Local rules for canonical cut and project tilings

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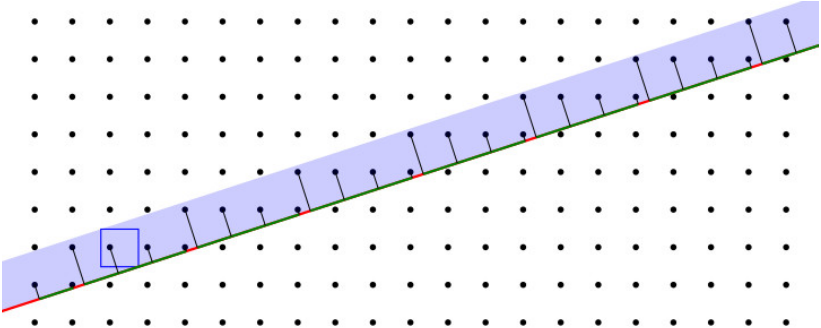
Outline

Outline

Cut the plane and project onto a line



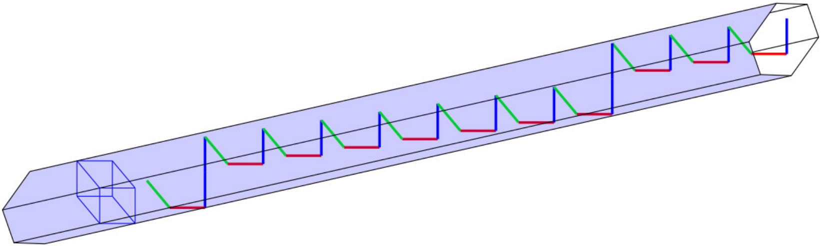
Cut $\mathbb{R}^n$ and project onto $\mathbb{R}^d$ (canonical cut and project)

Definition (Planar tiling)

Let E be a d-dim. affine space in $\mathbb{R}^n$ such that $E \cap \mathbb{Z}^n = \emptyset$.
Select the d-dim. faces with vertices in $\mathbb{Z}^n$ lying in $E + [0,1]^n$.
Project them onto E to get a so-called *planar* $n \to d$ *tiling*.

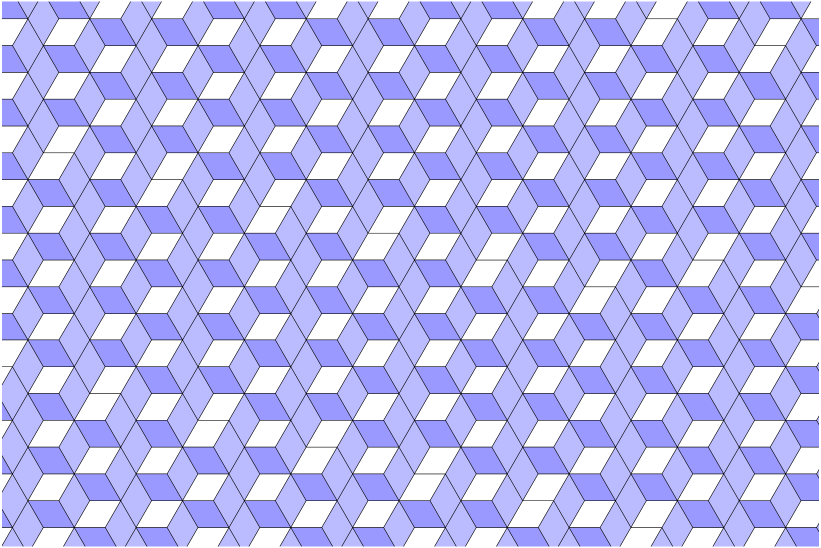
Q. Are such tilings periodic or not?

Cut the space and project onto a line

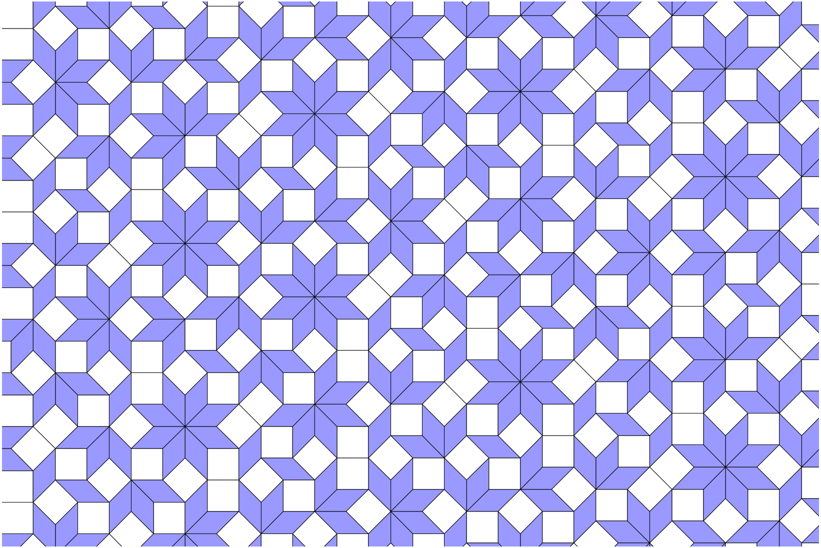


Billiard words are planar $3 \to 1$ tilings, but not the *Tribonacci word*.

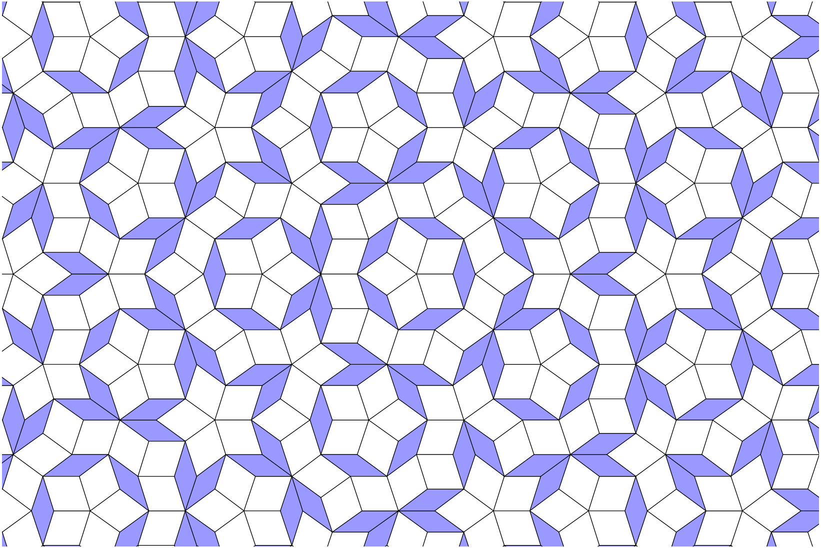
Cut the space and project onto a plane



Cut an higher dim. space and project onto a plane



And get a Penrose tiling (De Bruijn, 1981)



Outline

Multigrid

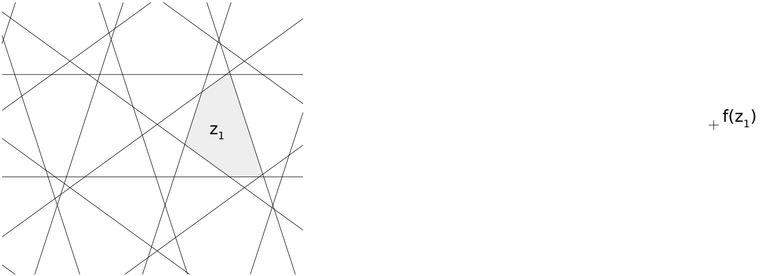
Definition (Multigrid)

The *multigrid* with shifts $s_1, \ldots, s_n$ in $\mathbb{R}$ and *grid vectors* $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^d$ is the set of n families of equally spaced parallel hyperplanes

$$H_i := \{\vec{x} \in \mathbb{R}^d \mid \langle \vec{x} | \vec{v}_i \rangle + s_i \in \mathbb{Z}\},$$

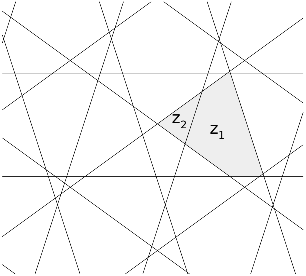
where at most d hyperplanes are assumed to intersect in a point.

