Bidimensional Sturmian Sequences and Substitutions

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Abstract. Substitutions are powerful tools to study combinatorial properties of sequences. There exist strong characterizations through substitutions of the Sturmian sequences that are S-adic, substitutive or a fixed-point of a substitution. In this paper, we define a bidimensional version of Sturmian sequences and look for analogous characterizations. We prove in particular that a bidimensional Sturmian sequence is always S-adic and give sufficient conditions under whose it is either substitutive or a fixed-point of a substitution.

Introduction

Substitutions generate infinite sequences by iteration, replacing a letter by a word. One of the most interesting property of sequences obtained in this way is that they are algorithmically easily generated and have a structure strongly ordered, though not restricted to the single periodic case.

The connection between substitutions and *Sturmian* sequences has been widely studied. Roughly speaking, a Sturmian sequence $\mathcal{S}_{\alpha,\rho}$ over the alphabet $\{1,2\}$ encodes the way the line $y = \alpha x + \rho$, α being irrational, crosses the unit squares of the lattice $\mathbb{Z}^2$ (see Fig. 1 and for more details [9, 11]).

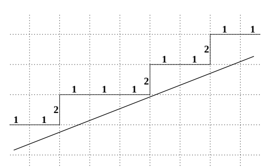

Fig. 1. The Sturmian sequence $\cdots 112111211211 \cdots$

A sequence is *S-adic* (see [13]) if it can be written as an infinite composition of a finite number of substitutions. So are Sturmian sequences, and more precisely:

$$\mathcal{S}_{\alpha,\rho} = \tau_0^{a_1-c_1} \circ \sigma_0^{c_1} \circ \tau_1^{a_2-c_2} \circ \sigma_1^{c_2} \circ \tau_0^{a_3-c_3} \circ \sigma_0^{c_3} \circ \tau_1^{a_4-c_4} \circ \sigma_1^{c_4} \circ \cdots$$

where (a_i) is the continued fraction expansion of α and (c_i) is the Ostrowski expansion of ρ (see [4]). However, not only Sturmian sequences are S-adic (see e.g. [8] for more details). Being more restrictive, one can consider the set of the *substitutives* sequences, introduced in [7], that are the sequences image under a morphism of a fixed-point of a (nontrivial) substitution:

$$\mathcal{S}_{\alpha,\rho} = \tau(\mathcal{S}') \quad \textit{and} \quad \mathcal{S}' = \sigma(\mathcal{S}').$$

It is proved that such sequences are exactly the Sturmian sequences $\mathcal{S}_{\alpha,\rho}$ with a *quadratic* irrational slope α and an intercept $\rho \in \mathbb{Q}(\alpha)$ (see [4]). If we furthermore require that $\mathcal{S}_{\alpha,\rho}$ *itself* is a fixed-point of a substitution, the previous characterization becomes that α is a *reduced* quadratic irrational, with some additional conditions on ρ (see e.g. [6, 14]). Let us recall (theorems of Lagrange and Galois) that an irrational number is quadratic (resp. reduced quadratic) iff its continued fraction expansion is *eventually periodic* (resp. *purely periodic*).

In this paper, we would like to proceed by analogy in the bidimensional case in order to obtain similar results. The first difficulty arises from the analogy itself, which is not so obvious and with whom we deal in the first three sections. Section 1 defines *bidimensional Sturmian sequences*, our analogue of Sturmian sequences. Sections 2 and 3 give the definition of, respectively, the bidimensional substitutions and the bidimensional continued fraction expansion we have chosen, namely the *generalized substitutions* introduced in [3] and the Brun's algorithm (see [5]). It is indeed a *choice* since there is no canonical multidimensional definition of a substitution or of a continued fraction expansion.

Our main results are given in Section 4. We here restricted ourselves to the case of *homogenous* bidimensional Sturmian sequences, which correspond to the Sturmian sequences $\mathcal{S}_{\alpha,\rho}$ for which $\rho = 0$. Theorem 3 proves that such bidimensional sequences are S-adic, while Theorem 4 gives a partial characterization very similar to the unidimensional case: a bidimensional Sturmian sequence is proved to be *substitutive* (resp. *fixed-point of a substitution*) if its parameters - the equivalent of the slope α of a Sturmian sequence - have an *eventually periodic* (resp. a *purely periodic*) bidimensional continued fraction expansion.

In Section 5, we examine the result of Section 4 from a more practical point of view: can we use the substitutions to *effectively* generate bidimensional Sturmian sequences? Though it does not completly solve the problem, Theorem 5 give a non trivial result in the substitutive case. We end the paper giving in Section 6 future extensions of the work presented here.

1 Stepped planes and bidimensional sequences

We here show how to associate to a plane a bidimensional sequence, by analogy to the one-dimensional case. This analogy also leads to define Sturmian bidimensional sequences. One denotes (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$.

The *face* (x, i^*), for $x \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$ is defined by (see Fig. 2):

$$(x, i^*) = \{x + re_j + te_k \mid 0 \leq r, t \leq 1 \text{ and } i \neq j \neq k\}.$$

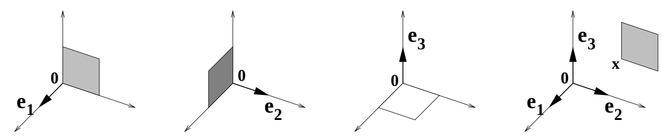

Fig. 2. From left to right: the faces $(0, i^*)$, $i = 1, 2, 3$ and $(x, 1^*) = x + (0, 1^*)$.

These faces generate the $\mathbb{Z}$-module of the formal sums of weighted faces $\mathcal{G} = \{\sum m_{x,i}(x, i^*) \mid m_{x,i} \in \mathbb{Z}\}$, on which the lattice $\mathbb{Z}^3$ acts by translation: $y + m_{x,i}(x, i^*) = m_{x,i}(y + x, i^*)$.

