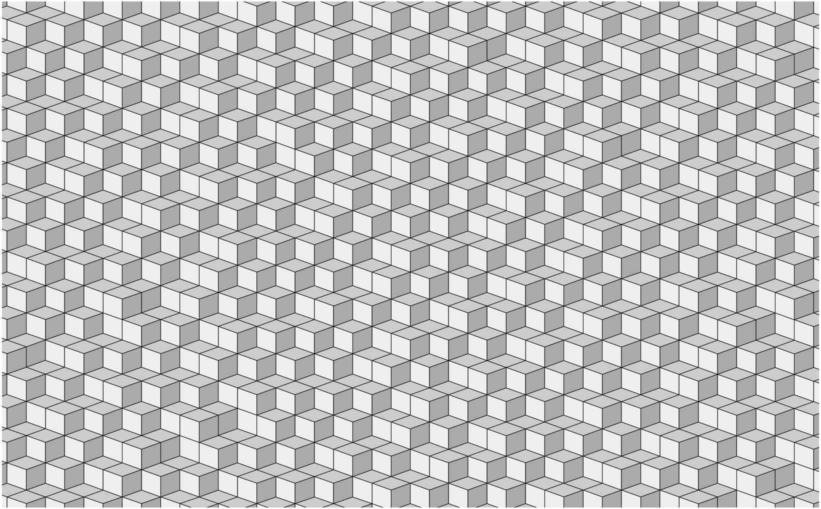

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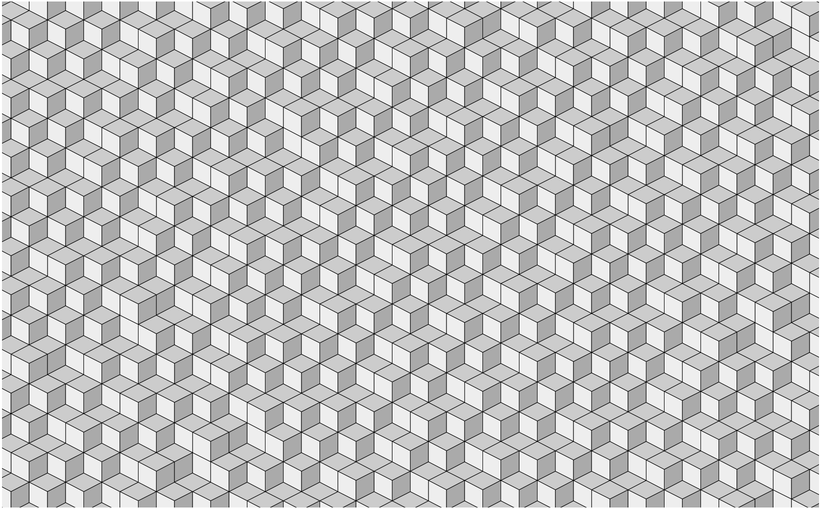

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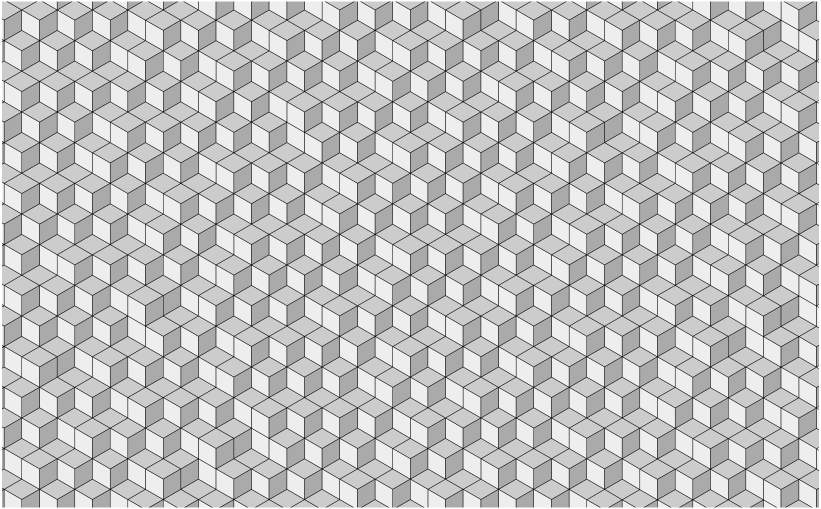
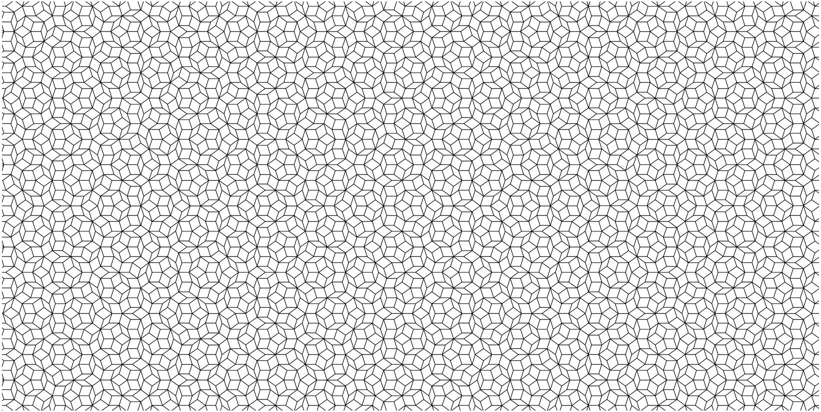