Dualization



- The grid hyperplanes divide the space into cells;
- To each cell z_i corresponds a vertex $f(z_i)$ of the tiling;
- If z_i and z_j are adjacent along $a + \vec{v}_k^{\perp}$, then $f(z_j) - f(z_i) = \vec{v}_k$

Dualization



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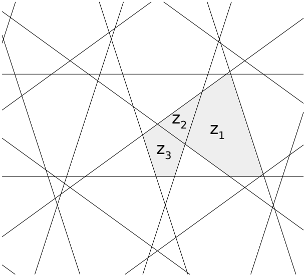

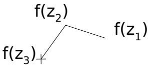

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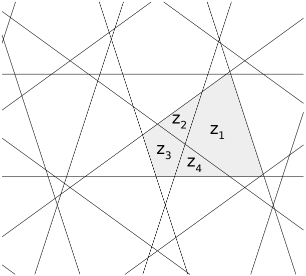

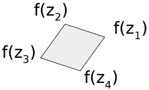

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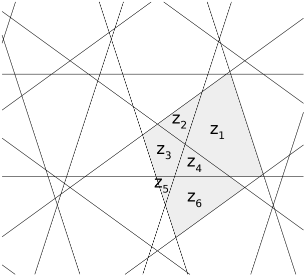

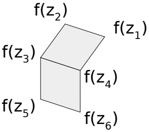

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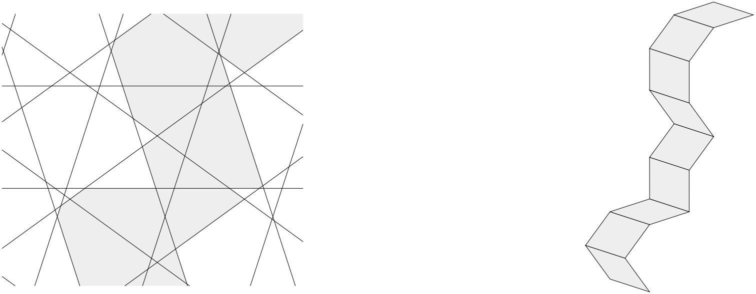

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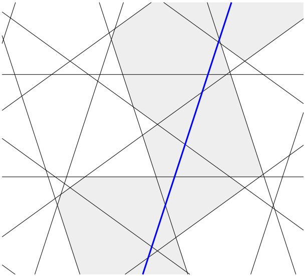

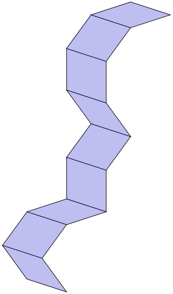

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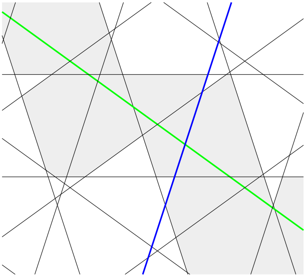

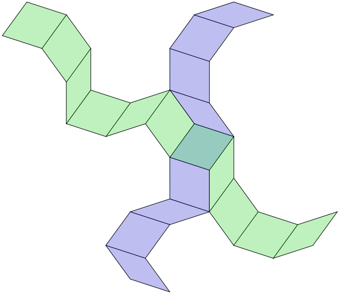

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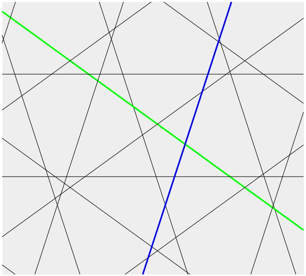

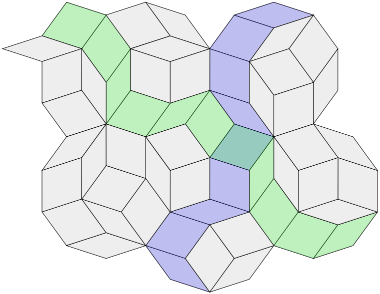

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Equivalence

Theorem (Gähler-Rhyner 1986)

Any multigrid dualization is a planar tiling, and conversely.

The grids are the intersection of the slope with the hyperplanes

$$G_i = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x} | \vec{e}_i \rangle \in \mathbb{Z}\}$$

Outline

Another way to define vectorial spaces

Definition (Grassmann coordinates)

The *Grassmann coordinates* of a vector space $\mathbb{R}\vec{u}_1 + \ldots + \mathbb{R}\vec{u}_d$ are the $d \times d$ minors of the matrix whose columns are the $\vec{u}_i$'s.

Q. How many Grassmann coordinates does have a subspace of $\mathbb{R}^n$?

Q. What are the Grassmann coordinates of a hyperplane?

Plücker relations

Theorem

A vector space is characterized by its Grassmann coordinates.

Q. What is the dimension of the set of d-dim. vector spaces of $\mathbb{R}^n$?

Plücker relations

Theorem

A vector space is characterized by its Grassmann coordinates.

Q. What is the dimension of the set of d-dim. vector spaces of $\mathbb{R}^n$?

Theorem

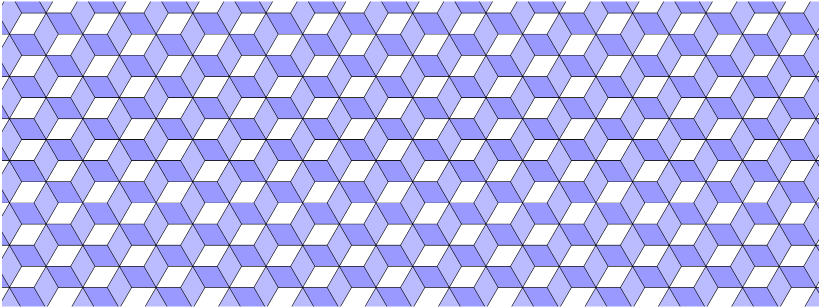
A non-zero real tuple $(G_{i_1,\ldots,i_d})$ are the Grassmann coordinates iff, for any $1 \leq k \leq n$ and any two d-tuples of indices they satisfy

$$G_{i_1,\ldots,i_d} G_{j_1,\ldots,j_d} = \sum_{l=1}^{d} \underbrace{G_{i_1,\ldots,i_d} G_{j_1,\ldots,j_d}}_{swap\ i_k\ and\ j_l}.$$

Link with planar tilings

Proposition

The tile generated by $\vec{v}_{i_1}, \dots \vec{v}_{i_d}$ has frequency $|G_{i_1,\dots,i_d}|$.



