On self-assembly of planar octagonal tilings of finite type

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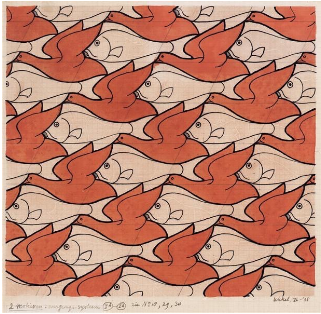
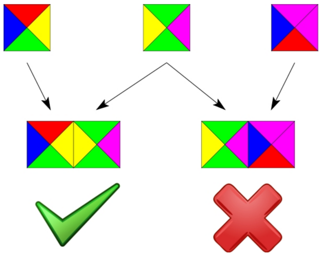
What is a tiling?



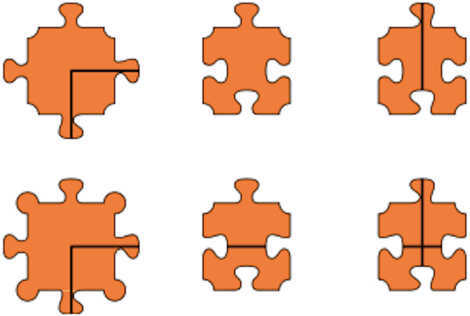
- *Tiling*: covering of the plane with copies of basic shapes without gaps and overlaps. The set of basic shapes is called a *prototile set* and the elements are called *tiles*.

Wang tiles (1961)



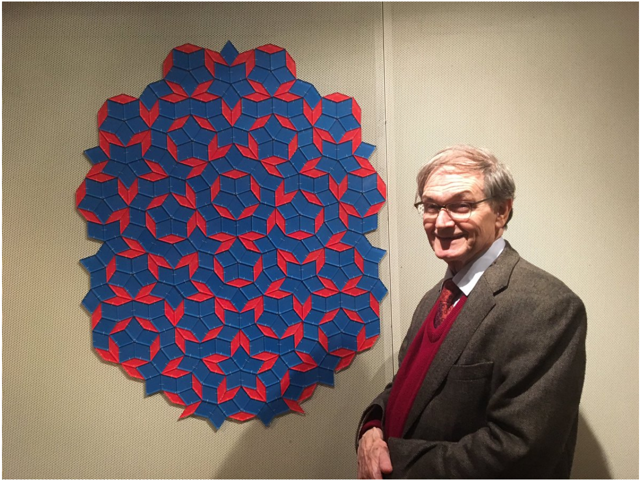
- A set of tiles is called *aperiodic* if copies of them can cover the whole plane but only in a non-periodic way.
- Berger in 1964 constructed the first aperiodic tileset: 20426 tiles!

Robinson tiling (1971)

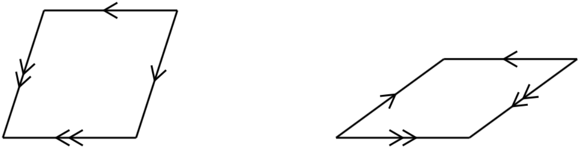


Prototiles of Robinson tiling

Penrose tiling (1974)

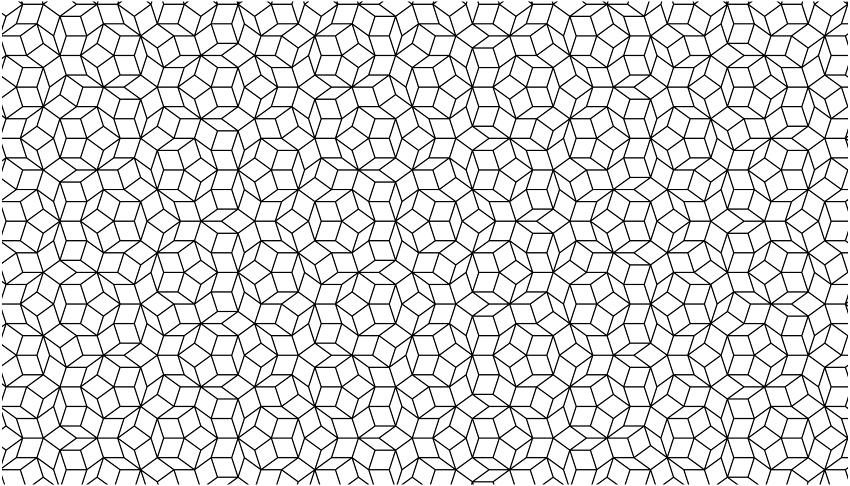


Penrose tiling (1974)



Prototileset of Penrose tilings

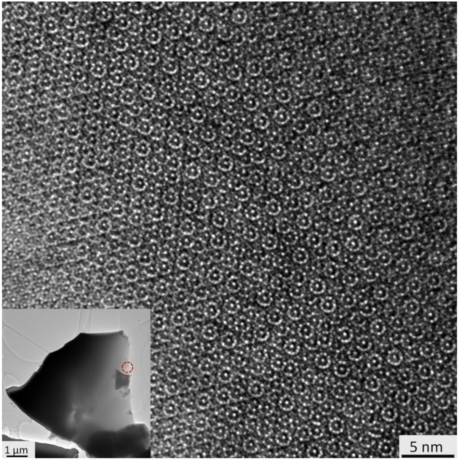
Penrose Tiling (1974)



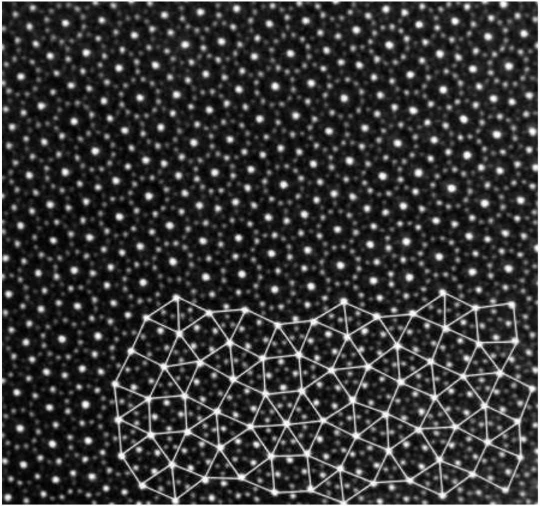
Motivation

- Rapid development of aperiodic tilings started after discovery of *quasicrystals* in 1982 by Dan Shechtman (Nobel prize in 2011);
- The atomic arrangement of a quasicrystal breaks the periodicity (no translational symmetry);
- Due to specific local structure of these materials the growth process of such crystals is still poorly understood.