One then uses faces to approximate planes of $\mathbb{R}^3$:

Definition 1. *Let $\mathcal{P}_{\alpha,\beta}$ be the homogenous plane of $\mathbb{R}^3$ defined by:*

$$\mathcal{P}_{\alpha,\beta} = \{x \in \mathbb{R}^3 \mid \langle x, {}^t(1, \alpha, \beta)\rangle = 0\}.$$

The stepped plane $\mathcal{S}_{\alpha,\beta}$ *associated to $\mathcal{P}_{\alpha,\beta}$ is defined by:*

$$\mathcal{S}_{\alpha,\beta} = \left\{(x, i^*) \mid \langle x, {}^t(1, \alpha, \beta)\rangle > 0 \text{ and } \langle x - e_i, {}^t(1, \alpha, \beta)\rangle \leq 0\right\},$$

and a patch *of $\mathcal{S}_{\alpha,\beta}$ is a finite subset of the faces of $\mathcal{S}_{\alpha,\beta}$.*

Notice that a patch of $\mathcal{S}_{\alpha,\beta}$ belongs to the $\mathbb{Z}$-module $\mathcal{G}$, but is geometric, that is, without multiple faces. According to the terminology introduced by Reveillès in [12], the stepped plane corresponds to the notion of *standard arithmetic plane* in discrete geometry.

We now recall from [1] (see also [2]) the way one can bijectively encode a stepped plane by a bidimensional sequence over three letters. We first define a one-to-one map from the faces of a stepped plane to its set of vertices:

Proposition 1 ([1]). *Let v be the map from the faces of $\mathbb{R}^3$ to the vertices of $\mathbb{Z}^3$ defined by (see Fig. 3, left):*

$$v : \begin{array}{l} (x, 1^*) \to x \\ (x, 2^*) \to x + e_1 \\ (x, 3^*) \to x + e_1 + e_2. \end{array}$$

Then v maps different faces of a same *stepped plane to different vertices.*

We then define a bijective map from the vertices of a stepped plane to $\mathbb{Z}^2$:

Proposition 2 ([1]). *Let $\mathcal{S}_{\alpha,\beta}$ be a stepped plane. The orthogonal projection π on the plane $x+y+z=0$ is a bijection from $\mathcal{S}_{\alpha,\beta}$ to the lattice $\mathbb{Z}\pi(e_1)+\mathbb{Z}\pi(e_2)$. Thus the map $\tilde{\pi}$ defined by $\tilde{\pi}(x)=(m,n)$ iff $\pi(x)=m\pi(e_1)+n\pi(e_2)$ is a bijection from $\mathcal{S}_{\alpha,\beta}$ to $\mathbb{Z}^2$. Moreover, one has the explicit formulas:*

$$\tilde{\pi}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = (p-r, q-r),$$

$$\tilde{\pi}^{-1}(m,n) = \begin{pmatrix} m \\ n \\ 0 \end{pmatrix} + \left(1 - \left\lceil \frac{m+\alpha n}{1+\alpha+\beta} \right\rceil\right) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

And finally we give the encoding $\mathcal{U}_{\alpha,\beta}$ of a stepped plane $\mathcal{S}_{\alpha,\beta}$ (where $\mathcal{U}_{\alpha,\beta}(m,n)$ is the letter at position (m,n) in the bidimensional sequence $\mathcal{U}_{\alpha,\beta}$):

Proposition 3 ([1]). *Let ϕ be the map which maps a stepped plane $\mathcal{S}_{\alpha,\beta}$ to the bidimensional sequence $\mathcal{U}_{\alpha,\beta}$ over the alphabet $\{1,2,3\}$ defined by:*

$$(x, i^*) \in \mathcal{S}_{\alpha,\beta} \quad \Leftrightarrow \quad \mathcal{U}_{\alpha,\beta}(\tilde{\pi} \circ v(x, i^*)) = i,$$

or equivalently:

$$\mathcal{U}_{\alpha,\beta}(m,n) = i \quad \Leftrightarrow \quad (v(\tilde{\pi}^{-1}(m,n), i^*), i^*) \in \mathcal{S}_{\alpha,\beta}$$

Then ϕ is one-to-one from the set $\{\mathcal{S}_{\alpha,\beta} \mid 0 < \alpha, \beta < 1\}$ to the set of the bidimensional sequences over $\{1,2,3\}$ (see Fig. 3). Notice that not all the bidimensional sequences over $\{1,2,3\}$ correspond to a stepped plane.

One then defines *Sturmian stepped planes* and *bidimensional Sturmian sequences* by analogy with the unidimensional case:

Definition 2. *A stepped plane $\mathcal{S}_{\alpha,\beta}$ is a* Sturmian stepped plane *if 1, α and β are linearly independent over $\mathbb{Q}$. A* bidimensional Sturmian sequence *is the image under ϕ of a Sturmian stepped plane.*

Thus, ϕ is a bijection between the Sturmian stepped planes and the bidimensional Sturmian sequences, for which we furthermore have explicit formulas.

2 Generalized substitutions

We here define substitutions that act on stepped planes (or, equivalently, on the bidimensional sequences corresponding to stepped planes). These substitutions are the *generalized substitutions*, introduced in [3] (see also [11], Chap. 8).

Let us recall that the *incidence matrix* M_σ of a (classic) substitution σ gives at position (i,j) the number of occurences of the letter i in the word $\sigma(j)$. Moreover, σ is said *unimodular* if $detM_\sigma = \pm 1$. We are now in a position to define the *generalized substitutions*:

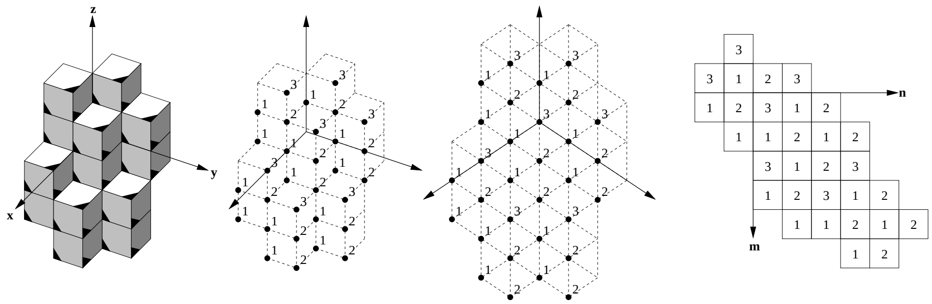

Fig. 3. From left to right: to each face corresponds a proper vertex (at its blacked corner); type 1, 2 or 3 of a vertex depends on the type of its corresponding face; the projection $\tilde{\pi}$ on the plane $x+y+z=0$ maps the vertices to a 2-dimensional lattice; we thus obtain a bidimensional sequence over $\{1,2,3\}$.