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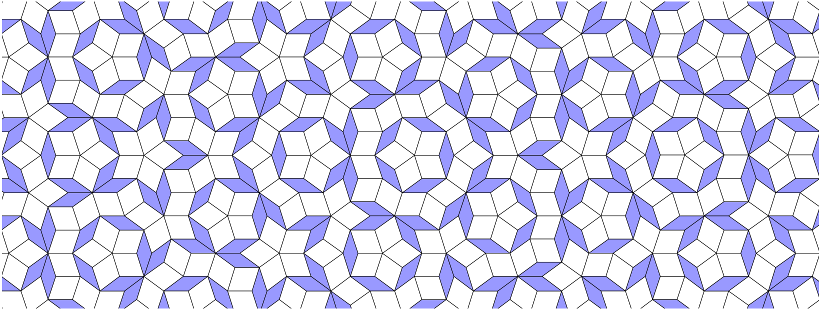
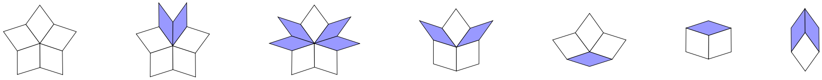


Outline

Pattern

Definition

A *pattern* of a tiling is a finite subset of the tiles of this tiling.



A *r-map* is a pattern formed by the tiles intersecting a closed r-ball.

The *r-atlas* of a tiling is th set of its r-maps.

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $2 \to 1$ planar tiling?

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $3 \to 1$ planar tiling?

Window

Definition

The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $4 \to 2$ planar tiling?

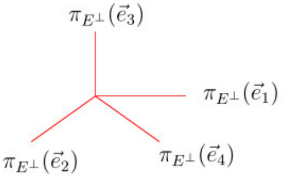
Window

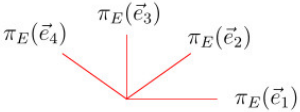
Definition

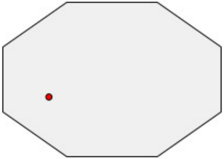
The *window* of a planar $n \to d$ tiling of slope E is the orthogonal projection of $E + [0,1]^n$ onto $E^\perp$.

Q. What is the window of a $5 \to 2$ planar tiling?

Tilings seen from the window







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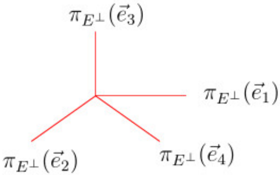

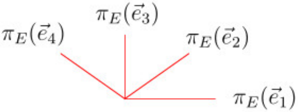

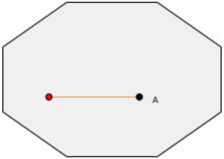

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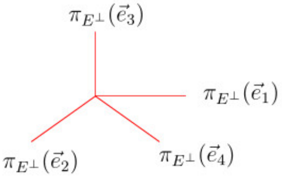

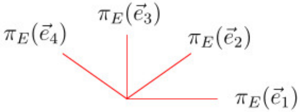

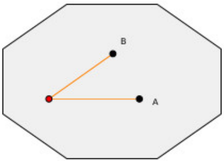

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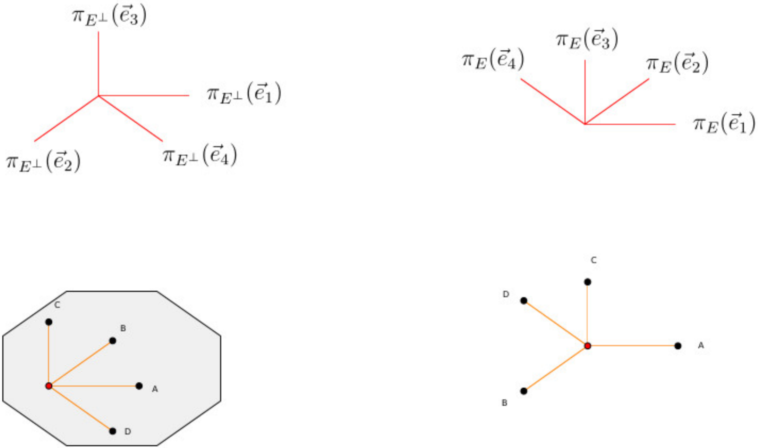

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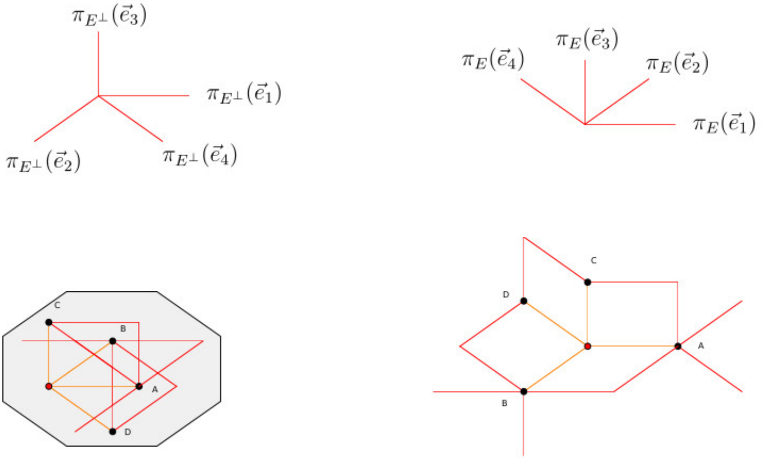

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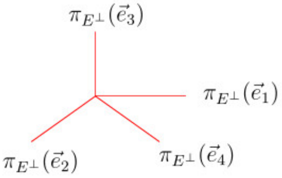

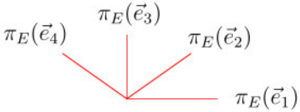

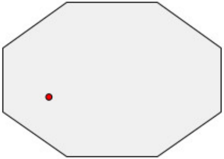

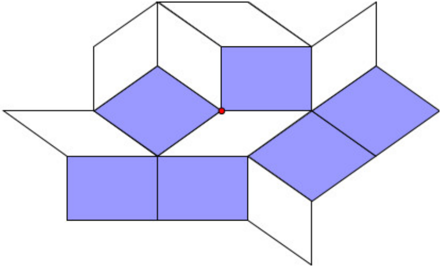

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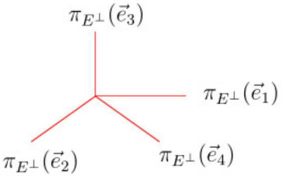

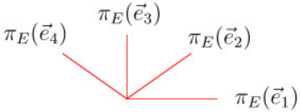

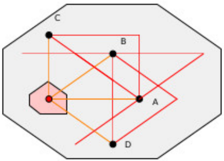

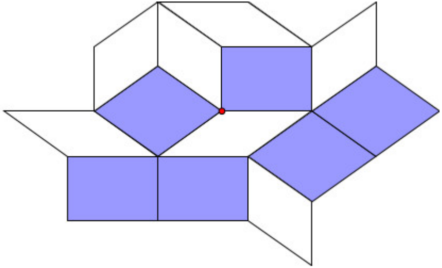

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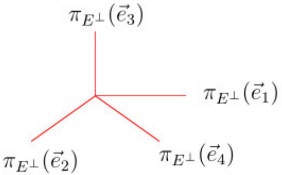

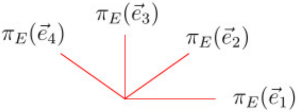

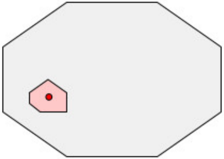

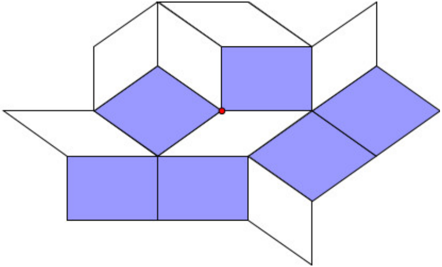

Counting patterns

Complexity: function which counts the size of the r-atlas.

