A Compact packings

Figure 17 acts as a map, showing the distribution of the 164 cases of Theorem 1. A periodic compact packing is then depicted for each case. The letter in brackets refers to the *type* of the compact packing, see Appendix B. The codings of a small and a medium corona from which the values of r and s can be computed is given top-right of each picture. Numbers 1–18 are large separated packings, number 19 is the unique which admits two small coronas and numbers 20–164 are those which admit a unique small corona (besides ssssss, as usual). In some cases, the small discs are very small and barely visible (numbers 32, 37–40, 41–44): the small corona however indicates where they are.

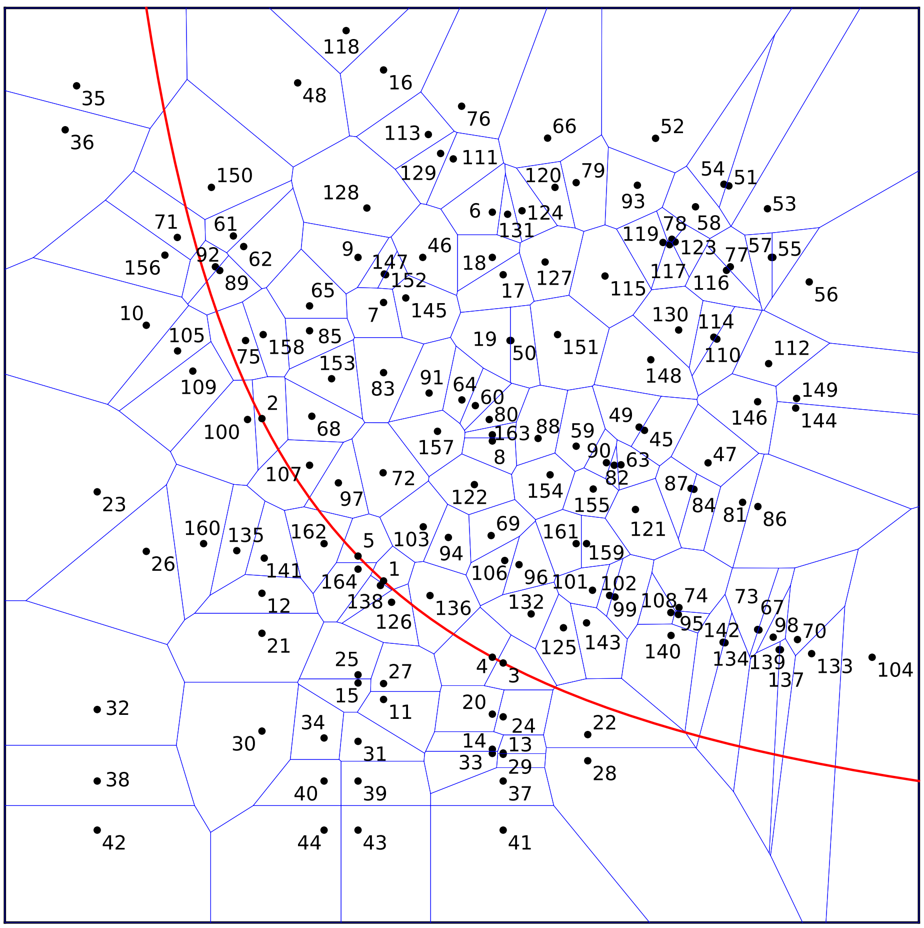
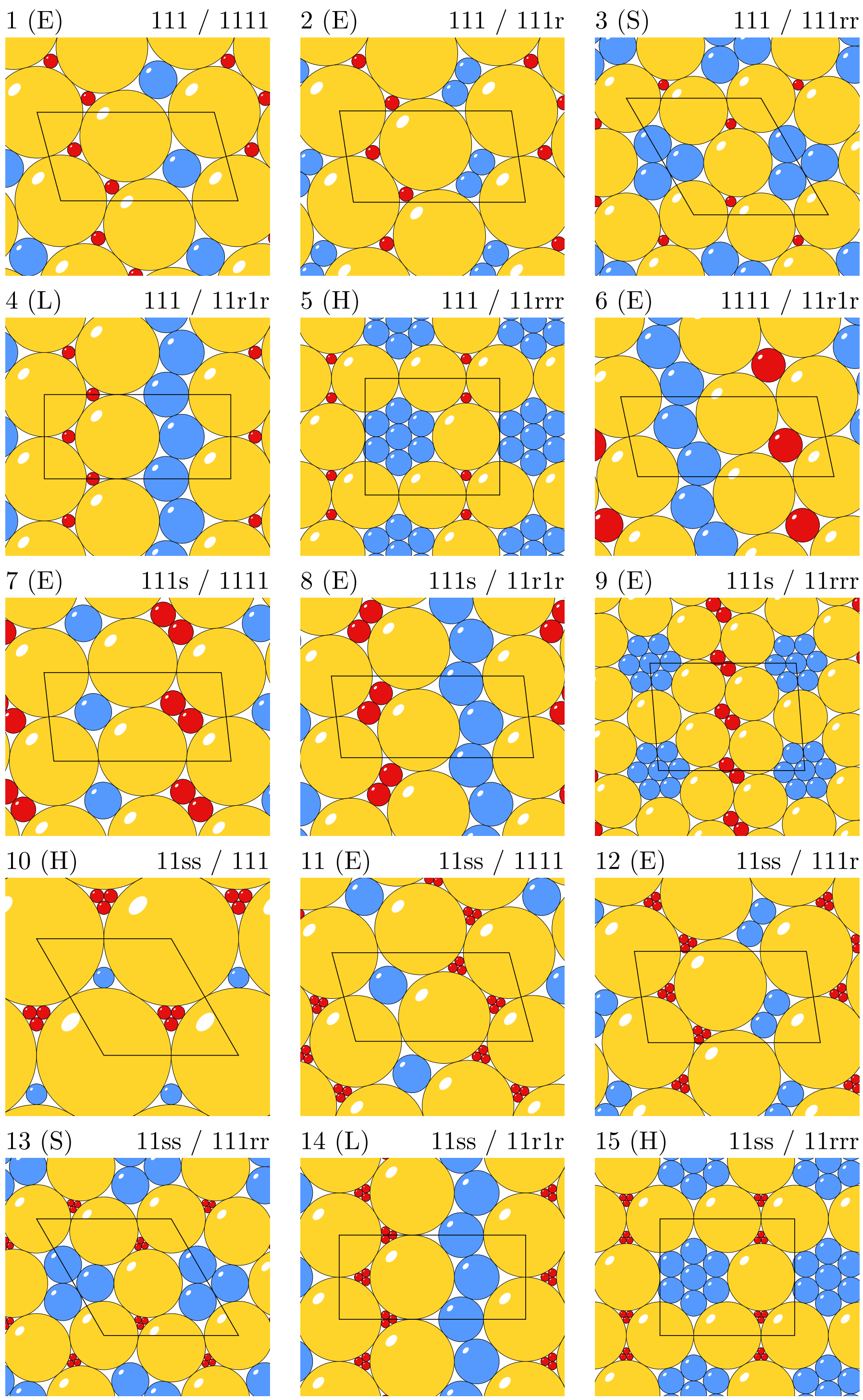
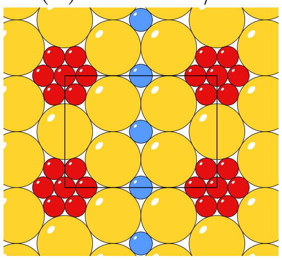


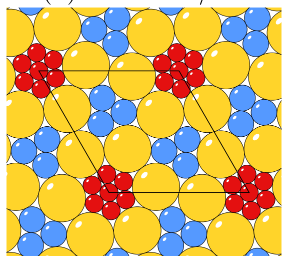
Figure 17: The 164 pairs (r, s), with abscissa r and ordinate $\frac{s}{r}$. Those below the hyperbola are such that an s-disc fits in the hole between three 1-discs (there are often derived from two disc packings). Voronoï cells just aim to give an idea of how close are two pairs.