Higher (co)dimensions



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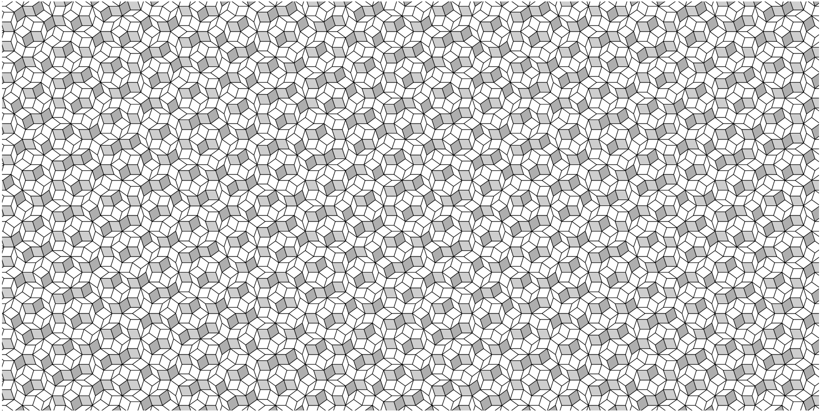

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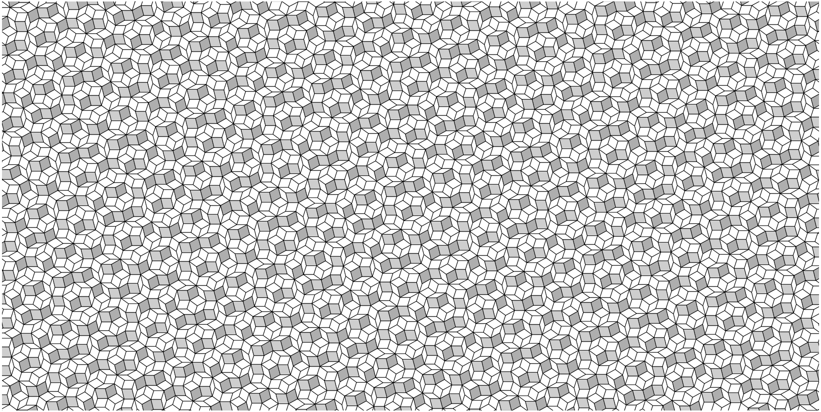

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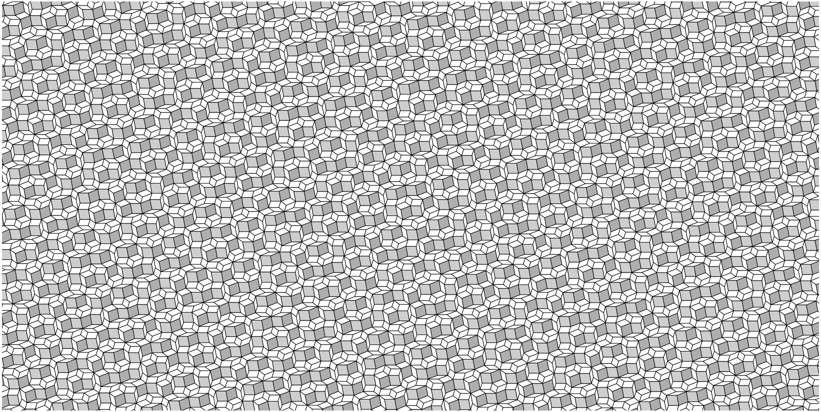

