The Ammann-Beenker Tilings Revisited

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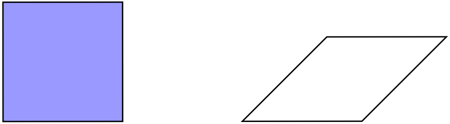
Arrowed tiles (Beenker, 1982)



Square and rhombus tiles form so-called *octagonal tilings*.

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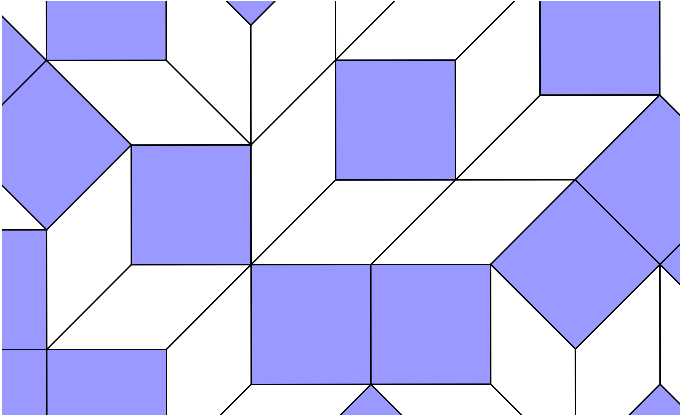

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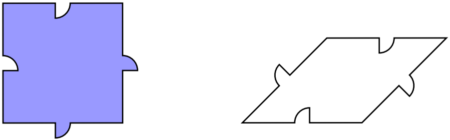

Which tilings do form *arrowed* square and rhombus tiles?

Arrowed tilings

Theorem

The arrowed tilings digitize the planes $(1, t, 1, 1, 2/t, 1)$, $t \in \widetilde{\mathbb{R}}$.

Corollary

The Ammann-Beenker tilings maximize the ratio rhombi/squares.

Arrowed tilings

Theorem

The arrowed tilings digitize the planes $(1, t, 1, 1, 2/t, 1)$*,* $t \in \widetilde{\mathbb{R}}$*.*

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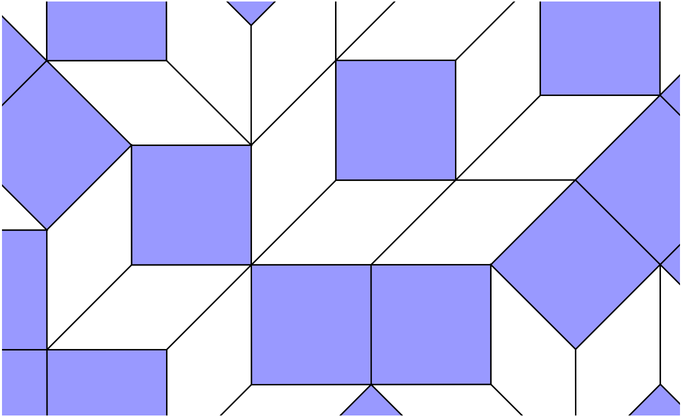
The Ammann-Beenker tilings maximize the ratio rhombi/squares.

Underlying idea

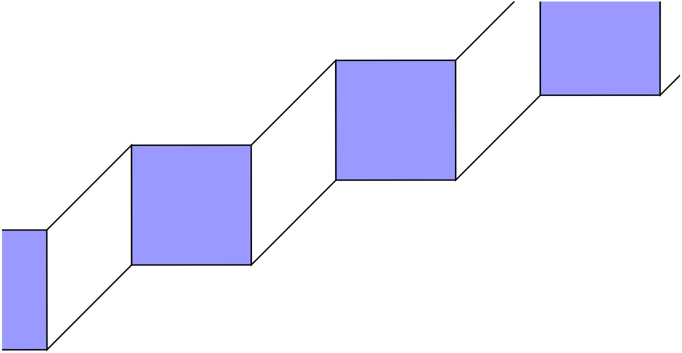
- rhombi = aluminium and squares = manganese (for example);
- Ammann-Beenker tiling = quasicrystal $\mathrm{Al}_{\sqrt{2}}\mathrm{Mn}_1$;
- $\mathrm{Al}_7\mathrm{Mn}_5$, $\mathrm{Al}_{41}\mathrm{Mn}_{29}$, $\mathrm{Al}_{239}\mathrm{Mn}_{169}$ = quasicrystal approximants.