Definition 3. *The* generalized substitution *associated to the unimodular substitution σ is the endomorphism Θ_σ of $\mathcal{G}$ defined by:*

$$\begin{cases} \forall i \in \mathcal{A}, & \Theta_\sigma(0,i^*) = \sum_{j=1}^{3} \sum_{s:\sigma(j)=p\cdot i\cdot s} \left(M_\sigma^{-1}(f(s)), j^*\right), \\ \forall x \in \mathbb{Z}^3,\ \forall i \in \mathcal{A}, & \Theta_\sigma(x,i^*) = M_\sigma^{-1}x + \Theta_\sigma(0,i^*), \\ \forall \sum m_{(x,i)}(x,i^*) \in \mathcal{G}, & \Theta_\sigma\left(\sum m_{(x,i)}(x,i^*)\right) = \sum m_{(x,i)}\Theta_\sigma(x,i^*), \end{cases}$$

where $f(w) = (|w|_1, |w|_2, |w|_3)$ and $|w|_i$ is the number of occurences of the letter i in w.

Example 1. Let us consider the *Rauzy substitution* σ:

$$\sigma \ :\ \begin{array}{l} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{array}, \qquad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

σ is unimodular, and one easily computes (see Fig. 4):

$$\Theta_\sigma \ :\ \begin{array}{l} (e_1,1^*) \mapsto (e_1,1^*) + (e_2,2^*) + (e_3,3^*), \\ (e_2,2^*) \mapsto (e_1 - e_3, 1^*), \\ (e_3,3^*) \mapsto (e_2 - e_3, 2^*). \end{array} .$$

We now define an especially interesting type of substitution:

Definition 4. *A substitution σ is of* Pisot type *if its incidence matrix M_σ has eigenvalues μ_1, μ_2 and λ satisfying $0 < |\mu_1|, |\mu_2| < 1 < \lambda$. The generalized substitution Θ_σ is then also said of Pisot type.*

If σ is of Pisot type and if ${}^t(1,\alpha,\beta)$ is the *left* eigenvector of M_σ for the dominant eigenvalue λ (that is, ${}^tM_\sigma {}^t(1,\alpha,\beta) = \lambda {}^t(1,\alpha,\beta)$), the plane $\mathcal{P}_{\alpha,\beta}$ is called the *contracting invariant plane* of σ and verifies:

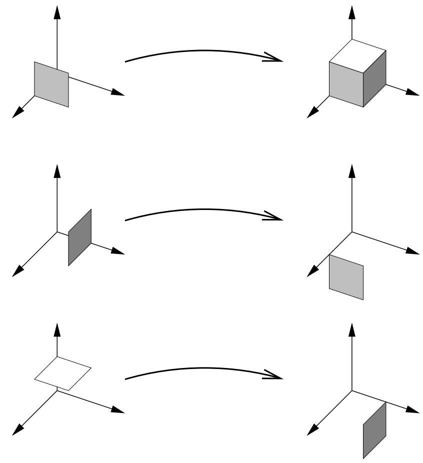

Fig. 4. The endomorphism Θ_σ for the Rauzy substitution: action on (e_i, i^*).

Proposition 4. *$\exists \mu$, $0 < \mu < 1$, such that if $x \in \mathcal{P}_{\alpha,\beta}$, then $M_\sigma x \in \mathcal{P}_{\alpha,\beta}$ and one has:*

$$||M_\sigma x|| \leq \mu ||x||.$$

The action of Θ_σ, when of Pisot type, on the stepped plane $\mathcal{S}_{\alpha,\beta}$ has some nice properties proved in [3]. Indeed, Θ_σ maps each patch of $\mathcal{S}_{\alpha,\beta}$ to a patch of $\mathcal{S}_{\alpha,\beta}$, the unit cube $\mathcal{U} = \{(e_i, i^*),\ i = 1, 2, 3\}$ is always a patch of $\mathcal{S}_{\alpha,\beta}$ and the sequence $(\Theta_\sigma^n(\mathcal{U}))$ is strictly increasing for inclusion and thus generates arbitrarily large patches of $\mathcal{S}_{\alpha,\beta}$ (see Fig. 5).

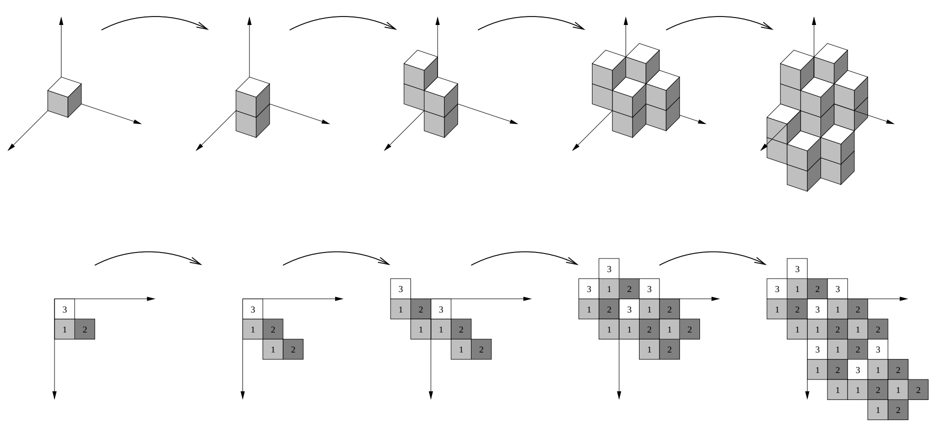

Fig. 5. $\Theta_\sigma^n(\mathcal{U})$ (top) and $(\phi \circ \Theta_\sigma \circ \phi^{-1})^n(\mathcal{U})$ (bottom) for the Rauzy substitution (which is of Pisot type), $n = 0, 1, 2, 3, 4$. Notice that the action of Θ_σ is not so obvious.