Quasicrystals



Quasicrystals



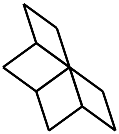
Question

Is it possible to grow an aperiodic tiling *locally*?

The meaning of the locality constraint:

- units of the growing cluster must be added one by one;
- decisions are local, i.e. according to tiles within a fixed distance;
- no information must be stored between the steps.

It is easy to make a mistake



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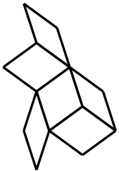

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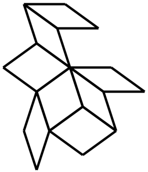

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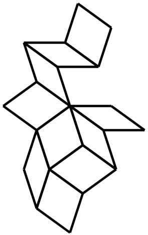

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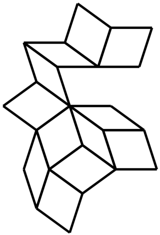

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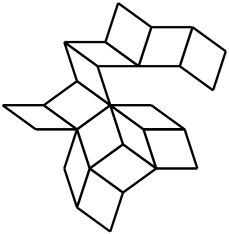

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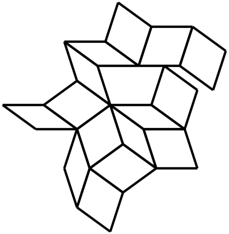
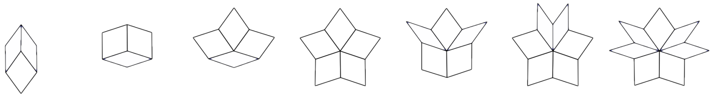


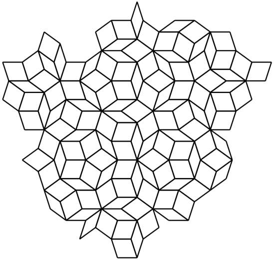
Vertex-atlas and Local rules

- Vertex-atlas $\mathcal{A}(r)$: all the patterns of radius r;
- Local rules: a finite set of patterns that characterize the tiling.



Main Obstacle: Deceptions

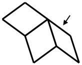
- Deceptions: patterns allowed by local rules which cannot be extended to a tiling of the entire plane;

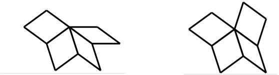


Theorem (Dworkin, Shieh, 1995)

Deceptions exist for all aperiodic tilesets.

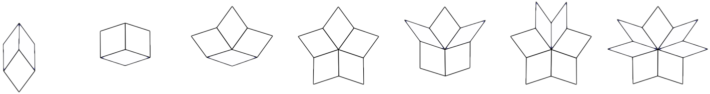
Possible Solution: avoid making choices





(a) (b)

- (a) is allowed;
- (b) is forbidden.



Self-Assembly Algorithm (Socolar, 1991)

- Start with a finite pattern of Penrose tiling;
- Keep adding the forced tiles one by one until it is possible;
- When there are none left, add a thick tile to a *special* site;
- Repeat.

Theorem (Socolar, 1991)

The algorithm can build any Penrose tiling.

Self-Assembly Algorithm (Socolar, 1991)

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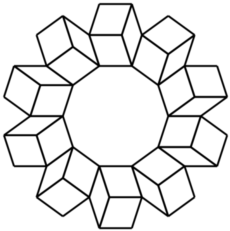
Theorem (Socolar, 1991)

The algorithm can build any Penrose tiling.

- However, this algorithm is *not* local.

Defective Seeds

With a *correct* seed it is impossible to get all the tiles, but with a *defective* seed one can grow a tiling of the entire plane except for a finite region!



The *decapod*, an example of such a seed for Penrose tiling.

Demonstration

Demonstration.

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Example: Planar $2 \to 1$ Tiling

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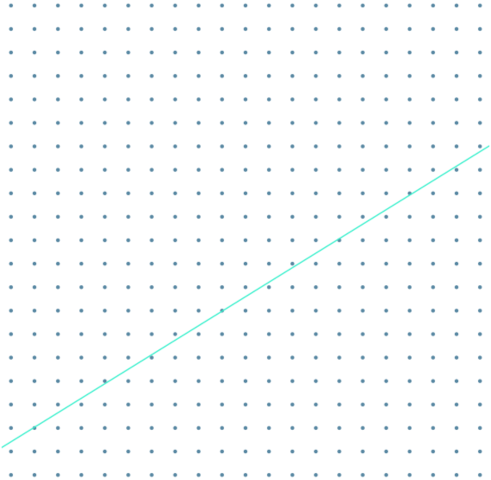
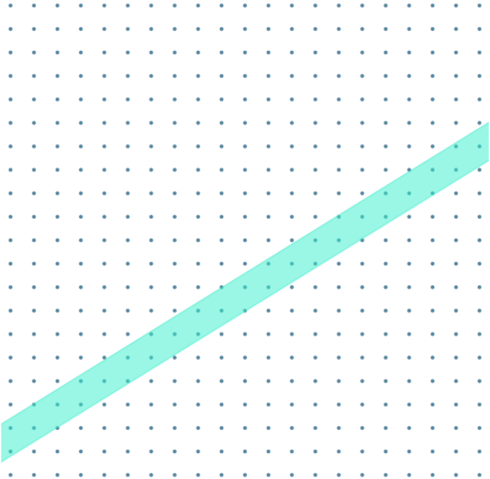
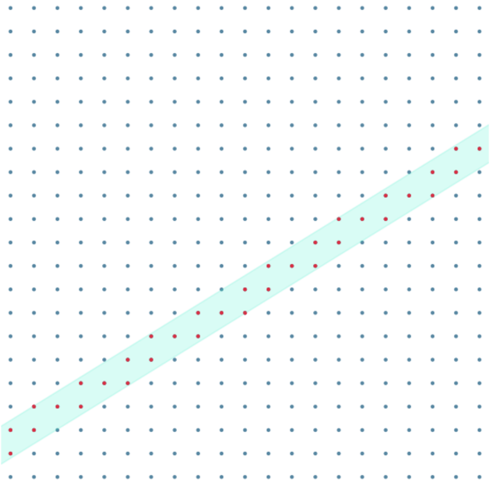