Alternating rhombi



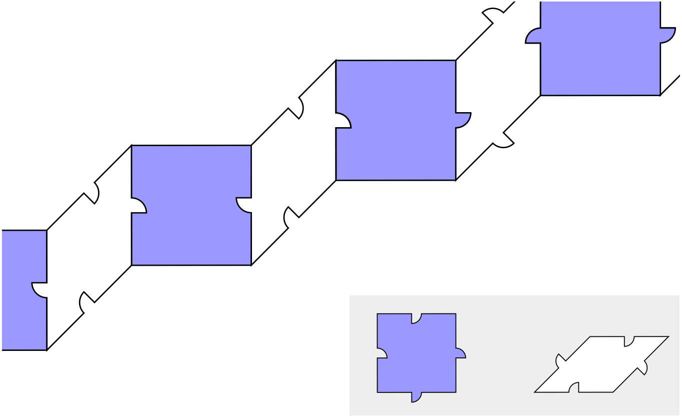
Consider an octagonal tiling. Assume it can be arrowed.

Alternating rhombi



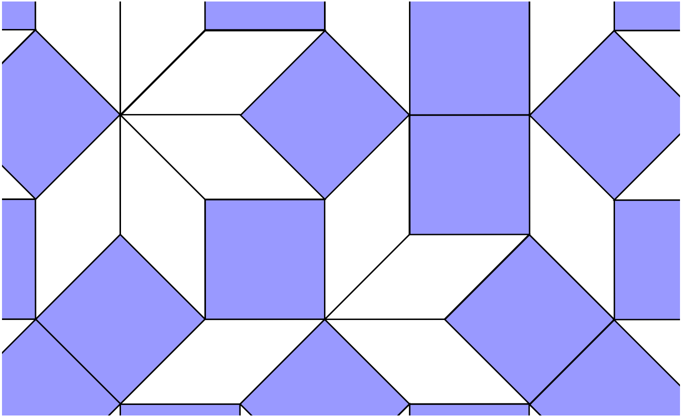
Consider a “stripe” of tiles (also called *Conway worms*).

Alternating rhombi



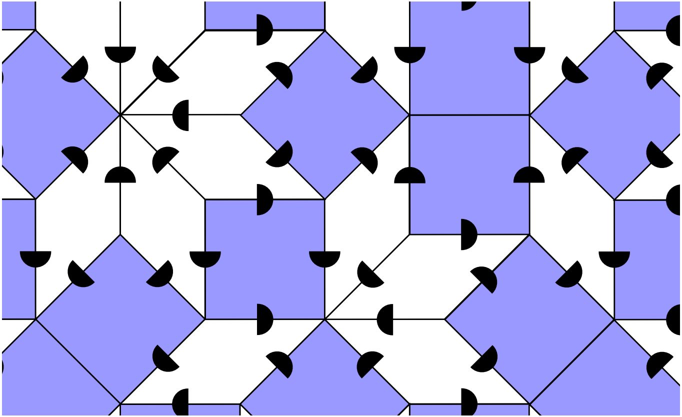
If rhombi do not alternate orientation, then tiles cannot be arrowed.

Alternating rhombi



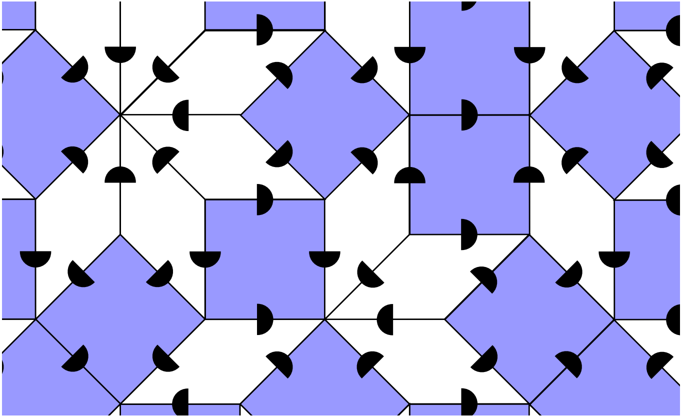
Conversely, consider an octagonal tiling where rhombi alternate.