3 Bidimensional continued fractions

Contrary to the unidimensional case with the Euclid's algorithm, there is no canonical continued fraction expansion in the bidimensional case. We thus fix here the expansion we will use further, that is the one produced by the *modified Jacobi-Perron algorithm*, which is a two-point extension of Brun's algorithm. Let us recall this algorithm (see e.g. [5] for more details):

Definition 5. *Let be $X = [0,1) \times [0,1)$ and T the map defined on $X \backslash (0,0)$ by:*

$$T(\alpha,\beta) = \begin{cases} \left(\frac{\beta}{\alpha}, \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor\right) & \text{if } \alpha \geq \beta, \\ \left(\frac{1}{\beta} - \left\lfloor \frac{1}{\beta} \right\rfloor, \frac{\alpha}{\beta},\right) & \text{if } \alpha < \beta. \end{cases}$$

For $n \geq 1$ and if possible (that is, while $\alpha_{n-1} \neq 0$), one denotes:

$$(\alpha_n, \beta_n) = T^n(\alpha,\beta),$$

and defines:

$$(a_n, \varepsilon_n) = \begin{cases} \left(\left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor, 0\right) & \text{if } \alpha_{n-1} \geq \beta_{n-1}, \\ \left(\left\lfloor \frac{1}{\beta_{n-1}} \right\rfloor, 1\right) & \text{if } \alpha_{n-1} < \beta_{n-1}. \end{cases}$$

The sequence $(a_n, \varepsilon_n)_{n \geq 1}$ is called the continued fraction expansion *of (α,β) (notice that $a_n \in \mathbb{N}^*$ and $\varepsilon_n \in \{0,1\}$). This sequence is infinite iff 1, α and β are linearly independent over $\mathbb{Q}$*

Let us give a matricial point of view on this algorithm. For $a \in \mathbb{N}^*$, one defines the substitutions:

$$\sigma_{(a,0)} \;:\; \begin{array}{l} 1 \to \overbrace{11\cdots1}^{a\ times}3 \\ 2 \to 1 \\ 3 \to 2 \end{array}, \qquad \sigma_{(a,1)} \;:\; \begin{array}{l} 1 \to \overbrace{11\cdots1}^{a\ times}2 \\ 2 \to 3 \\ 3 \to 1 \end{array},$$

whose incident matrices are:

$$A_{(a,0)} = \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_{(a,1)} = \begin{pmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So that, with $(\alpha_0, \beta_0) = (\alpha, \beta)$ and $\eta_k = \max(\alpha_{k-1}, \beta_{k-1})$, one has for $n \geq 1$:

$$\eta_n {}^t A_{(a_n,\varepsilon_n)} \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix}.$$

We can give an expanded formulation of the previous equality:

Proposition 5. *Let* 1, α *and* β *be linearly independent over* $\mathbb{Q}$ *and let* (a_n, ε_n) *be the continued fraction expansion of* (α, β). *Then there exists a sequence* (α_n, β_n) *of couples in* $[0,1)^2$ *such that:*

$$\forall n \in \mathbb{N}, \qquad \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = (\eta_1 \eta_2 \cdots \eta_n)\, {}^t A_{(a_1,\varepsilon_1)} {}^t A_{(a_2,\varepsilon_2)} \cdots {}^t A_{(a_n,\varepsilon_n)} \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix}.$$

4 Substitutions and bidimensional Sturmian sequences

The previous sections have successively defined the bidimensional Sturmian sequences (or, equivalently, the Sturmian stepped planes), substitutions acting on these sequences and a bidimensional continued fraction expansion. We thus are now in a position to try to extend in the bidimensional case the results for (unidimensional) Sturmian sequences given in the introduction. Our results are first given in terms of stepped plane, and are them summed up in terms of bidimensional Sturmian sequences at the end of the section.

Let us first define a generalized substitution which plays a specific role:

Definition 6. *The generalized substitution associated to the unimodular substitution of Pisot type* $\sigma_{(a,\varepsilon)}$ *introduced in Section 3 is denoted* $\Theta_{(a,\varepsilon)}$. *Such a generalized substitution is said of* Brun type.

We then have the following fundamental theorem (proved in Appendix):

Theorem 1. $\Theta_{(a_n,\varepsilon_n)}$ *is a bijection from* $\mathcal{S}_{\alpha_n,\beta_n}$ *onto* $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$.

We shall stress that there is no contradiction between Theorem 1 and the results recalled at the end of Section 2, which would yield here that $\Theta_{(a_n,\varepsilon_n)}$ is a one-to-one map from the stepped plane associated to its contracting invariant plane to itself. Indeed, neither $\mathcal{P}_{\alpha_n,\beta_n}$ nor $\mathcal{P}_{\alpha_{n-1},\beta_{n-1}}$ are invariant planes of $\Theta_{(a_n,\varepsilon_n)}$ (except if the expansion (a_n, ε_n) is purely periodic of period 1, in which case all these planes are identical).

As we did in Proposition 5, we can give an expanded formulation of Theorem 1:

Theorem 2. *Let* $\mathcal{S}_{\alpha,\beta}$ *be a Sturmian stepped plane and* (a_n, ε_n) *be the continued fraction expansion of* (α, β). *Then there exists a sequence* $(\mathcal{S}_{\alpha_n,\beta_n})$ *of Sturmian stepped planes such that:*

$$\forall n \in \mathbb{N}, \qquad \mathcal{S}_{\alpha,\beta} = \Theta_{(a_1,\varepsilon_1)} \circ \Theta_{(a_2,\varepsilon_2)} \circ \cdots \circ \Theta_{(a_n,\varepsilon_n)} (\mathcal{S}_{\alpha_n,\beta_n}).$$

We thus obtain for $\mathcal{S}_{\alpha,\beta}$ an equation - called *expansion* - which looks like the classic S-adic expansion of a Sturmian sequence (see e.g. [13] for more details on S-adicity), though the number of different substitutions of our expansion may be unbounded. We will fix this last point thanks to the following proposition:

Proposition 6. *Let us define the substitutions:*

$$\sigma_0 \ : \ \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 13 \end{array} \ , \qquad \gamma_0 \ : \ \begin{array}{l} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{array} , \qquad \sigma_1 \ : \ \begin{array}{l} 1 \to 1 \\ 2 \to 12 \\ 3 \to 3 \end{array} , \qquad \gamma_1 \ : \ \begin{array}{l} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{array} .$$

These substitutions are unimodular and verify:

$$\forall (a, \varepsilon) \in \mathbb{N} \times \{0, 1\}, \qquad \sigma_{(a,\varepsilon)} = \sigma_\varepsilon^a \circ \gamma_\varepsilon.$$

An induction easily proves Proposition 6. Let Σ_ε and Γ_ε be the generalized substitutions associated to σ_ε and γ_ε. A computation yields $\Theta_{\sigma\circ\sigma'} = \Theta_{\sigma'} \circ \Theta_\sigma$, and Proposition 6 allows us to rewrite Theorem 2 in the following way:

Theorem 3. *Let $\mathcal{S}_{\alpha,\beta}$ be a Sturmian stepped plane and (a_n, ε_n) be the continued fraction expansion of (α, β). Then there exists a sequence $(\mathcal{S}_{\alpha_n,\beta_n})$ of Sturmian stepped planes such that:*

$$\forall n \in \mathbb{N}, \qquad \mathcal{S}_{\alpha,\beta} = \Gamma_{\varepsilon_1} \circ \Sigma_{\varepsilon_1}^{a_1} \circ \Gamma_{\varepsilon_2} \circ \Sigma_{\varepsilon_2}^{a_2} \circ \cdots \circ \Gamma_{\varepsilon_n} \circ \Sigma_{\varepsilon_n}^{a_n} \left(\mathcal{S}_{\alpha_n,\beta_n}\right),$$

where Γ_ε and Σ_ε are associated to the substitutions defined in Proposition 6.

We now consider the case of periodic expansions. Let us recall that a sequence (u_n) is *eventually periodic with period p and preperiod d* if $n > d \ \Rightarrow \ u_{n+p} = u_n$. If moreover $d = 0$, the sequence is said *purely periodic*. In this case, one has:

Theorem 4. *Let $\mathcal{S}_{\alpha,\beta}$ be a Sturmian stepped plane and (a_n, ε_n) be the continued fraction expansion of (α, β). If this expansion is eventually periodic, then there exist two generalized substitutions Θ_d and Θ_p, and a stepped plane $\mathcal{S}_p$ such that:*

$$\mathcal{S}_{\alpha,\beta} = \Theta_d(\mathcal{S}_p), \quad \textit{with} \quad \mathcal{S}_p = \Theta_p(\mathcal{S}_p).$$

And if the expansion is purely periodic, one has simply:

$$\mathcal{S}_{\alpha,\beta} = \Theta_p(\mathcal{S}_{\alpha,\beta}).$$

Proof. It follows easily from Theorem 2 with:

$$\begin{aligned} \Theta_d &= \Theta_{(a_1,\varepsilon_1)} \circ \Theta_{(a_2,\varepsilon_2)} \circ \cdots \circ \Theta_{(a_d,\varepsilon_d)}, \\ \Theta_p &= \Theta_{(a_{d+1},\varepsilon_{d+1})} \circ \Theta_{(a_{d+2},\varepsilon_{d+2})} \circ \cdots \circ \Theta_{(a_{d+p},\varepsilon_{d+p})}, \\ \mathcal{S}_p &= \mathcal{S}_{\alpha_d,\beta_d}, \end{aligned}$$

where p is the period of the expansion of (α, β) and d its preperiod. □

According to the terminology used in the introduction, Theorem 3 and 4 state that a bidimensional Sturmian sequence $\mathcal{U}_{\alpha,\beta}$ has always a S-adic expansion, and is substitutive (resp. a fixed-point of a substitution) if the expansion of (α, β) is eventually periodic (resp. purely periodic). Notice that, contrary to the unidimensional case, we do not yet obtain a complete characterization of bidimensional Sturmian sequences that are substitutive or fixed-point of a substitution. We will discuss this more carefully in the last section.

5 Effective generation of stepped planes

It is to notice that successive applications of generalized substitutions on a **finite** initial patch do not necessarily cover, to infinity, the **whole** stepped plane but only an infinite subset of it (think for example about a non simply-connected subset or a cone ...). Such a problem, that we investigate in this section, can however be of great practical interest, for example to effectively generate standard arithmetic plane in discrete geometry.

The following lemma, proved in Appendix, deals with the "almost" expansivity of a generalized substitution of Pisot type:

Lemma 1. *Let σ be a unimodular substitution σ of Pisot type (all the notations are those of Section 2). Then there exist $k \in [0,1)$ and $C \in \mathbb{R}^+$ such that:*

$$\begin{cases} (x, i^*) \in \mathcal{S}_{\alpha,\beta} \\ (y, j^*) \in \Theta_\sigma(x, i^*) \\ ||y|| \geq C \end{cases} \Rightarrow \quad ||x|| \leq k||y||.$$

It provides us a case in which one we can generate the whole stepped plane:

Theorem 5. *Let $\mathcal{S}_{\alpha,\beta}$ be a substitutive stepped plane, that is, a stepped plane such that there exist two generalized substitutions Θ_d and Θ_p verifying:*

$$\mathcal{S}_{\alpha,\beta} = \Theta_d(\mathcal{S}_p), \quad \textit{with} \quad \mathcal{S}_p = \Theta_p(\mathcal{S}_p).$$

If Θ_p is of Pisot type and bijective on $\mathcal{S}_p$, then there exists a finite patch P of $\mathcal{S}_p$ such that:

$$\mathcal{S}_{\alpha,\beta} = \Theta_d \left(\lim_{n \to +\infty} \Theta_p^n(P) \right).$$