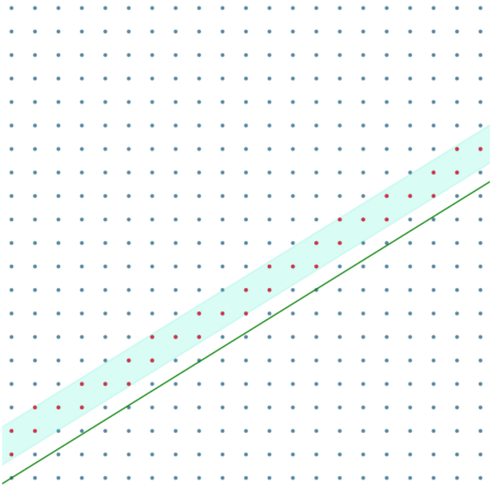
Example: Planar $2 \to 1$ Tiling



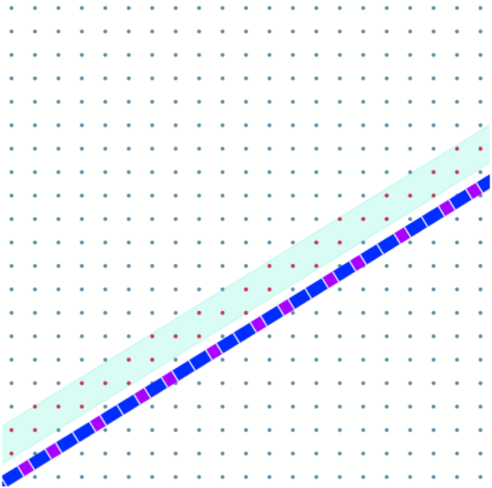
Example: Planar 2 $\to$ 1 Tiling



Example: Planar $2 \to 1$ Tiling



Example: Planar 2 $\to$ 1 Tiling



Example: Planar $2 \to 1$ Tiling

Cut-and-project tilings

Definition

Let E be a d-dim. affine space in $\mathbb{R}^n$ called the slope.
Select the d-dim. faces with vertices in $\mathbb{Z}^n$ lying in $E + [0,1]^n$.
Project them onto E to get a so-called *planar* $n \to d$ tiling.

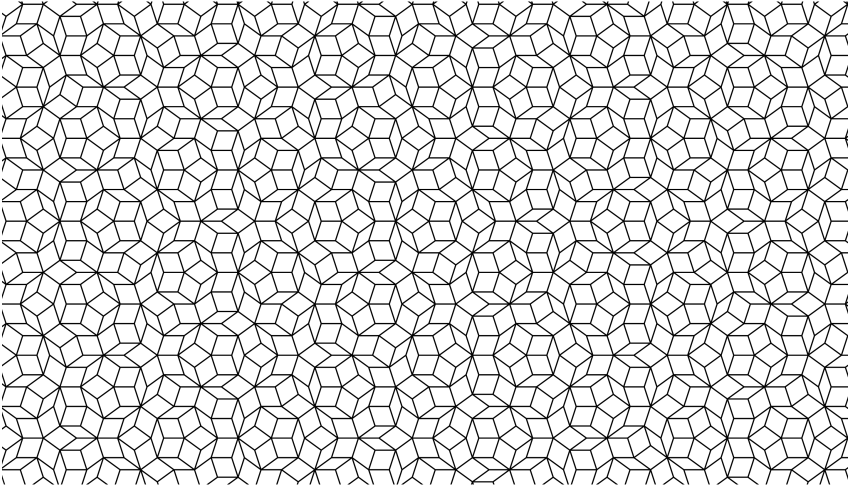
Cut-and-project tilings

Theorem (De Bruijn, 1981)

Penrose tiling is planar $5 \to 2$ *with the slope generated by*

$$u = \begin{pmatrix} 1 \\ \cos(2\pi/5) \\ \cos(4\pi/5) \\ \cos(6\pi/5) \\ \cos(8\pi/5) \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ \sin(2\pi/5) \\ \sin(4\pi/5) \\ \sin(6\pi/5) \\ \sin(8\pi/5) \end{pmatrix}$$

Example: Penrose Tiling

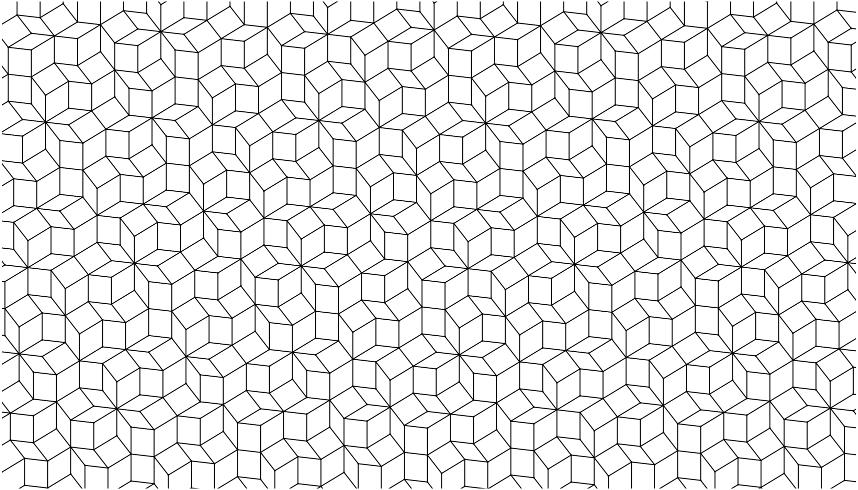


Example: Golden-Octagonal

Golden-Octagonal tiling is planar $4 \to 2$ with the slope generated by

$$u = \begin{pmatrix} -1 \\ 0 \\ \phi \\ \phi \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 1 \\ \phi \\ 1 \end{pmatrix}$$