Theorem (Julien 2010)

A generic planar $n \to d$ tiling has complexity $\Theta(r^{d(n-d)})$.

Q. What is the complexity of a Fibonacci word?

Q. What is the complexity of a Penrose tiling?

Quasiperiodicity

Definition (quasiperiodic or repetitive or minimal)

A tiling is *quasiperiodic* if whenever a pattern occurs somewhere, it reoccurs at uniformly bounded distance from any point.

Q. Is it true that periodic tilings are quasiperiodic?

Q. Is it true that non-periodic tilings are quasiperiodic?

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Theorem

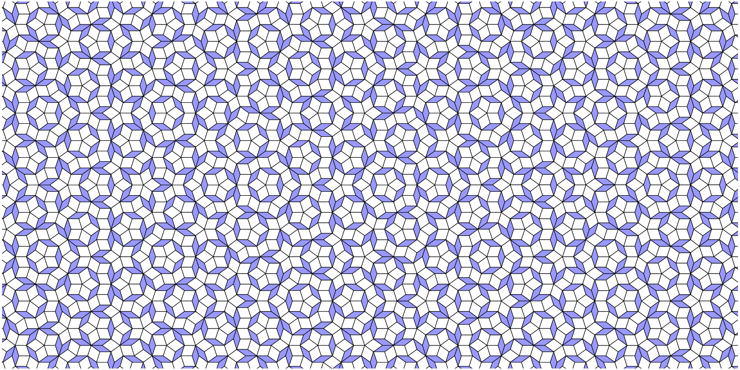
Planar tilings are quasiperiodic.

Patterns even have *frequencies*, related to the area of their regions.

Slope shift

Proposition

If a slope $a + E$ is in the smallest rational space containing $b + E$, then the planar tilings with these slopes have the same patterns.



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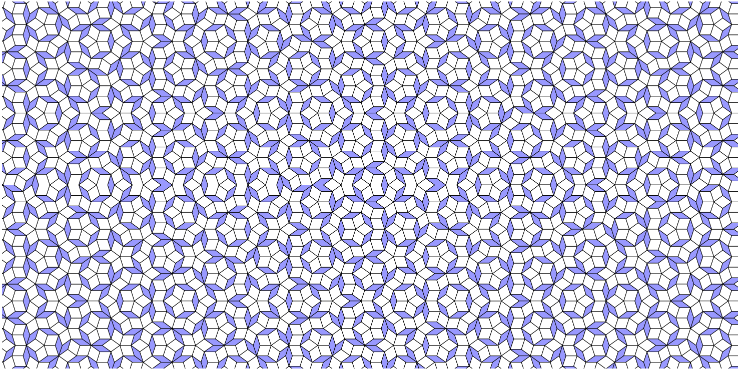

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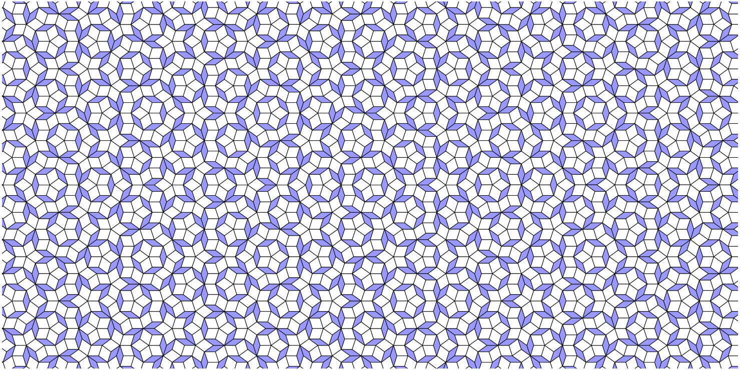

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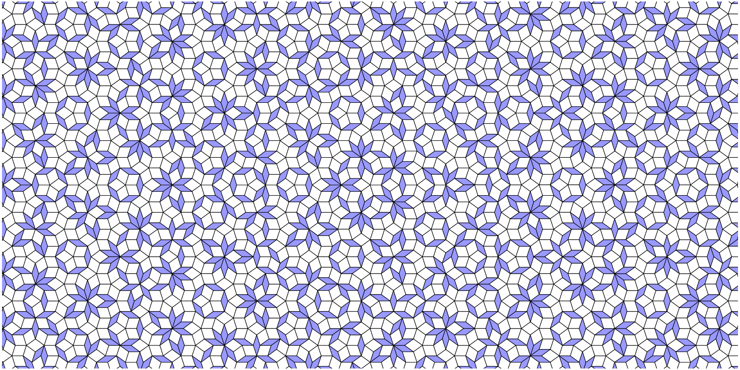

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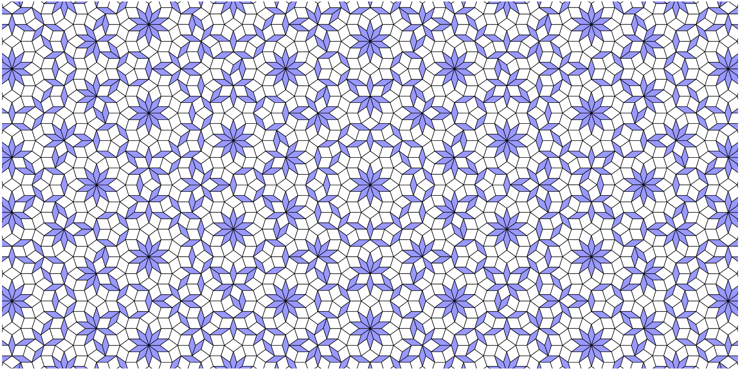
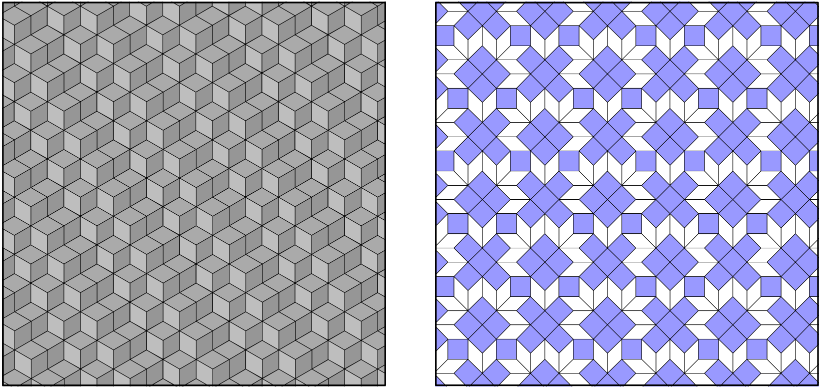