Proof. Let C and k be the constants of Lemma 1 for the substitution Θ_p and let P be the patch formed by the faces (x, i^*) of $\mathcal{S}_p$ such that $||x|| \leq C$.
Let (y, j^*) be a face of $\mathcal{S}_p$. Consider the sequence $(y_m, j_m^*)_{m \geq 1}$ such that $(y_1, j_1^*) = (y, j^*)$ and $\Theta_p(y_{m+1}, j_{m+1}^*) = (y_m, j_m^*)$. This sequence is well defined since Θ_p is bijective. While $||y_m|| \geq C$, Lemma 1 yields $||y_{m+1}|| \leq k||y_m||$, with $k < 1$. Hence for m large enough, one has $||y_m|| \leq C$, that is, $(y_m, j_m^*) \in P$ and $\Theta_p^m(y_m, j_m^*) = (y, j^*)$. □

In particular, by Theorem 1 and 4, the previous theorem holds if (α, β) has an eventually periodic expansion of period p and preperiod d, **under the hypothesis** that $\Theta_p = \Theta_{(a_{d+1}, \varepsilon_{d+1})} \circ \Theta_{(a_{d+2}, \varepsilon_{d+2})} \circ \cdots \circ \Theta_{(a_{d+p}, \varepsilon_{d+p})}$ is of Pisot type (what can be false since for example $\Theta_{(1,1)} \circ \Theta_{(1,0)}$ is not of Pisot type). Proposition 6 can be used in practice to iterate on P only four different generalized substitutions, whatever Θ_p may be.

Example 2. Let (α, β) have the purely periodic expansion $[(1,1),(1,0),(1,0)]^*$ (of period 3). One computes $A_{(1,0)}A_{(1,0)}A_{(1,1)} = M_\sigma^2$, where σ is the Rauzy substitution introduced in Section 2. Thus $\Theta_p = \Theta_{(1,1)} \circ \Theta_{(1,0)} \circ \Theta_{(1,0)}$ is of Pisot type and $\mathcal{S}_{\alpha,\beta}$, fixed-point of Θ_p, can be generated applying Θ_p to a finite patch.

Notice that it is easy to see in the previous example that Θ_p and Θ_σ have the same invariant plane, and thus both generate, starting from P, patches of the same stepped plane. But we shall stress that Theorem 1 yields the bijectivity of Θ_p **as product of substitutions of Brun type** (see Definition 6). The Rauzy substitution - or any substitution with the same incidence matrix M_σ - might do not be bijective (more precisely not *onto*) on $\mathcal{S}_{\alpha,\beta}$. We can certainly claim that such a substitution generates arbitrarily large patches of $\mathcal{S}_{\alpha,\beta}$, but not necessarily the whole plane to infinity. In fact, explicit examples of substitutions of Pisot type which do not cover the whole plane are known (though the Rauzy substitution is not one of them).

6 Perspectives

This paper has defined the bidimensional Sturmian sequences (or, equivalently, the Sturmian stepped planes), on which act the generalized substitutions introduced in [3]. We have proved that every bidimensional Sturmian sequence is S-adic (according to the terminology of [13]), what extends to the bidimensional case the analogous result already known for unidimensional Sturmian sequences.

Similarly, the sufficient condition (on eventually or purely periodic continued fraction expansions), for unidimensional Sturmian sequences to be substitutive or fixed-point of a substitution, has been here extended to an analogous condition (on Brun's expansions), for bidimensional Sturmian sequences. However, we did not prove that our condition is also necessary, though it holds in the unidimensional case. A way to fix that, could be to extend the notion of *return word* - introduced in [7] and used to prove the unidimensional case - to some suitable bidimensional analogous notion of "return pattern". Such a bidimensional extension of return word have already been done in [10]. We hence have good hopes to complete our characterization of substitutive bidimensional Sturmian sequences.

As noticed in the introduction, we focused on the *homogenous* case, that is the analogous of the unidimensional Sturmian sequences with intercept equals to zero (that is, $\mathcal{S}_{\alpha,\rho}$ with $\rho = 0$, according to the notation of the introduction). Indeed, instead of the plane $\mathcal{P}_{\alpha,\beta}$ of Definition 1, we should consider the general case of a plane $\mathcal{P}_{\alpha,\beta,\rho} = {}^t(0,0,\rho) + \mathcal{P}_{\alpha,\beta}$. In the unidimensional case, taking into account an intercept ρ just leads to additional conditions that are, roughly speaking, conditions on the Ostrowski expansion of ρ similarly to the conditions on the continued fraction expansion of α (see [4]). It remains to give and prove some similar conditions on the intercept in the bidimensional case.

Last, we could carry out some improvements to the more practical results of Section 5. Indeed, starting from a *finite* initial patch to iterate a substitution is certainly more convenient than starting from the whole plane. But it is not so easy to compute this finite patch. Could not the unit cube $\mathcal{U}$, which is proved to be a patch of *any* stepped plane, suffices to generate the whole plane, as it is the case for the Rauzy substitution? Some counter-examples prove that the answer is in general negative, but it would be interesting to characterize the "good" cases. Similarly, conditions to have the substitution Θ_p of Theorem 5 of Pisot type (and thus, suitable to generate the plane) would be interesting.

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Appendix

Proof of Theorem 1 results of the three following lemmas:

Lemma 2. *$\Theta_{(a_n,\varepsilon_n)}$ maps a face of $\mathcal{S}_{\alpha_n,\beta_n}$ to faces of $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$.*

Proof. Let (x,i^*) be a face of $\mathcal{S}_{\alpha_n,\beta_n}$. By definition:

$$\langle x, {}^t(1,\alpha_n,\beta_n)\rangle > 0 \quad \text{and} \quad \langle x-e_i, {}^t(1,\alpha_n,\beta_n)\rangle \leq 0.$$

A face (y,j^*) of the image $\Theta_{(a_n,\varepsilon_n)}(x,i^*)$ can be written:

$$y = A^{-1}_{(a_n,\varepsilon_n)}(x+f(s)), \quad \text{with} \quad \sigma_{(a_n,\varepsilon_n)}(j) = p\cdot i\cdot s.$$

It suffices to prove that (y,j^*) belongs to $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$, that is:

$$\langle y, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle > 0 \quad \text{and} \quad \langle y-e_j, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \leq 0.$$