1 (E) 111 / 1111
2 (E) 111 / 111r
3 (S) 111 / 111rr
4 (L) 111 / 11r1r
5 (H) 111 / 11rrr
6 (E) 1111 / 11r1r
7 (E) 111s / 1111
8 (E) 111s / 11r1r
9 (E) 111s / 11rrr
10 (H) 11ss / 111
11 (E) 11ss / 1111
12 (E) 11ss / 111r
13 (S) 11ss / 111rr
14 (L) 11ss / 11r1r
15 (H) 11ss / 11rrr

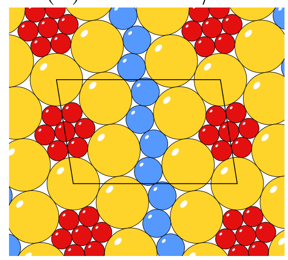
16 (E) 11sss / 1111



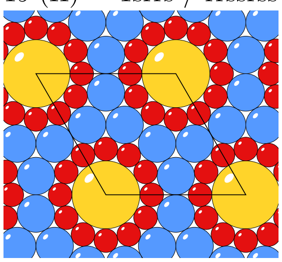
17 (S) 11sss / 111rr



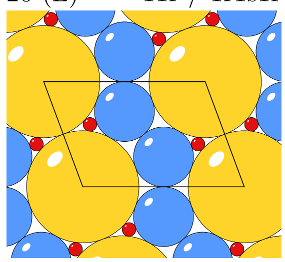
18 (L) 11sss / 11r1r



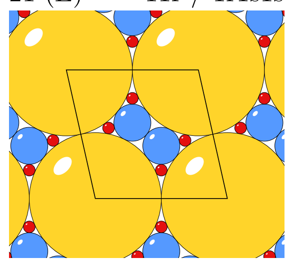
19 (H) 1srrs / rrssrss



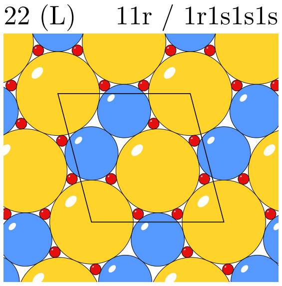
20 (L) 11r / 1r1s1r



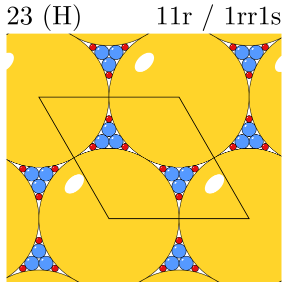
21 (E) 11r / 1r1s1s



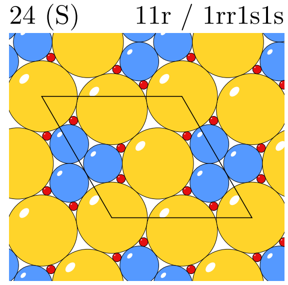
22 (L) 11r / 1r1s1s1s