Alternating rhombi



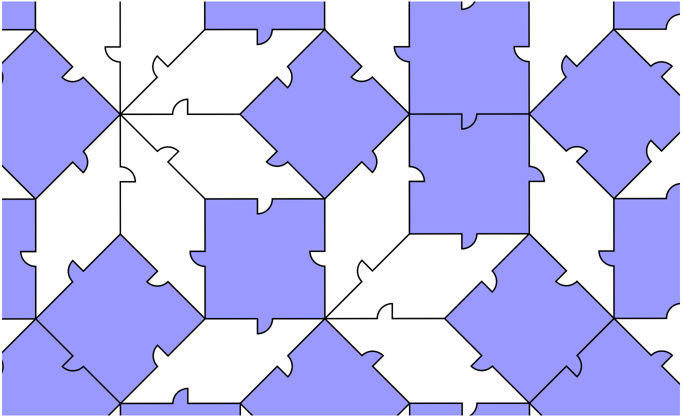
Endow rhombi with arrows pointing towards the acute angles.

Alternating rhombi



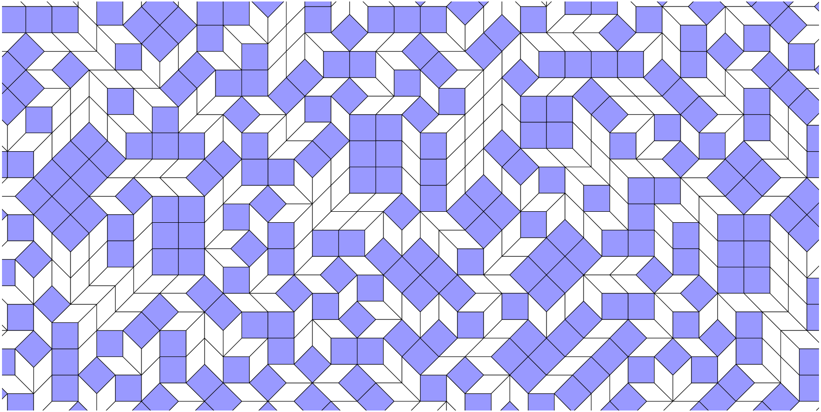
Endow squares with parallel arrows being equally oriented.

Alternating rhombi



Gluing each arrow with the tile on its left yields arrowed tiles.

Planar octagonal tilings



Lift: homeomorphism which maps rhombi on 2-faces of unit 4-cubes.

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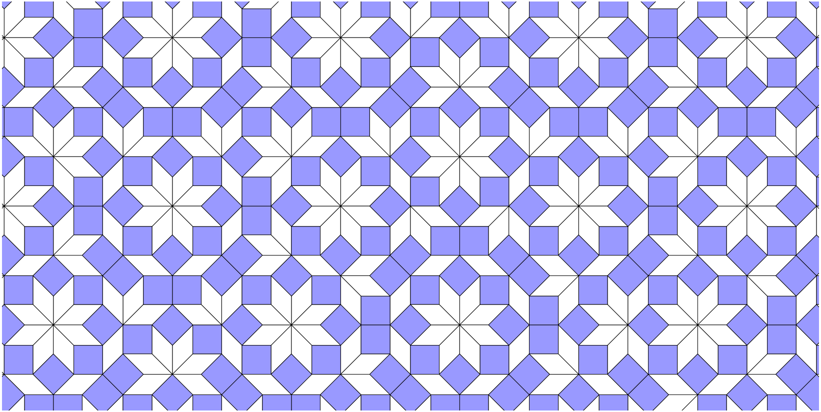
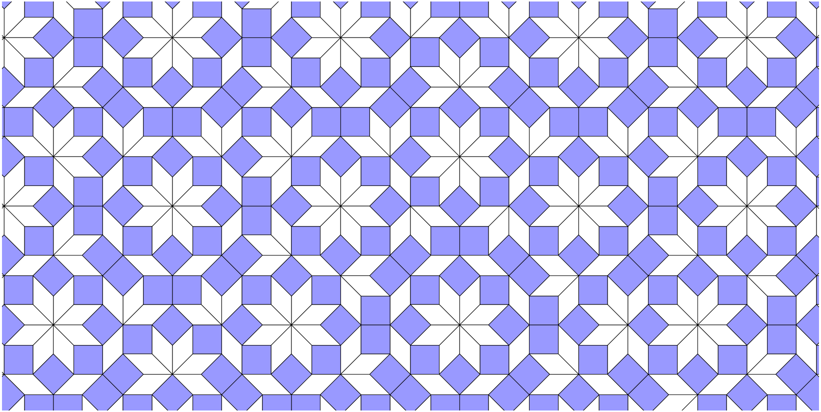