Example: Golden-Octagonal

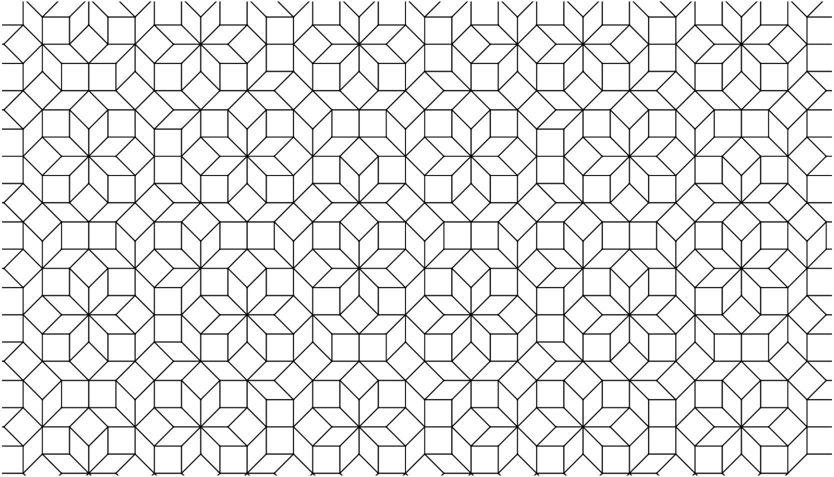


Example: Ammann-Beenker

Ammann-Beenker tiling is planar $4 \to 2$ with the slope generated by

$$u = \begin{pmatrix} 1 \\ \cos(\pi/4) \\ \cos(2\pi/4) \\ \cos(3\pi/4) \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ \sin(\pi/4) \\ \sin(2\pi/4) \\ \sin(3\pi/4) \end{pmatrix}$$

Example: Ammann-Beenker



Local Rules

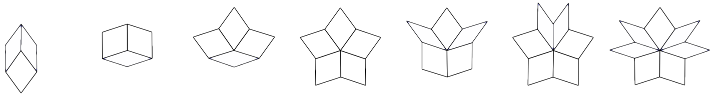
Definition (Local rules)

A d-plane $E \subset \mathbb{R}^n$ is said to admit *local rules* if there exists a vertex-atlas $\mathcal{A}(r)$ so that any $n \to d$ tiling with the same atlas is planar with the slope parallel to E.

Theorem (Bedaride, Fernique, 2017)

A planar $4 \to 2$ tiling admits local rules if and only if it is determined by its subperiods (easily checked on the generating vectors).

Examples



- Penrose tillings have local rules.
- Golden-Octagonal tilings have local rules.
- Ammann-Beenker tilings do not have local rules!

Proposition

In order to have a local self-assembly algorithm for a planar tiling it is necessary for the slope of the tiling to admit local rules.

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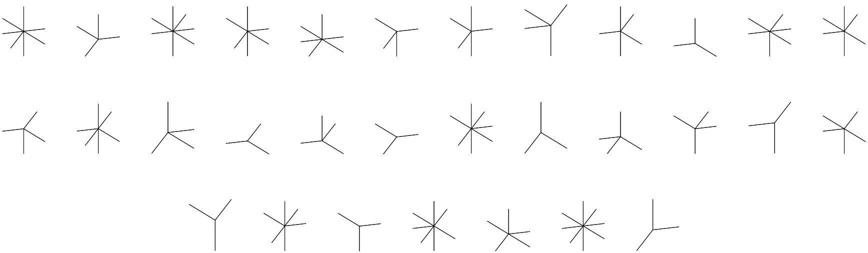
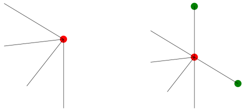
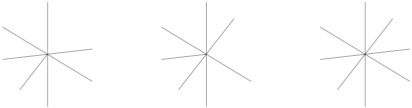
1-Atlas of Golden-Octagonal Tilings



Forced Vertex Example:





Local Algorithm

Given $r > 0$, a vertex-atlas $\mathcal{A}(r)$ and a finite pattern S:

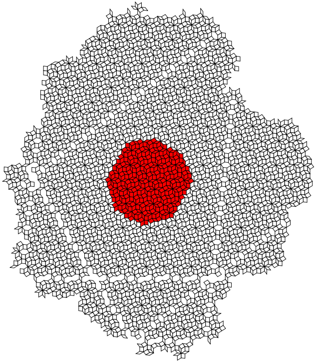
- pick at random a vertex v in S and let $P(v, r)$ be the subpattern of radius r and center v;
- consider the set F of all the elements in the vertex-atlas $\mathcal{A}(r)$ that *matches* with the subpattern $P(v, r)$;
- add to S all the vertices that appear in every pattern of F;
- Repeat.

Main Conjecture

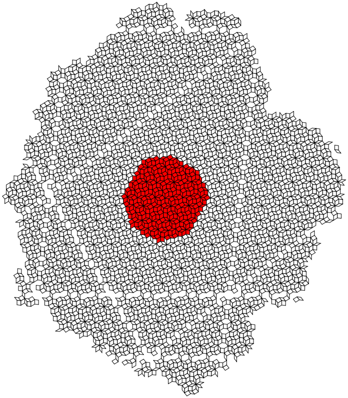
Conjecture

For a planar tiling $\mathcal{T}$ with local rules, a seed S, and a big enough vertex-atlas, the algorithm generates the intersection of all the tilings with slopes parallel to the slope of $\mathcal{T}$ which have S as a subset.