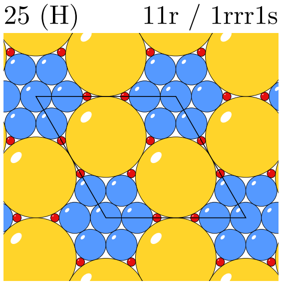
23 (H) 11r / 1rr1s



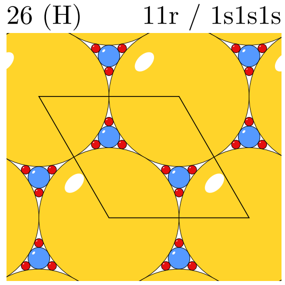
24 (S) 11r / 1rr1s1s



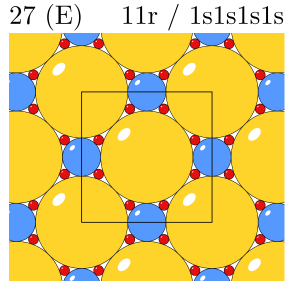
25 (H) 11r / 1rrr1s



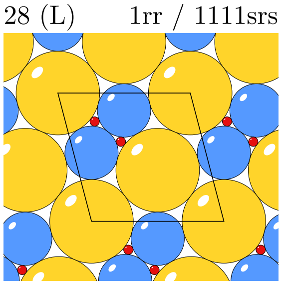
26 (H) 11r / 1s1s1s



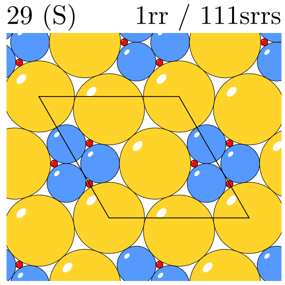
27 (E) 11r / 1s1s1s1s



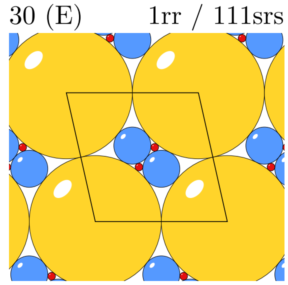
28 (L) 1rr / 1111srs



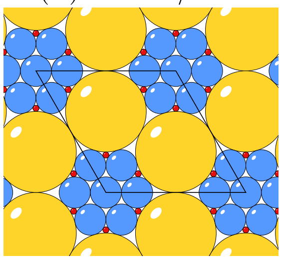
29 (S) 1rr / 111srrs



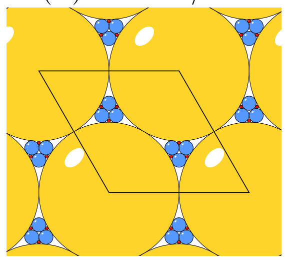
30 (E) 1rr / 111srs



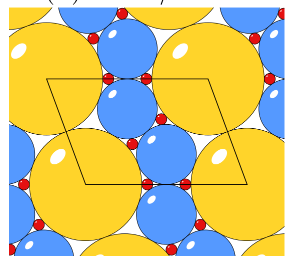
31 (H) 1rr / 11srrrs



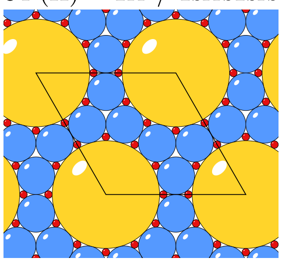
32 (H) 1rr / 11srrs



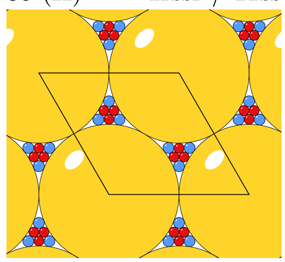
33 (L) 1rr / 11srs1srs



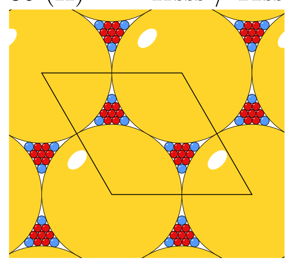