Planar: lift in $E + [0, t]^4$, where E is the slope and t the thickness.

Shadows and subperiods



Shadow: orthogonal projection of the lift along a basis vector.

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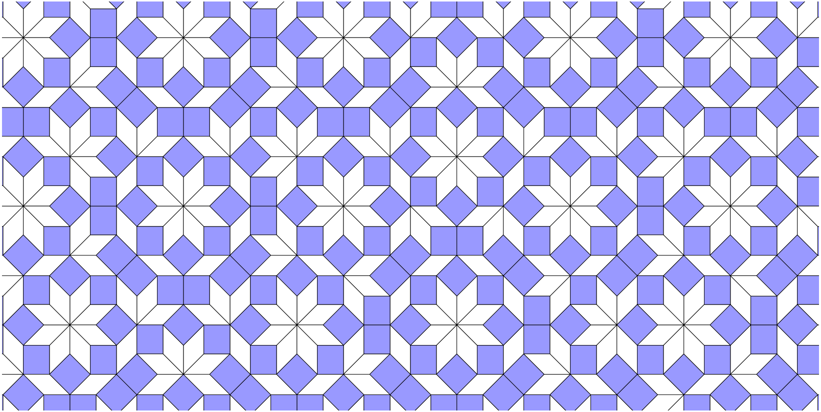

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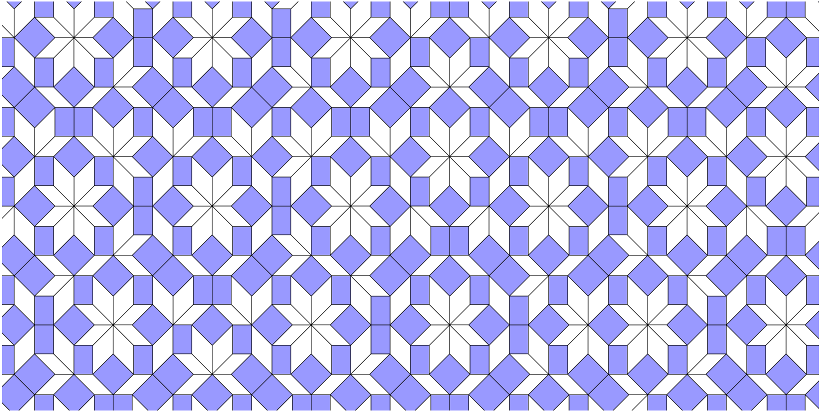