One has:

$$\begin{aligned}
\langle y, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle &= \langle A^{-1}_{(a_n,\varepsilon_n)}(x+f(s)), {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \\
&= \langle x+f(s), {}^tA^{-1}_{(a_n,\varepsilon_n)}\,{}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \\
&= \langle x+f(s), \eta_n\,{}^t(1,\alpha_n,\beta_n)\rangle \\
&= \eta_n \underbrace{\langle x, {}^t(1,\alpha_n,\beta_n)\rangle}_{>0} + \eta_n \underbrace{\langle f(s), {}^t(1,\alpha_n,\beta_n)\rangle}_{\geq 0 \text{ since } f(s),\alpha_n,\beta_n \geq 0}
\end{aligned}$$

So $\langle y, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle > 0$. Similarly:

$$\begin{aligned}
\langle y-e_j, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle &= \langle A^{-1}_{(a_n,\varepsilon_n)}(x+f(s))-e_j, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \\
&= \langle x+f(s)-A_{(a_n,\varepsilon_n)}e_j, {}^tA^{-1}_{(a_n,\varepsilon_n)}\,{}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \\
&= \eta_n\langle x+f(s)-A_{(a_n,\varepsilon_n)}e_j, {}^t(1,\alpha_n,\beta_n)\rangle
\end{aligned}$$

But it holds, by definition of the incidence matrix and of f:

$$A_{(a_n,\varepsilon_n)}e_j = f(\sigma_{(a_n,\varepsilon_n)}(j)) = f(p)+e_i+f(s) \geq e_i+f(s),$$

from what follows $\langle y-e_j, {}^t(1,\alpha_{n-1},\beta_{n-1})\rangle \leq 0$. $\square$

Lemma 3. *$\Theta_{(a_n,\varepsilon_n)}$ is one-to-one from $\mathcal{S}_{\alpha_n,\beta_n}$ to $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$.*

Proof. Let (x,i^*) and (x',i'^*) be two faces of $\mathcal{S}_{\alpha_n,\beta_n}$ and suppose that there is a face (y,j^*) both in $\Theta_{(a_n,\varepsilon_n)}(x,i^*)$ and $\Theta_{(a_n,\varepsilon_n)}((x',i'^*)$. By Lemma 2, we already know that (y,j^*) lies in $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$. One can write y in two ways:

$$y = A^{-1}_{(a_n,\varepsilon_n)}(x+f(s)) = A^{-1}_{(a_n,\varepsilon_n)}(x'+f(s')),$$

where $\sigma_{(a_n,\varepsilon_n)}(j) = p\cdot i\cdot s = p'\cdot i'\cdot s'$. We thus have $x+f(s) = x'+f(s')$.
If $x = x'$, since s and s' are both suffixes of the word $\sigma_{(a_n,\varepsilon_n)}(j)$, it yields $s = s'$,

then $i = i'$ and hence $(x, i^*) = (x', i'^*)$.
If $x \neq x'$, s and s' are suffixes of different lengths of $\sigma_{(a_n,\varepsilon_n)}(j)$. Suppose for example that s' is shorter: one can write $s = v \cdot i' \cdot s'$: $x + f(s) = x' + f(s')$ yields $x + f(v) + f(i') = x'$, that is, $x' - e_{i'} \geq x$ since $f(i') = e_{i'}$ and $f(v) \geq 0$. Then $\langle x' - e_{i'}, {}^t(1, \alpha_n, \beta_n) \rangle \geq \langle x, {}^t(1, \alpha_n, \beta_n) \rangle > 0$, what contradicts that (x', i'^*) is a face of $\mathcal{S}_{\alpha_n,\beta_n}$. So $(x, i^*) = (x', i'^*)$ and the proof is completed. $\square$

Lemma 4. *$\Theta_{(a_n,\varepsilon_n)}$ is onto from $\mathcal{S}_{\alpha_n,\beta_n}$ on $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$.*

Proof. Let (y, j^*) be a face of $\mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$. We search for a face (x, i^*) of $\mathcal{S}_{\alpha_n,\beta_n}$ such that $(y, j^*) \in \Theta_{(a_n,\varepsilon_n)}(x, i^*)$, that is, we search for x and a word s such that:

$$\begin{cases} \sigma_{(a_n,\varepsilon_n)}(j) = p \cdot i \cdot s, \\ \\ y = A^{-1}_{(a_n,\varepsilon_n)}(x + f(s)) \\ \\ \langle x, {}^t(1, \alpha_n, \beta_n) \rangle > 0 \quad \text{and} \quad \langle x - e_i, {}^t(1, \alpha_n, \beta_n) \rangle \leq 0. \end{cases}$$

Let us write $x = A_{(a_n,\varepsilon_n)} y - f(s) = A_{(a_n,\varepsilon_n)} y - (A_{(a_n,\varepsilon_n)} e_j - f(p) - e_i)$. A computation similar to those effectued in the proof of Lemma 2 easily yields:

$$\langle x, {}^t(1, \alpha_n, \beta_n) \rangle = u + r_{|p|} \quad \text{and} \quad \langle x - e_i, {}^t(1, \alpha_n, \beta_n) \rangle = u + t_{|p|},$$

where:

$$\begin{aligned} u &= \frac{1}{\eta_n} \langle y, {}^t(1, \alpha_{n-1}, \beta_{n-1}) \rangle, \\ r_{|p|} &= \langle f(p) + e_i, {}^t(1, \alpha_n, \beta_n) \rangle - \frac{1}{\eta_n} \langle e_j, {}^t(1, \alpha_{n-1}, \beta_{n-1}) \rangle, \\ t_{|p|} &= r_{|p|} - \langle e_i, {}^t(1, \alpha_n, \beta_n) \rangle. \end{aligned}$$

Hence it suffices to prove that one can choose the prefix p of $\sigma_{(a_n,\varepsilon_n)}(j)$ such that $u + t_{|p|} \leq 0 < u + r_{|p|}$ (that is, such that $(x, i^*) \in \mathcal{S}_{\alpha_n,\beta_n}$).