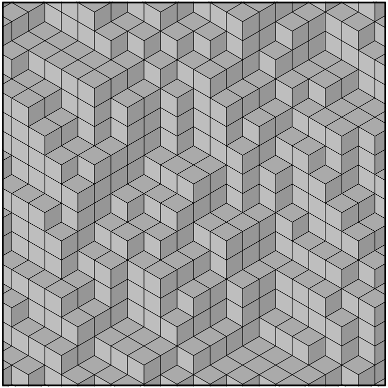
Outline

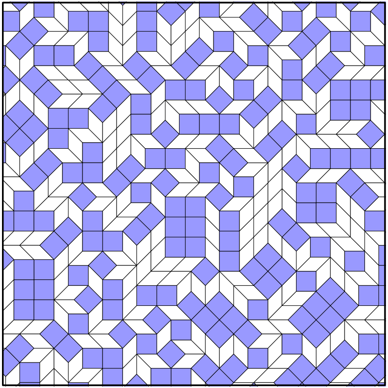
General $n \to d$ tilings



Planar tilings are well ordered. . .

General $n \to d$ tilings



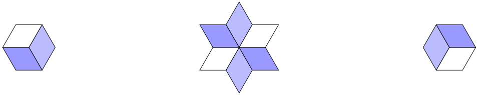


Planar tilings are well ordered. . . but they can easily be messed up!

Local rules

Definition (Local rules)

A planar tiling of slope E has *diameter r* and *thickness t local rules* if any tiling with a smaller or equal r-atlas lifts into $E + [0, t]^n$.



Main Open Question

Which planar tilings do admit local rules?

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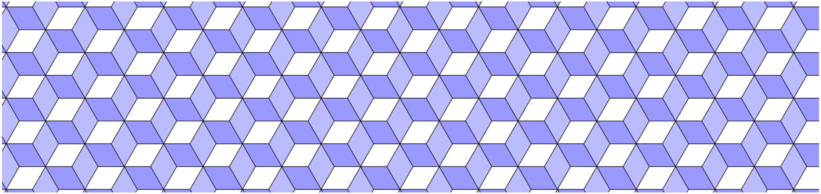

Main Open Question

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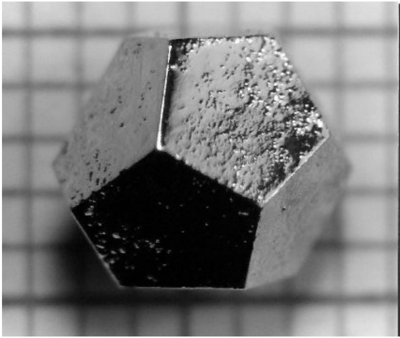
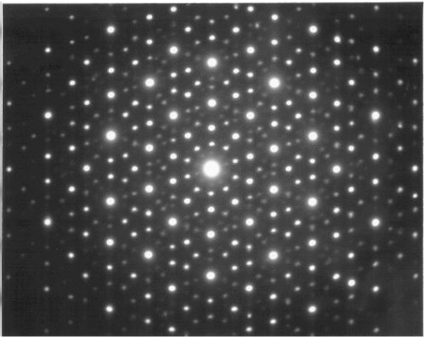
Link with quasicrystals

Planar $n \to d$ tilings aim to model the *structure* of *quasicrystals*.





Local rules aim to model their *stability* (*i.e.*, energetic interactions).

Outline

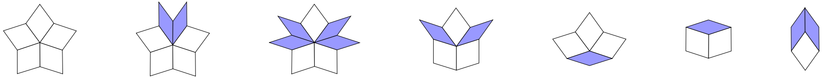
Penrose tilings

Definition (Penrose tiling)

A *Penrose tiling* is a planar $5 \to 2$ tiling with slope

$$\frac{1}{5}(1,1,1,1,1) + \mathbb{R}\left(\cos\frac{2k\pi}{5}\right)_{0\leq k\leq 4} + \mathbb{R}\left(\sin\frac{2k\pi}{5}\right)_{0\leq k\leq 4}.$$

It is the dualization of the multigrid with vectors $e^{\frac{2ik\pi}{5}}$ and shifts $\frac{1}{5}$.



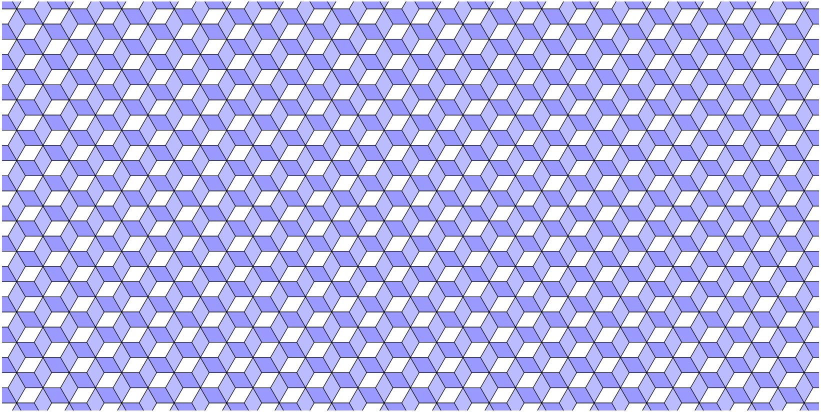
Theorem (de Bruijn, 1981)

Penrose tilings have local rules of diameter 0 and thickness 1.

n-fold tilings

Definition (n-fold tiling)

A *n-fold tiling* is a planar $n \to 2$ tiling which has the same finite patterns as its image under a rotation by $2\pi/n$.



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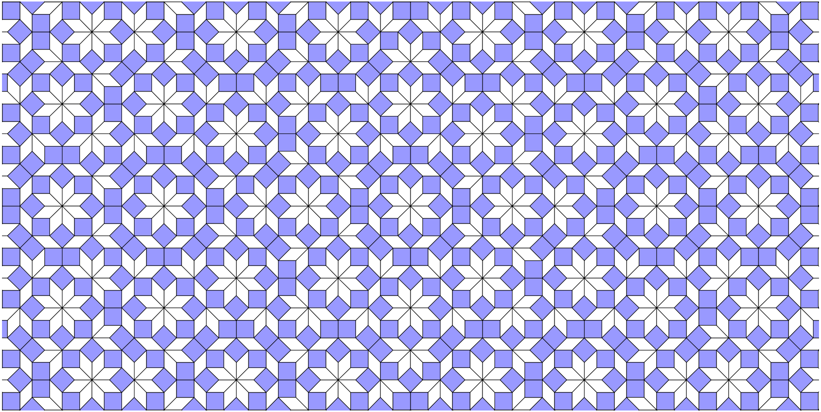