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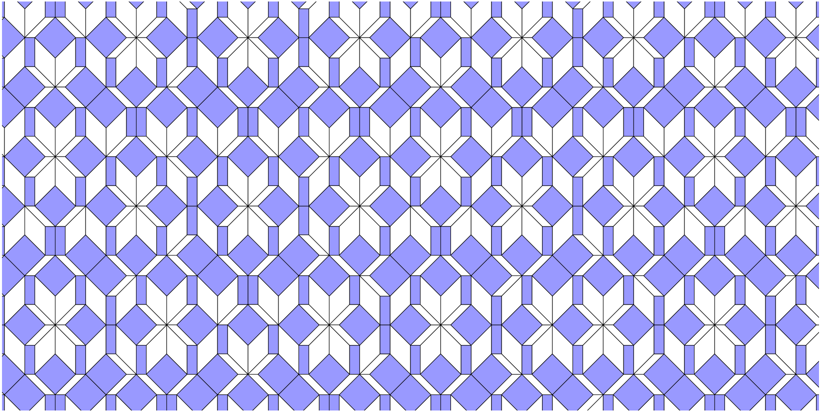

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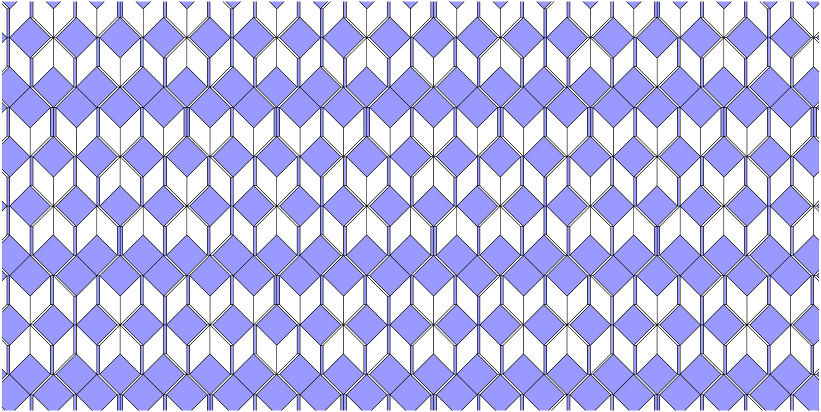

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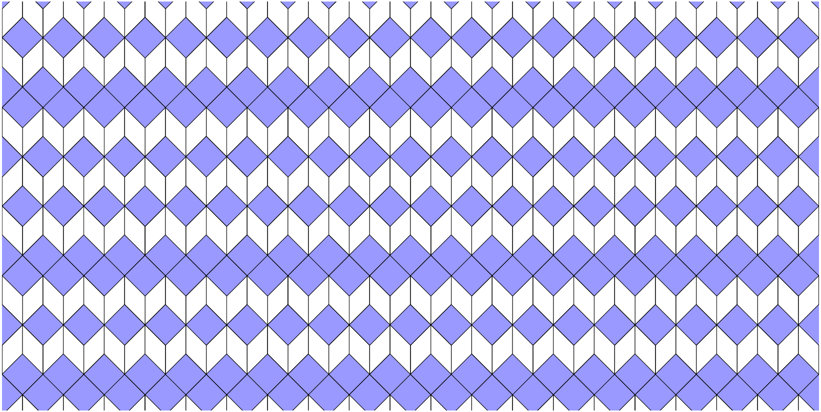

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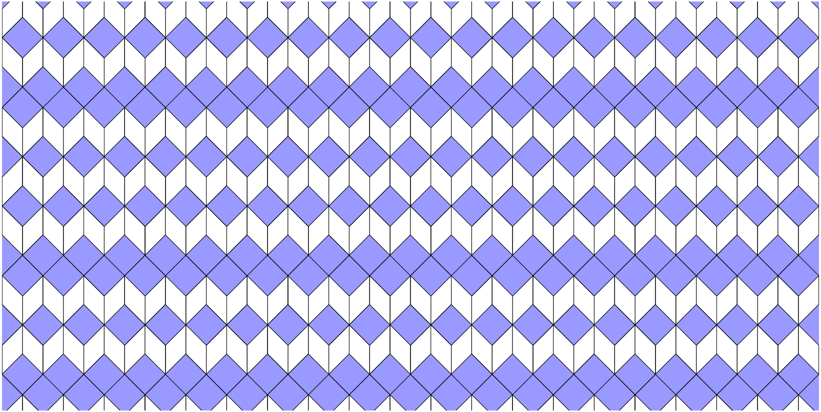

Subperiod: shadow period. Rhombus alternation forces simple ones.