Let us assume that $\varepsilon_n = 0$ (the case $\varepsilon_n = 1$ is similar). Then $(y, j^*) \in \mathcal{S}_{\alpha_{n-1},\beta_{n-1}}$ implies $0 < u \leq \frac{1}{\alpha_{n-1}} \langle e_j, {}^t(1, \alpha_{n-1}, \beta_{n-1}) \rangle$, and by Definition 5:

$$a_n = \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor, \quad \alpha_n = \frac{\beta_{n-1}}{\alpha_{n-1}}, \quad \beta_n = \frac{1}{\alpha_{n-1}} - \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor \quad \textit{and} \quad \eta_n = \alpha_{n-1}.$$

If $j = 2$, $\sigma_{(a_n,0)}(2) = 1$ forces $i = 1$ and $p = s = \epsilon$ (the empty word). One then computes $r_0 = \langle e_1, {}^t(1, \alpha_n, \beta_n) \rangle - \frac{1}{\alpha_{n-1}} \langle e_2, {}^t(1, \alpha_{n-1}, \beta_{n-1}) \rangle = 0$ and $t_0 = r_0 - \langle e_1, {}^t(1, \alpha_n, \beta_n) \rangle = -1$. Since $0 < u \leq \frac{1}{\alpha_{n-1}} \langle e_2, {}^t(1, \alpha_{n-1}, \beta_{n-1}) \rangle = 1$, one has $u + t_0 = 0 < u + r_0$ and thus $(y, 2^*) = \Theta_{(a_n,0)}(x, 1^*)$, with $(x, 1^*) = (A_{(a_n,0)} y, 1^*) \in \mathcal{S}_{\alpha_n,\beta_n}$.

The case $j = 3$ is similar. $\sigma_{(a_n,0)}(3) = 2$ forces $i = 2$ and $p = s = \epsilon$. One then computes $r_0 = \alpha_n - \frac{\beta_{n-1}}{\alpha_{n-1}} = 0$ and $t_0 = -\alpha_n$. Since $0 < u \leq \frac{\beta_{n-1}}{\alpha_{n-1}} = \alpha_n$, one has $u + t_0 = 0 < u + r_0$ and thus $(y, 3^*) = \Theta_{(a_n,0)}(x, 2^*)$, with $(x, 2^*) = (A_{(a_n,0)}y, 2^*) \in \mathcal{S}_{\alpha_n,\beta_n}$.

The case $j = 1$ is slightly more difficult since p can be choosen among the $a_n + 1$ words 1^k, $k = 0 \dots a_n$, that are prefixes of $\sigma_{(a_n,0)}(1)$. One has to prove that there exists $k \in 0, 1, \dots, a_n$ such that $u + t_k \leq 0 < u + r_k$. One easily computes:

$$\begin{aligned}
&r_k = t_k + 1 = t_{k+1}, \quad \text{for } k = 0 \dots a_n - 1,\\
&r_{a_n} = 0 \quad \text{and} \quad t_{a_n} = \left\lfloor \frac{1}{\alpha_{n-1}} \right\rfloor - \frac{1}{\alpha_{n-1}} \in (-1, 0],\\
&t_0 = -\frac{1}{\alpha_{n-1}}.
\end{aligned}$$

Moreover, one has $0 < u \leq \frac{1}{\alpha_{n-1}}$. Hence $\tilde{k} = \max\{k \leq a_n \mid u + t_k \leq 0\}$ is well defined (indeed $\tilde{k} = \left\lfloor \frac{1}{\alpha_{n-1}} - u \right\rfloor$) and verifies $u + r_{\tilde{k}} > 0$. Since $|p| = \tilde{k}$ implies $s = 1^{a_n - \tilde{k}}3$, that is, $f(s) = (a_n - \tilde{k})e_1 + e_3$, one has $(y, 1^*) = \Theta_{(a_n,0)}(x, l^*)$, with $x = A_{(a_n,0)}y - f(s) \in \mathcal{S}_{\alpha_n,\beta_n}$ and $l = 1$ if $\tilde{k} < a_n$, $l = 3$ otherwise. $\square$

Proof of Lemma 1. One can write:

$$(y, j^*) = (M_\sigma^{-1}(x + f(s)), j^*), \quad \text{with} \quad \sigma(j) = p \cdot i \cdot s.$$

Roughly speaking, since (y, i^*) is in $\mathcal{S}_{\alpha,\beta}$, y is not far from the contracting invariant plane $\mathcal{P}_{\alpha,\beta}$ of M_σ, and since x is almost equal to $M_\sigma y$, $||x||$ result mainly of the contraction of $||y||$ by M_σ.
Let us write it neatly. Let u denote ${}^t(1, \alpha, \beta)$, the left eigenvector of M_σ for its dominant eigenvalue, and let $\mu < 1$ be the constant of Proposition 4. One has:

$$\begin{aligned}
x &= M_\sigma y - f(s)\\
&= M_\sigma\left(\langle y, u\rangle u + (y - \langle y, u\rangle u)\right) - f(s)\\
&= \langle y, u\rangle M_\sigma u + M_\sigma(y - \langle y, u\rangle u) - f(s).
\end{aligned}$$

We then use that $y - \langle y, u\rangle u$ belongs to $\mathcal{P}_{\alpha,\beta}$, plane on which M_σ is μ-contracting, that $0 < |\langle y, u\rangle| \leq \max(1, \alpha, \beta)$ since $(y, i^*) \in \mathcal{S}_{\alpha,\beta}$, and that $||f(s)|| \leq f_\sigma := ||f(\sigma(1)) + f(\sigma(2)) + f(\sigma(3))||$:

$$\begin{aligned}
||x|| &\leq ||\langle y, u\rangle M_\sigma u|| + \mu||y - \langle y, u\rangle u|| + f_\sigma\\
&\leq ||\langle y, u\rangle M_\sigma u|| + \mu||y|| + \mu||\langle y, u\rangle u|| + f_\sigma\\
&\leq \underbrace{\max(1, \alpha, \beta)(||M_\sigma u|| + \mu||u||) + f_\sigma}_{B} + \mu||y||.
\end{aligned}$$

Since B depends only on σ, for any k such that $\mu < k < 1$ and for $C = \frac{B}{k-\mu} \in \mathbb{R}^+$, it holds: $||y|| \geq C \Rightarrow ||x|| \leq k||y||$. $\square$