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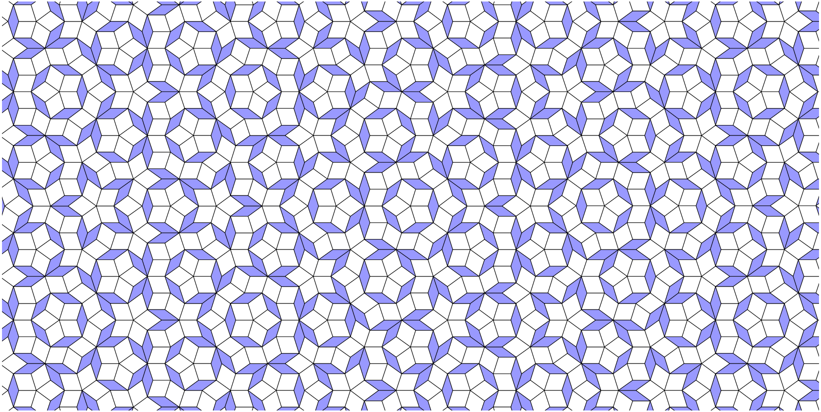

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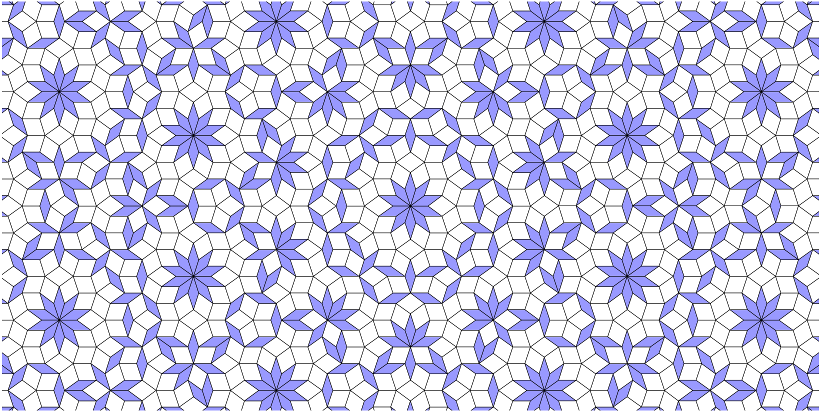

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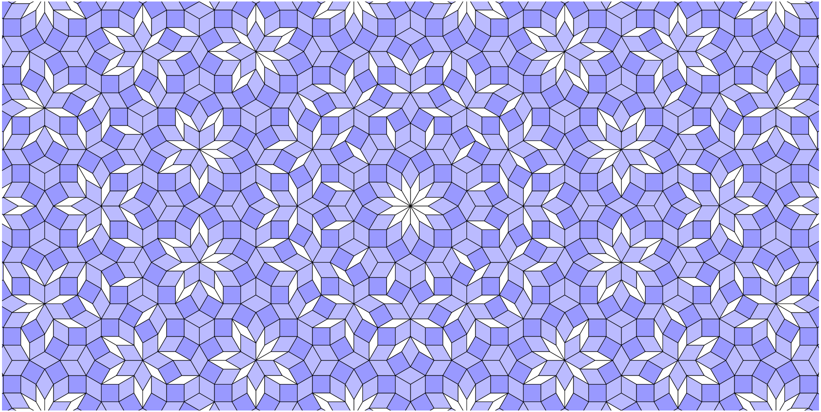

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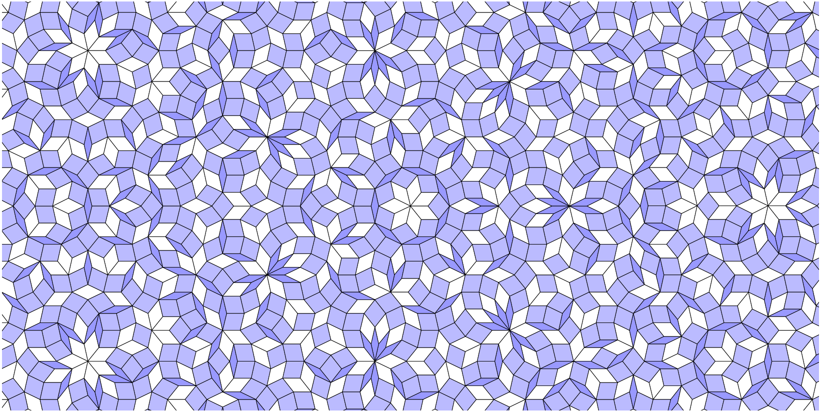

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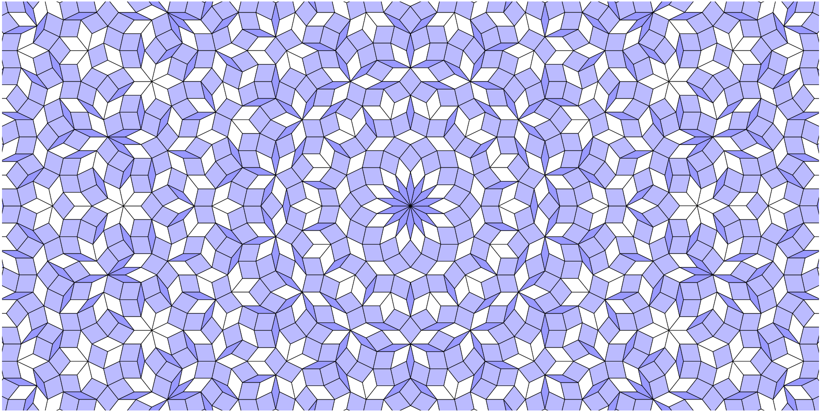

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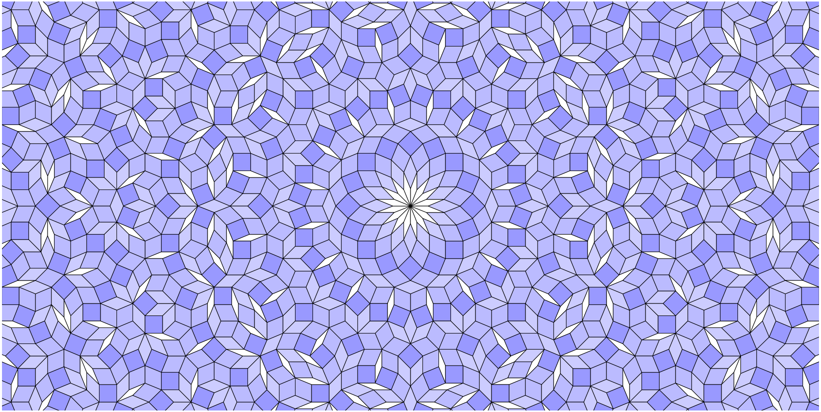
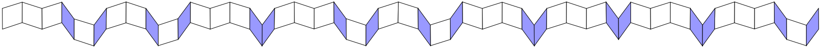


Local rules for n-fold tilings

Theorem (Socolar 1990)

An n-fold tiling has local rules when n is not a multiple of 4.

Local rules actually enforce an *alternation condition*:



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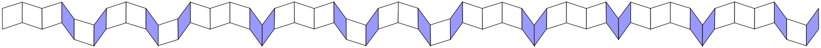

When n is a multiple of 4, there are square tiles...

Subperiods

Definition (Subperiod)

A planar $n \to d$ tiling has a *subperiod* if one gets a periodic tiling by an orthogonal projection onto $d + 1$ well-chosen basis vectors.

For example, a Penrose tiling has 10 subperiods (video).

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For example, a Penrose tiling has 10 subperiods (video).