Golden-Octagonal Growth



Golden-Octagonal Growth



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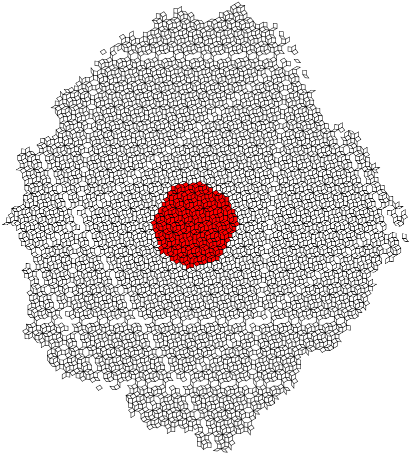
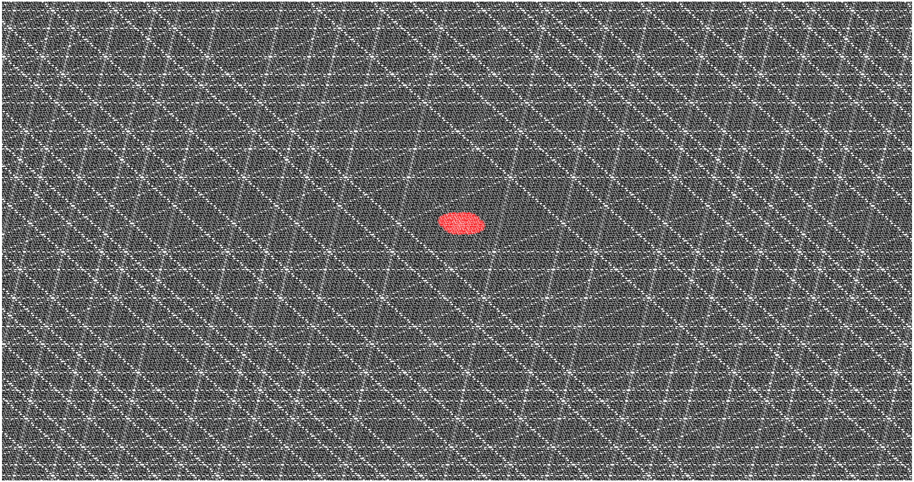
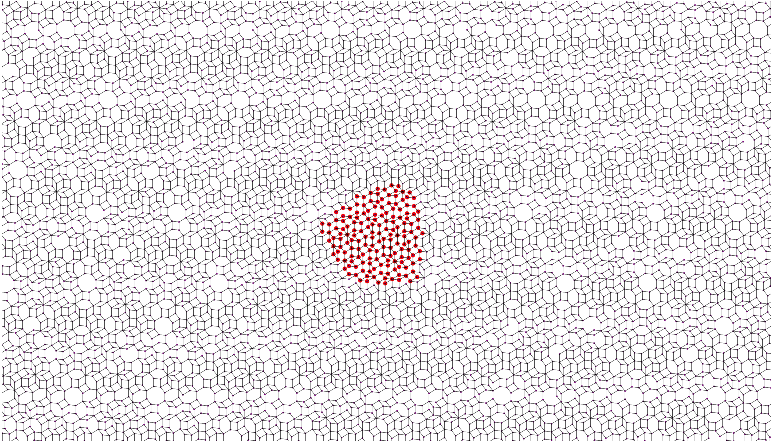


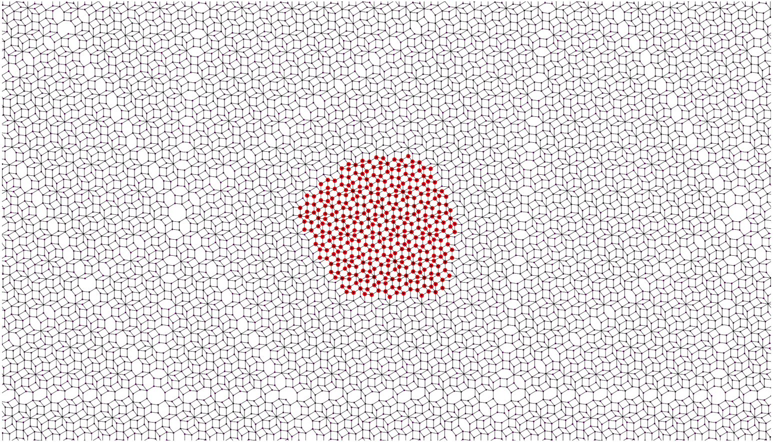
Golden-Octagonal Growth



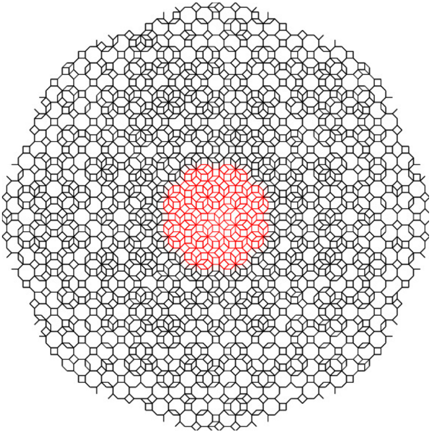
Smaller Seed



Bigger Seed



Ammann-Beenker



Ammann-Beenker tiling does not have local rules and will not grow.

Conclusions

- Infinite growth is observed for infinite family of tilings
- Algorithm permits to jump over undefined tiles and avoid being stuck
- The algorithm is local but it misses some tiles (*Conway worms*)
- Bigger seed $\rightarrow$ bigger proportion of the plane covered

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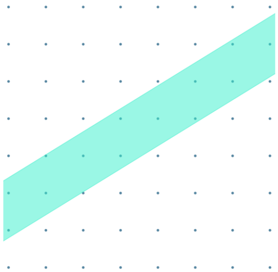
Window

Definition (Window)

The *window* W of a planar tiling with a slope $E \subset \mathbb{R}^n$ is the orthogonal projection of $[0,1]^n$ onto $E^\perp$, where $E^\perp$ is a complementary space to E

$$W = \pi^\perp([0,1]^n).$$



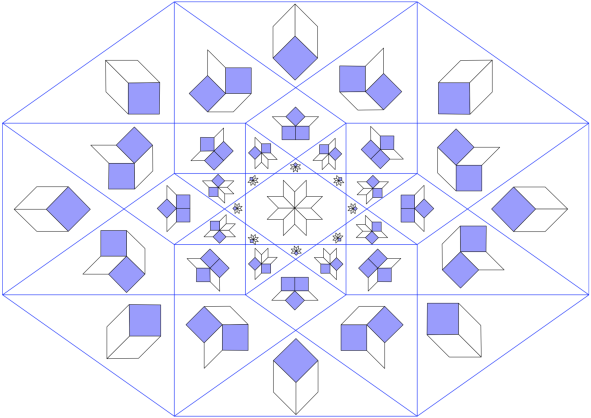
Regions in the Window

Proposition

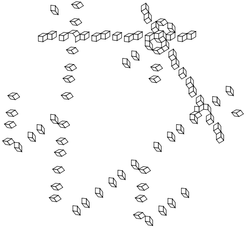
To every pattern of a tiling we can assign a region in the window:

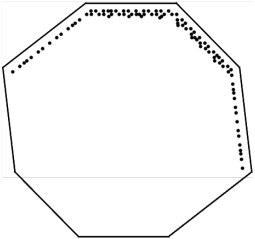
$$R(P) = \bigcap_{x:\pi(x)\in P} (W - \pi^{\perp}(x)).$$