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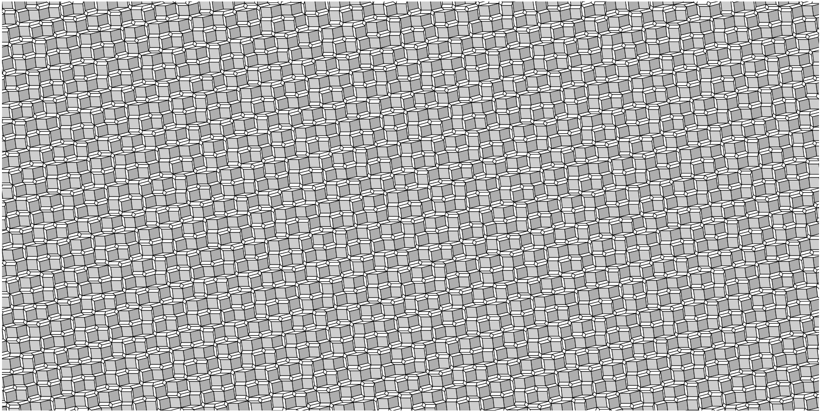

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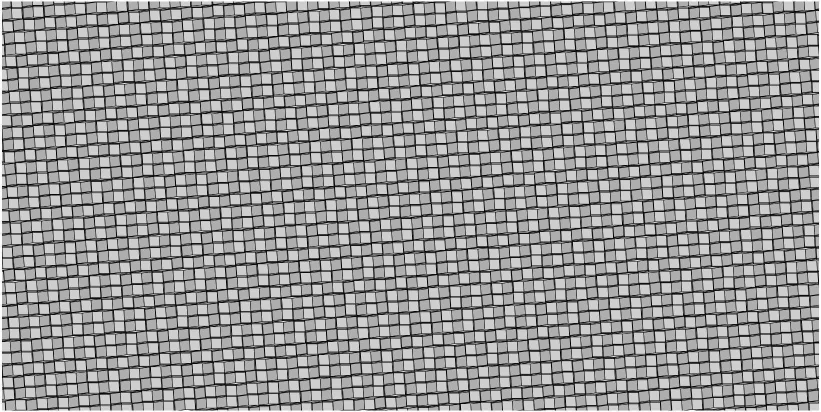

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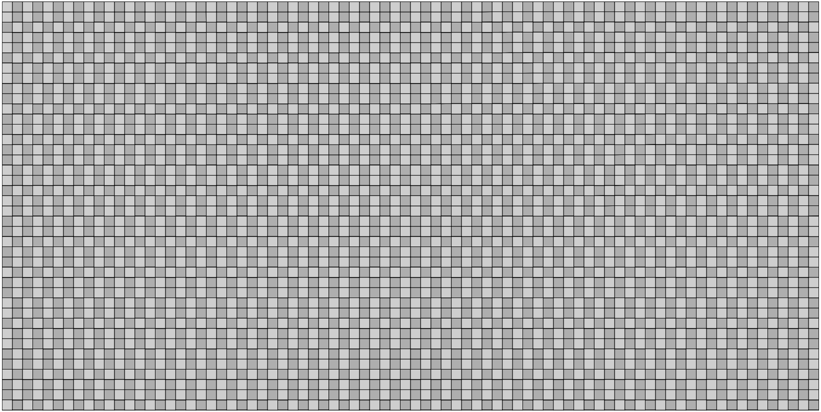

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Conclusion/Perspectives

Decorated local rules

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Natural undecorated local rules

Only algebraic slopes can have natural undecorated rules (Le '95).
Even fewer slopes can have strong undecorated rules (Levitov '88).
There is yet no complete characterization of these slopes.