Plücker coordinates

Definition (Plücker, 1865)

$E = \mathbb{R}\vec{u} + \mathbb{R}\vec{v} \subset \mathbb{R}^4$ has coord. $(G_{ij})_{ij} = (u_i v_j - u_j v_i)_{ij} \in \mathbb{P}^5(\mathbb{R})$.

Proposition

The tile proportions of planar tilings are given by the Plücker coord.

Example

The Ammann-Beenker tilings are the planar tilings of thickness 1 and slope $(1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$; they have $\sqrt{2}$ rhombi for 1 square.

Linear and quadratic relations

Proposition

Subperiods of planar tilings yield linear relations on Plücker coord.

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Subperiods forced by arrowed tiles yield: $G_{12} = G_{14} = G_{23} = G_{34}$.

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Lemma

The planar arrowed tilings have slope $(1, t, 1, 1, 2/t, 1)$, $t \in \widetilde{\mathbb{R}}$.

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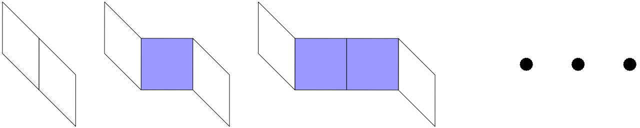
Remark

Arrowing tiles amouts to forbidding arbitrarily big patterns.



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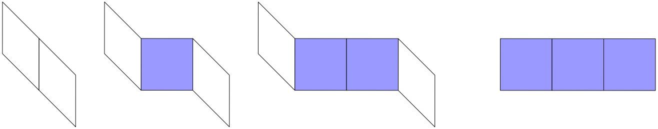

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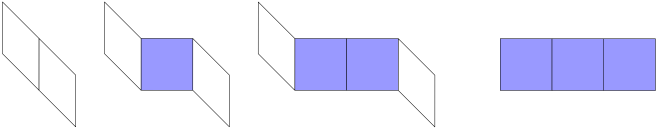

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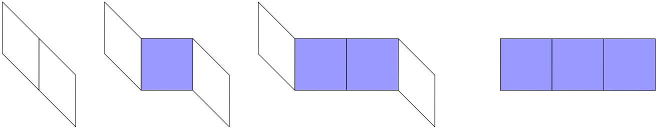

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Thus not $G_{13} = G_{24}$, i.e., equiprobable orientations of squares.

Beyond Ammann-Beenker tilings

Here, subperiods characterize a family of slopes and the planarity.

When they characterize finitely many slopes, the planarity follows.

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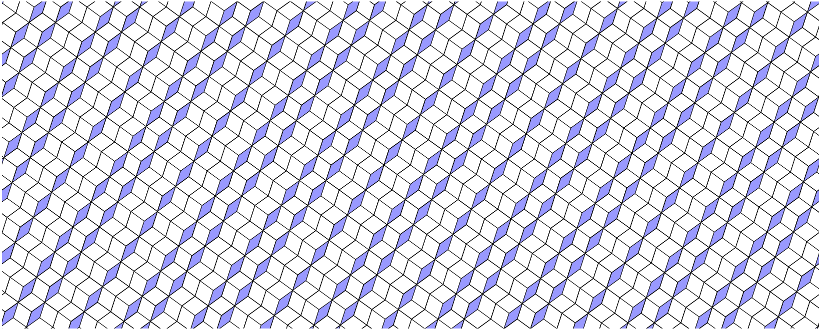

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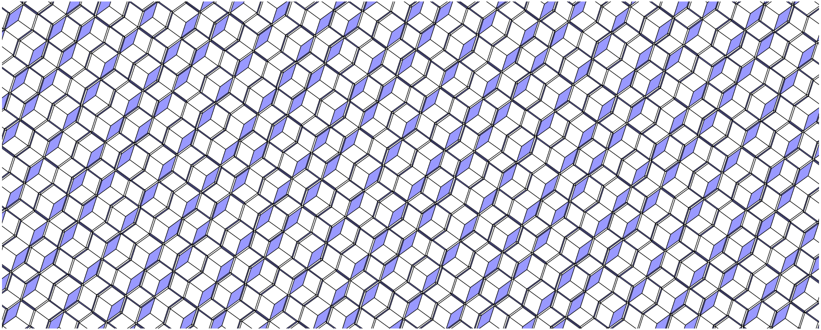