This translates in linear rational dependencies between Grassmann coordinates over $d + 1$ indices. For Penrose:

$$G_{12} = G_{23} = G_{34} = G_{45} = G_{51}, \qquad G_{13} = G_{35} = G_{52} = G_{24} = G_{41}.$$

Planarity issues

Proposition

The subperiods of a planar tiling can be enforced by local rules.

But these local rules may not suffice to enforce planarity. . .

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But these local rules may not suffice to enforce planarity...

Theorem (Bédaride-Fernique 2015)

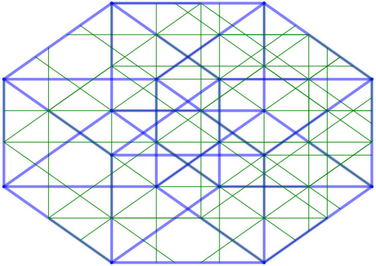
A planar $4 \to 2$ tiling has local rules iff its slope is characterized by its subperiods. In particular the slope is quadratic (or rational).

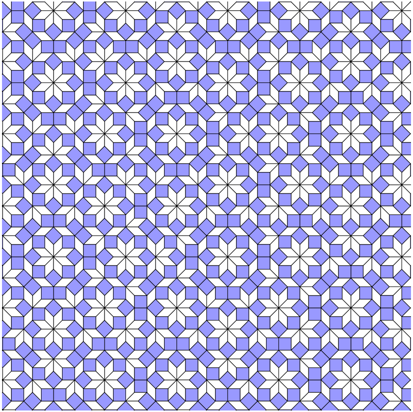
Outline

4p-fold tilings

Theorem (Bédaride-Fernique 2015)

The 4p-fold tilings do not have local rules.





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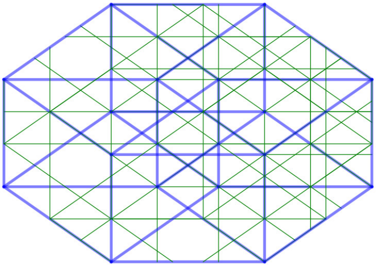

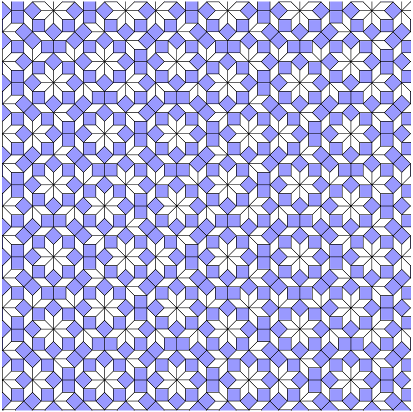

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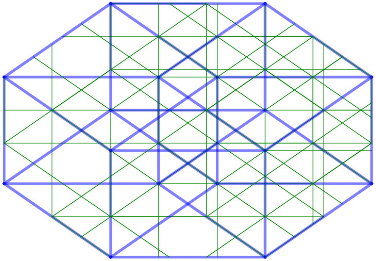

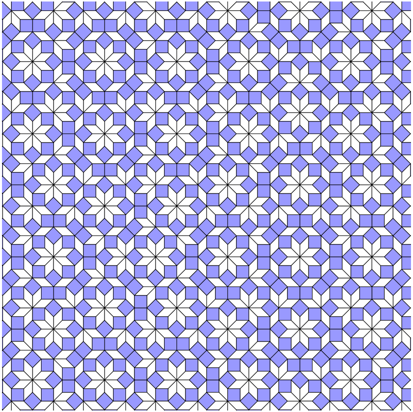

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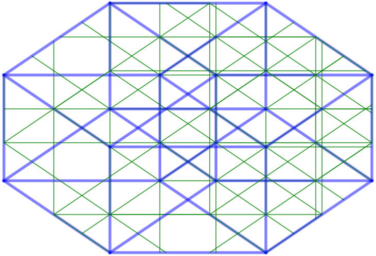

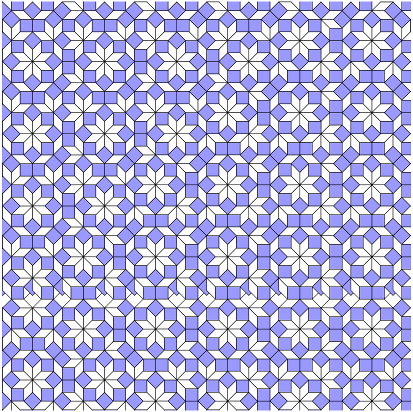

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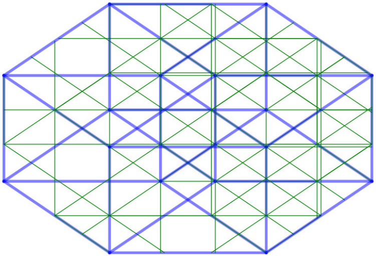
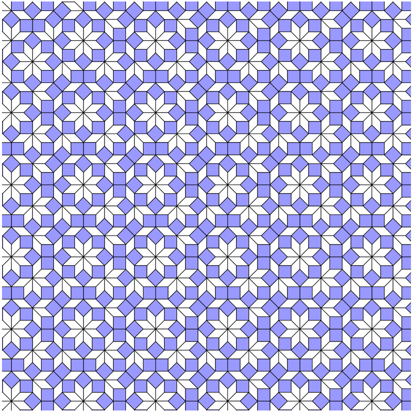




Local rules of thickness 1

Full subperiods: any projection on $d+1$ basis vector is periodic.

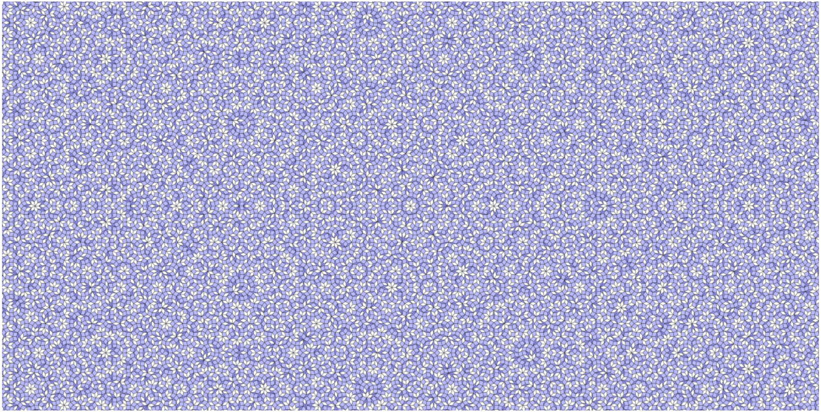
Theorem (Levitov 1988)

A planar tiling with thickness 1 *local rules has full subperiods.*

For n-fold tilings, this yields $n \in \{4, 6, 8, 10, 12\}$.

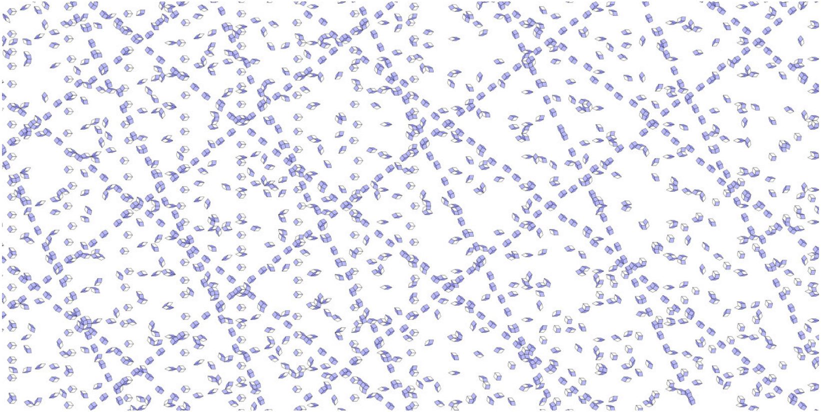
These are the only symmetries yet observed in real quasicrystals...