34 (H) 1rr / 1srrs1srs



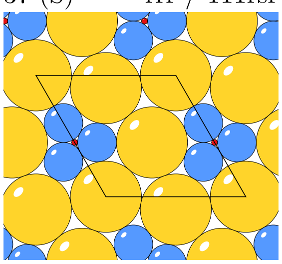
35 (H) 1rssr / 11ss



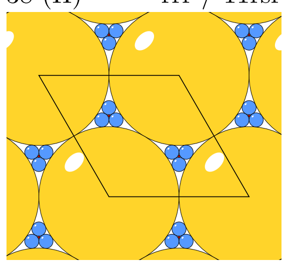
36 (H) 1rsss / 11ss



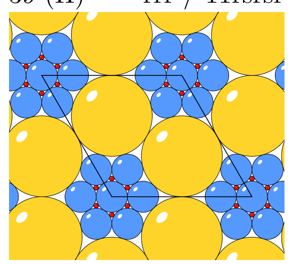
37 (S) rrr / 111rsr



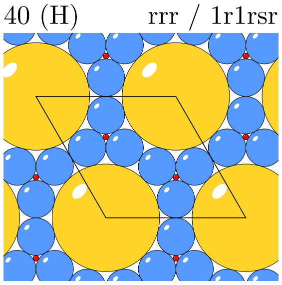
38 (H) rrr / 11rsr



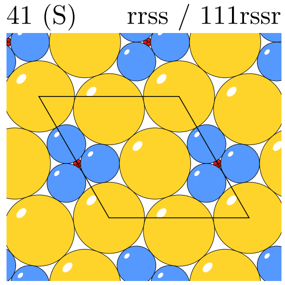
39 (H) rrr / 11rsrsr



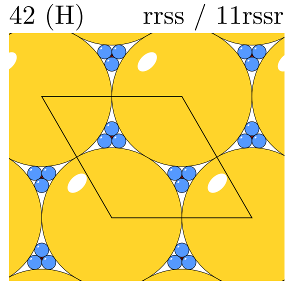
40 (H) rrr / 1r1rsr



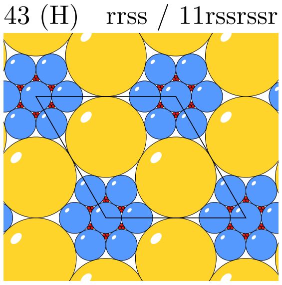
41 (S) rrss / 111rssr



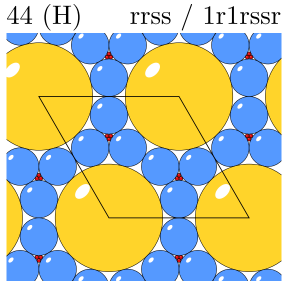
42 (H) rrss / 11rssr



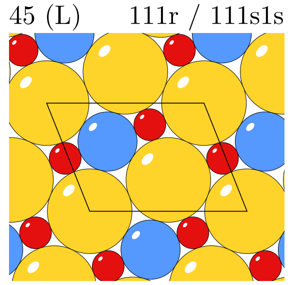
43 (H) rrss / 11rssrssr



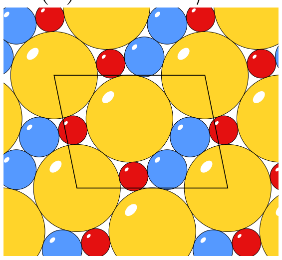
44 (H) rrss / 1r1rssr



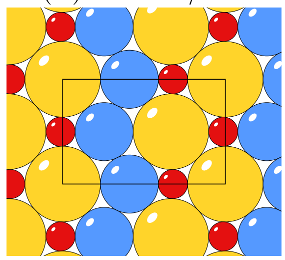
45 (L) 111r / 111s1s