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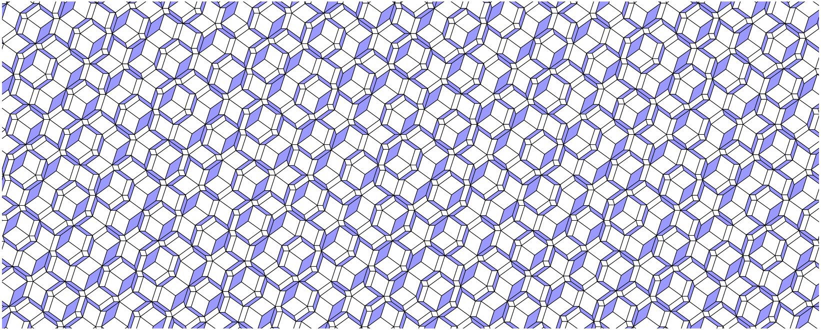

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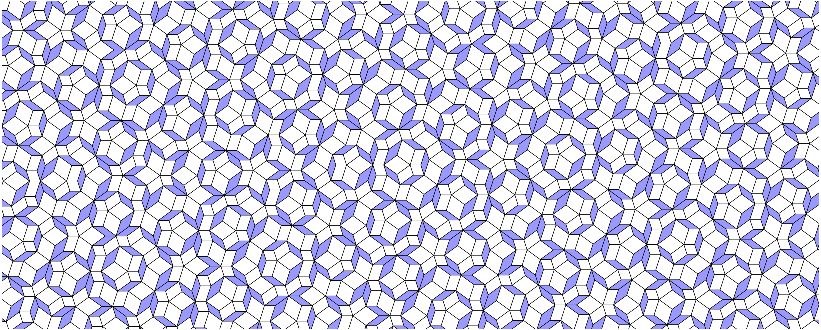

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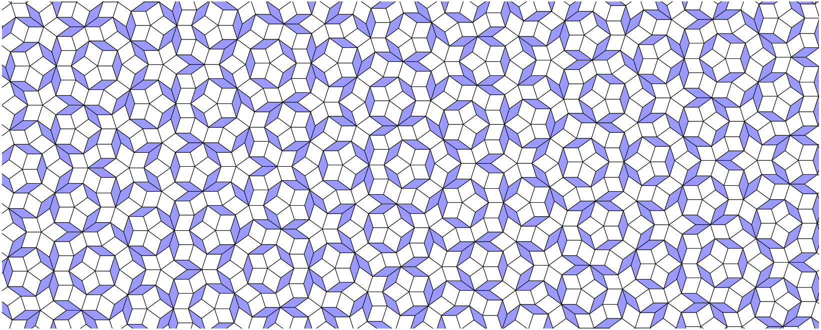

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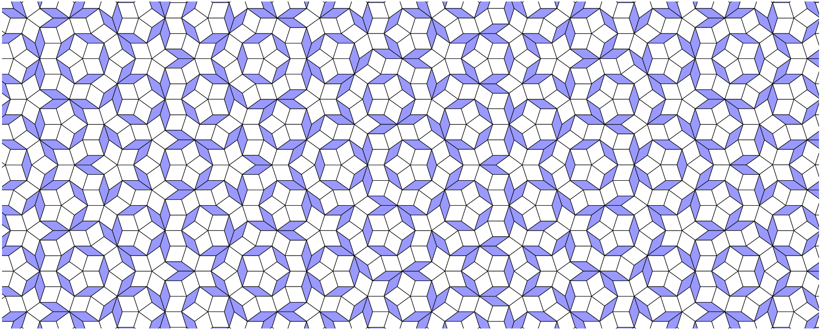

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Thank you for your attention