Subregions in the window



Golden-Octagonal





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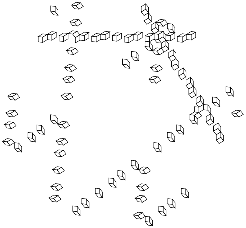

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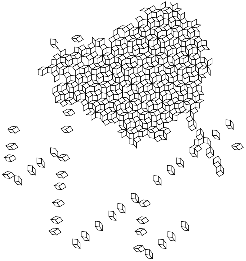

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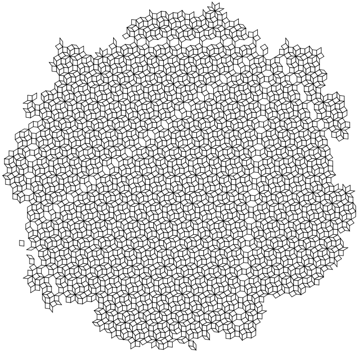

Conclusions

- Empty stripes consist of patterns which are close to the border of the window when projectied to the perpendicular space
- Bigger seed → more information about the position of the window → bigger proportion of the plane covered

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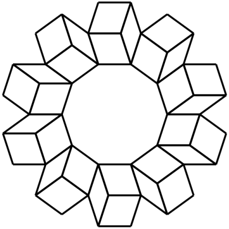
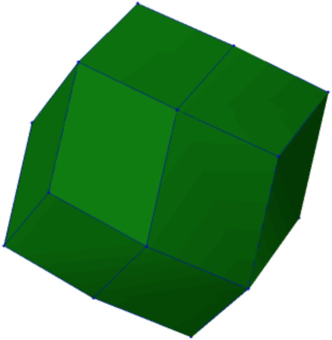
Reminder: Defective Seed for Penrose Tilings



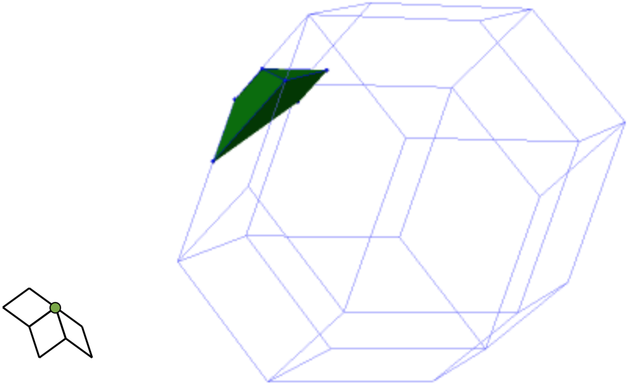
- Growth starting from the decapod covers the entire plane except for finite (and untileable) region in the center

Window

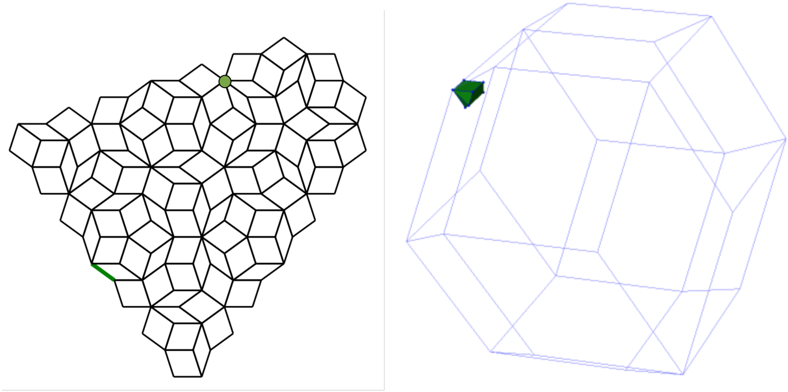


The window for Penrose tiling.

Examples



Examples

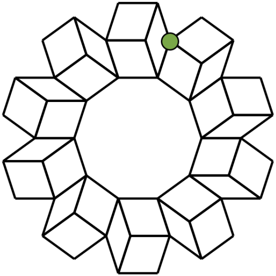


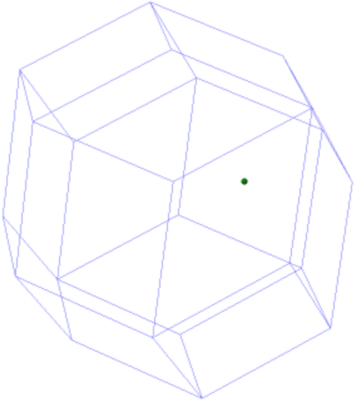
Examples

$$R(tiling) = \{point\}.$$

Examples

$$R(decapod) = \{point\}$$





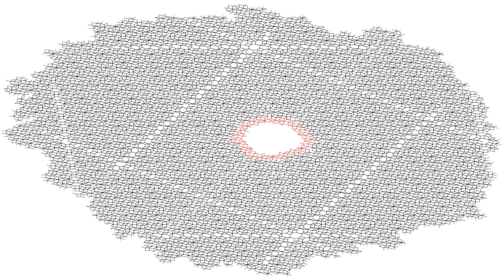
Defective Seeds For Tilings with Local Rules

Lemma

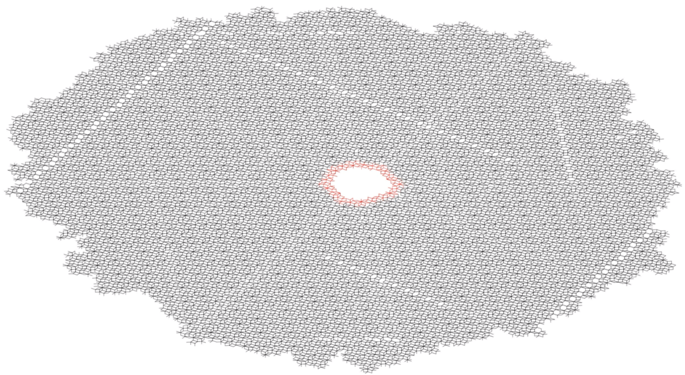
For any tiling with local rules $\mathcal{T}$ and for any $R > \lceil \max(||p_i||_1) \rceil$, where $\{p_i\}$ is the set of subperiods of $\mathcal{T}$, there exist a seed D with following properties:

- *every subpattern of D of radius R is correct (i.e. it is a subset of a tiling with the same slope)*
- $R(D) = \{point\}$

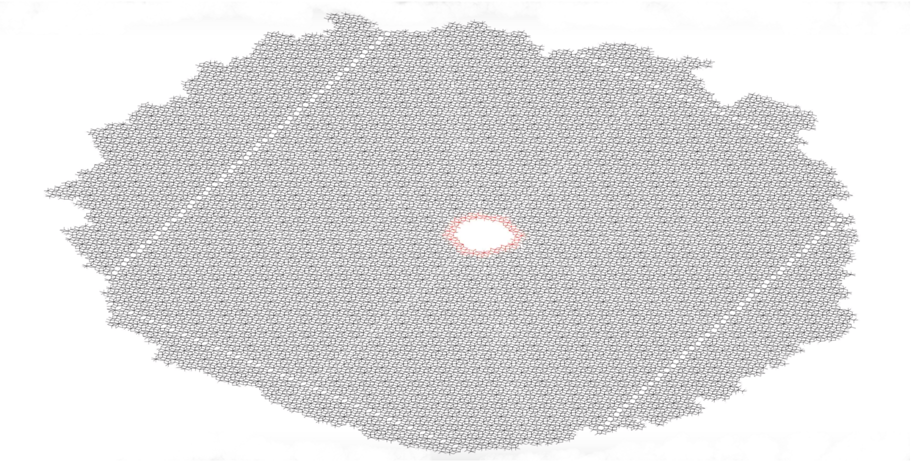
Golden-Octagonal Defective Seed



Golden-Octagonal Defective Seed



Golden-Octagonal Defective Seed



Defective Seeds

Conjecture

For all the planar tilings with local rules there is a set of defective seeds such that the growth with such seeds will produce a tiling of the entire plane except for a finite region.

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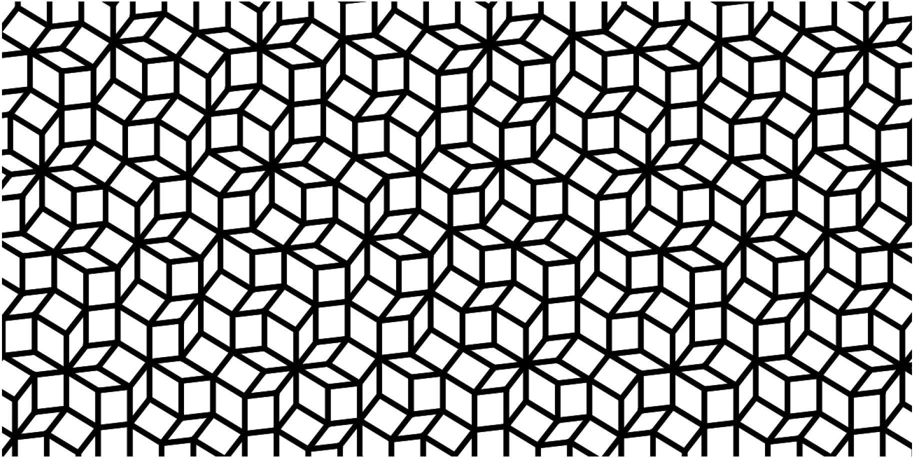
Shadows



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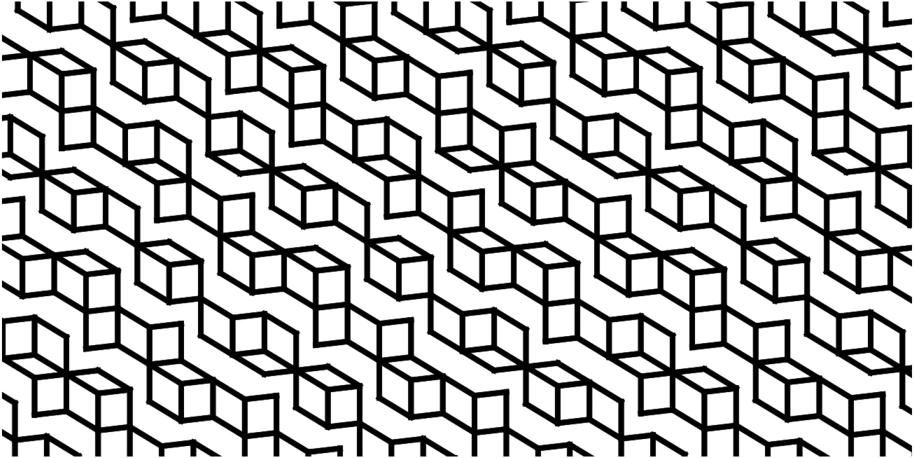

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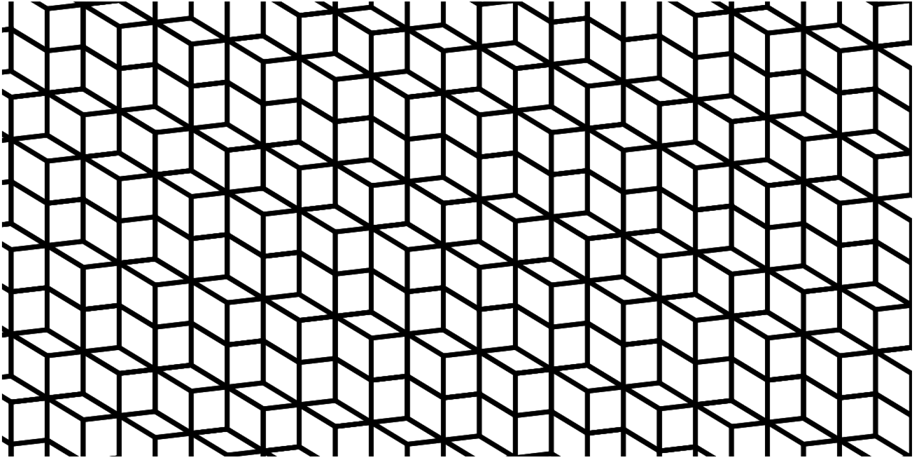
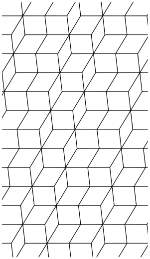


Shadows

Definition

The *ijk*-shadow of a $4 \to 2$ planar tiling is the orthogonal projection of its *lift* to the space generated by e_i, e_j and e_k.