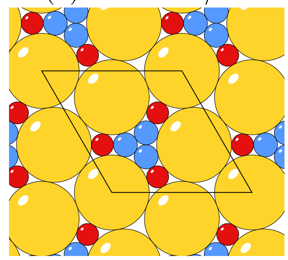
46 (L) 111r / 11r1s



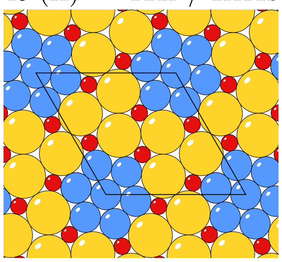
47 (H) 111r / 1r1r1s



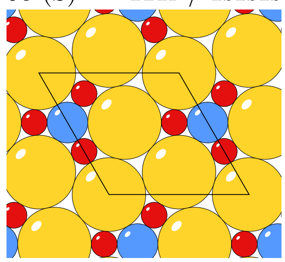
48 (S) 111r / 1rr1s



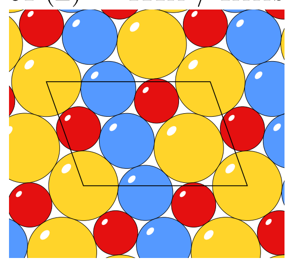
49 (H) 111r / 1rrr1s



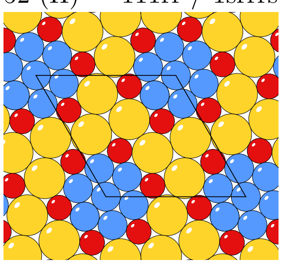
50 (S) 111r / 1s1s1s



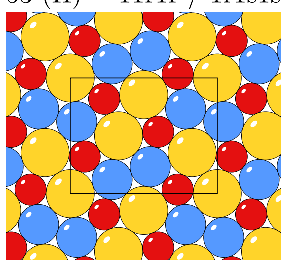
51 (L) 111rr / 1rrrrs



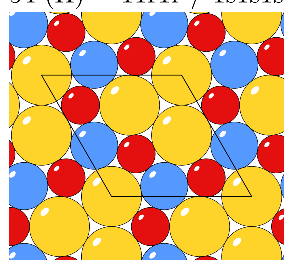
52 (H) 111rr / 1srrrs



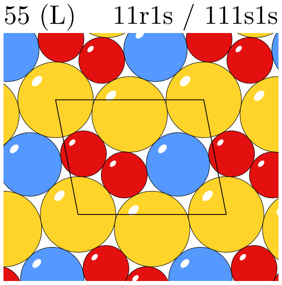
53 (H) 11r1r / 1r1s1s



54 (H) 11r1r / 1s1s1s



55 (L) 11r1s / 111s1s



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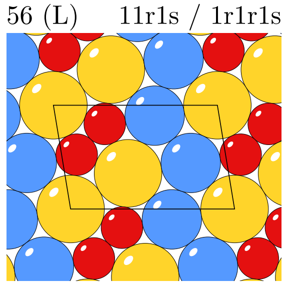
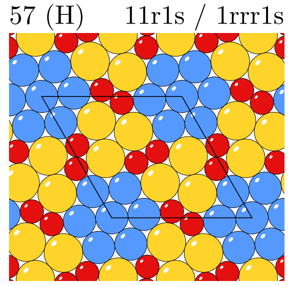
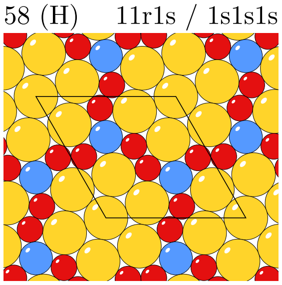