Proof sketch



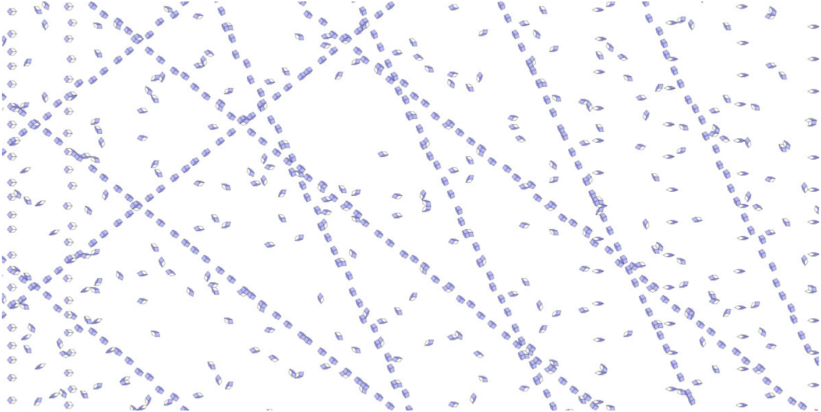
Consider a planar tiling which does not have full subperiods.

Proof sketch



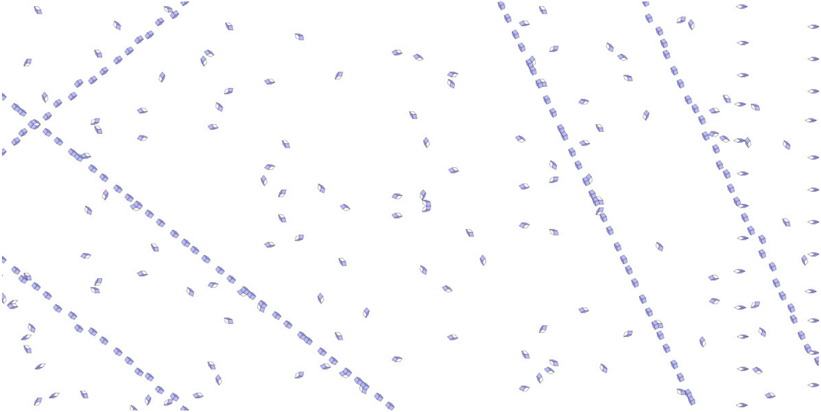
Shifting the slope creates *flips*. We shift without creating patterns.

Proof sketch



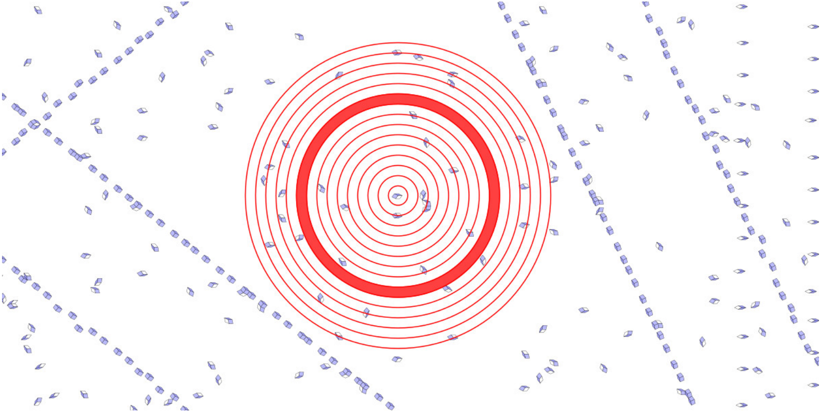
There are lines of flips (corresp. to subperiods) and isolated flips.

Proof sketch



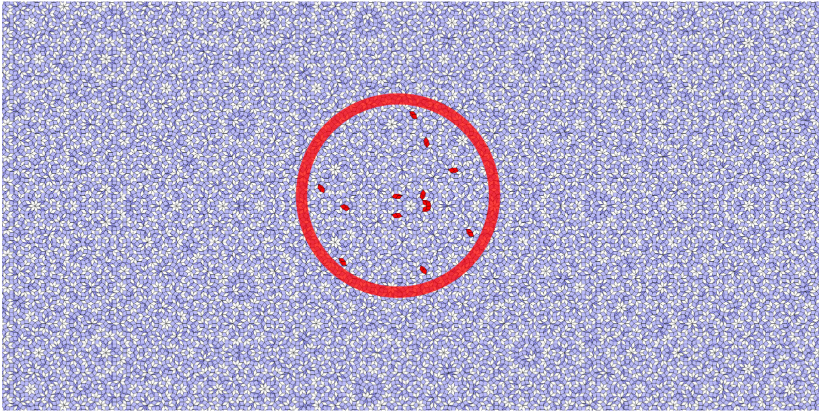
The smaller the shift is, the sparser these flips are.

Proof sketch



Given r, we eventually find a ring of thickness r without any flip.

Proof sketch

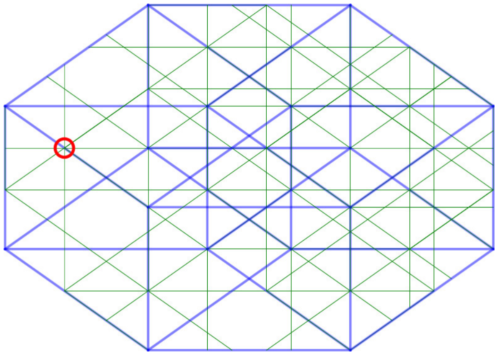


This yields a planar tiling of thickness $t > 1$ with the same r-atlas.

Algebraic obstruction

Theorem (Le 1995)

The slope of a planar tiling with local rules is algebraic.



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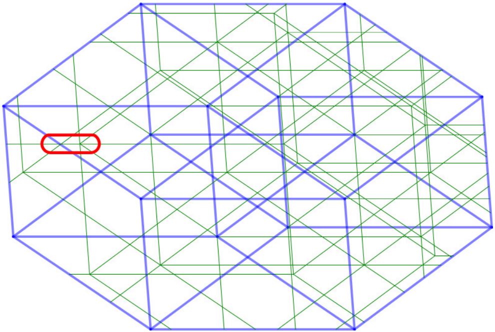

Some open questions

Planar $4 \to 2$ tilings have LR iff subperiods charaterize the slope.
This also holds for planar $n \to n-1$ tilings.
Does this hold for any planar $n \to d$ tiling?
If it does, the algebraic degree would be at most $\lfloor n/d \rfloor$. Tight?

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When? Which sets of slopes can be obtained in this way?

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We considered only uncolored tiles, *i.e.*, tiling spaces of finite type.
What if we add colors, *i.e.*, if we consider sofic tiling spaces?
$\rightsquigarrow$ see Mathieu Sablik's lecture.