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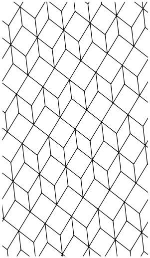

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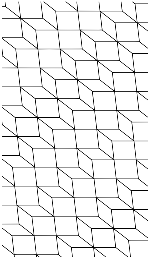

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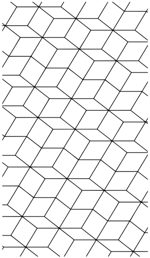

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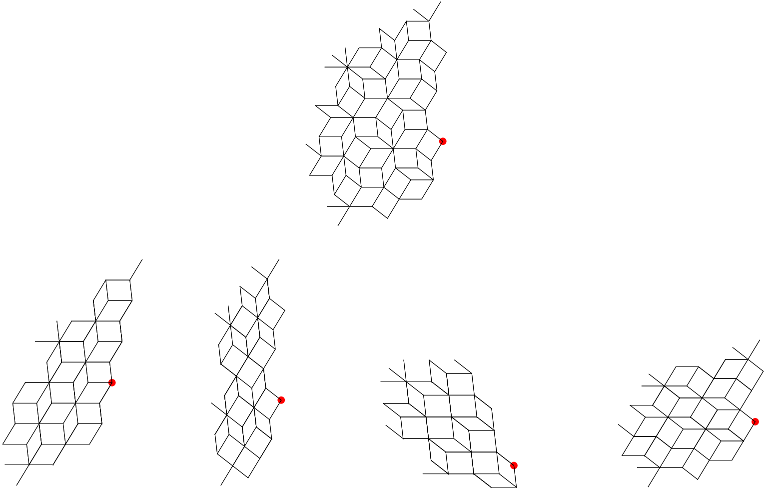
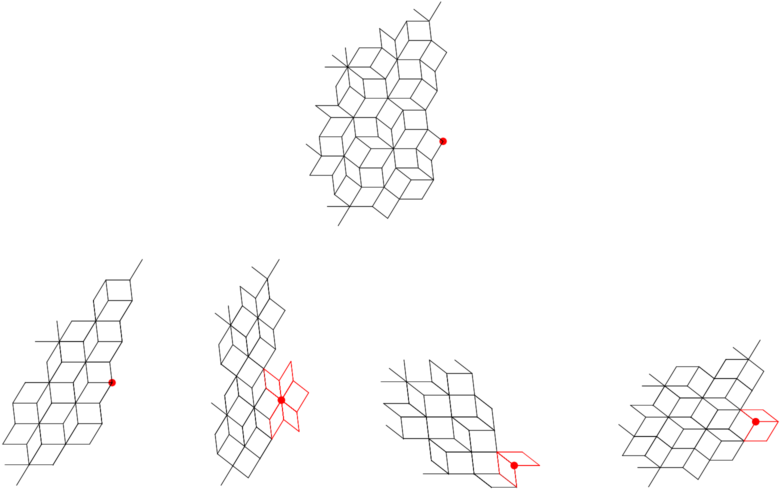
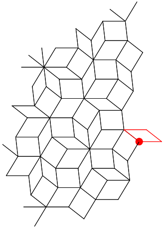
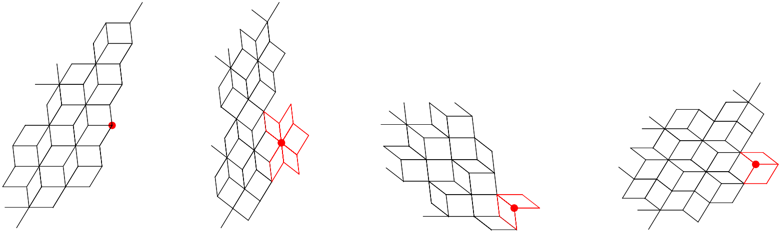


Shadows Can Vote!



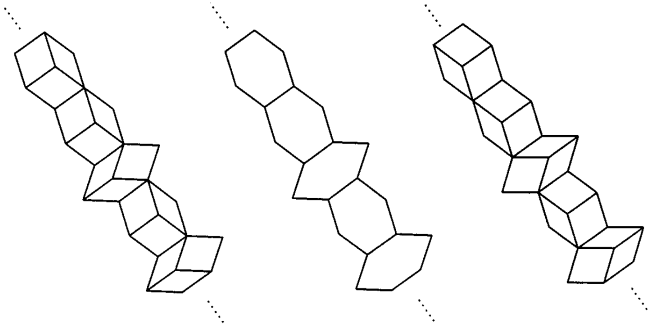
Shadows Can Vote!



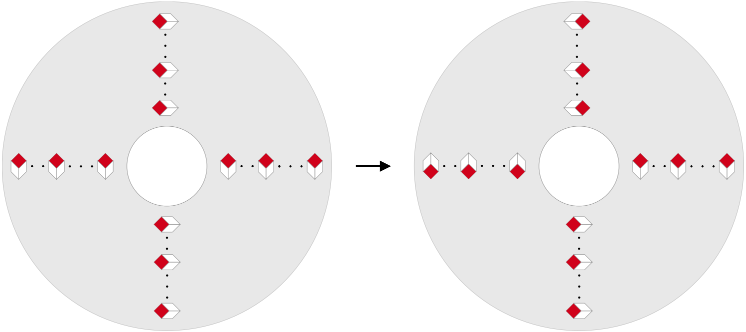


Thank you for your attention!

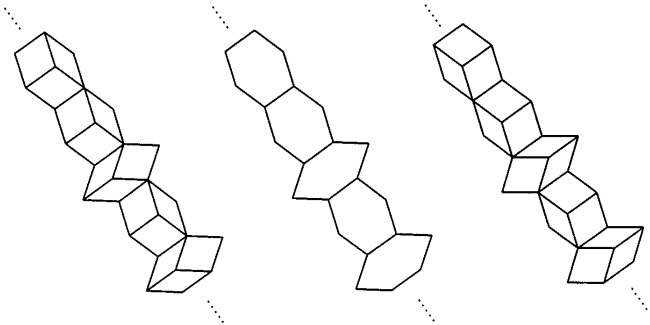
Conway worms



now to Construct The Defective Seeds?



Conway worms



Marginal sites