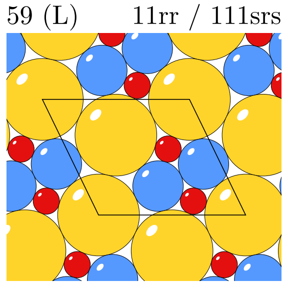
57 (H) 11r1s / 1rrr1s



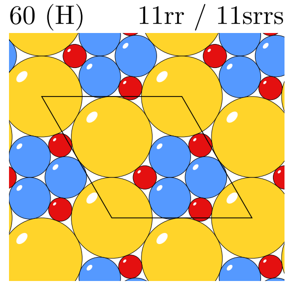
58 (H) 11r1s / 1s1s1s



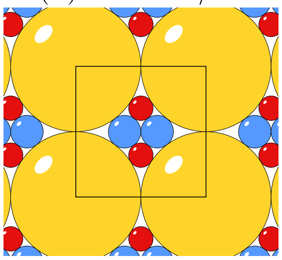
59 (L) 11rr / 111srs



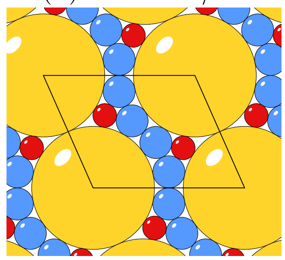
60 (H) 11rr / 11srrs



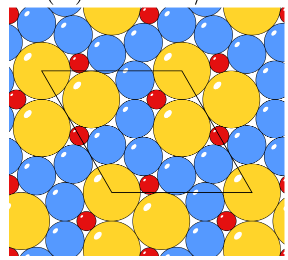
61 (E) 11rr / 11srs



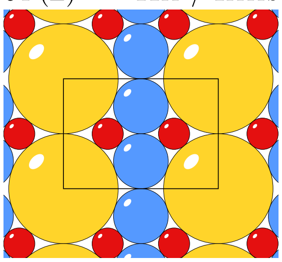
62 (L) 11rr / 1r1rs



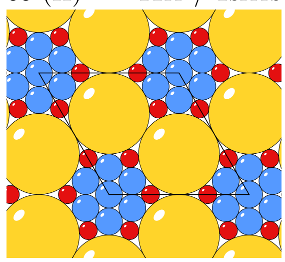
63 (H) 11rr / 1rr1rs



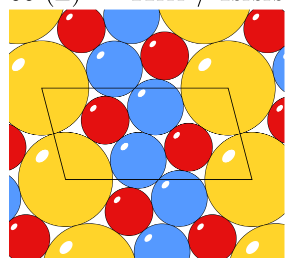
64 (L) 11rr / 1rrrrs



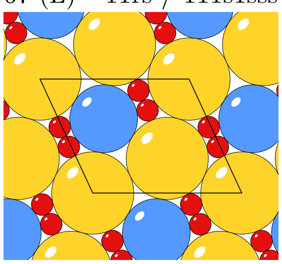
65 (H) 11rr / 1srrrs



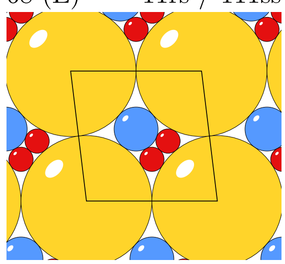
66 (L) 11rrr / 1srsrs



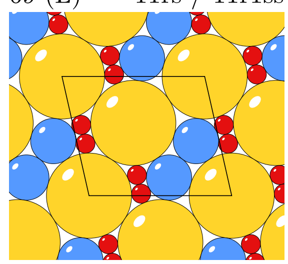
67 (L) 11rs / 111s1sss



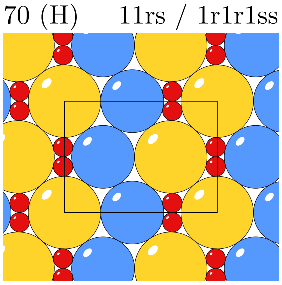
68 (E) 11rs / 111ss



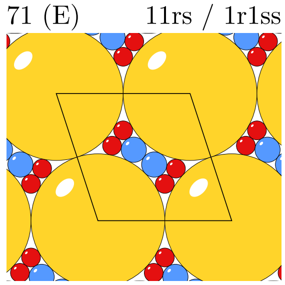
69 (L) 11rs / 11r1ss



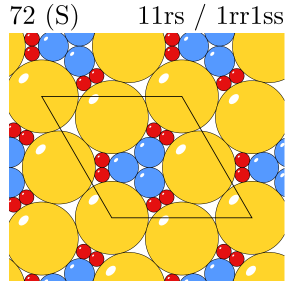
70 (H) 11rs / 1r1r1ss



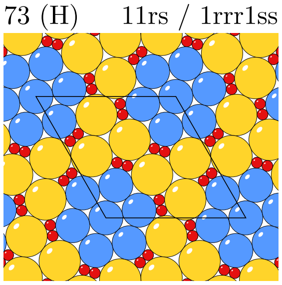
71 (E) 11rs / 1r1ss



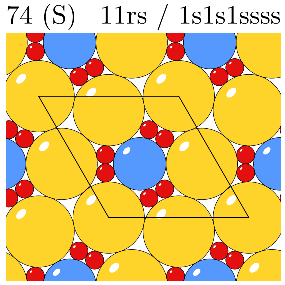
72 (S) 11rs / 1rr1ss



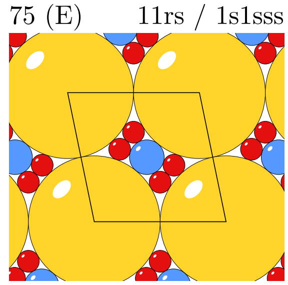
73 (H) 11rs / 1rrr1ss



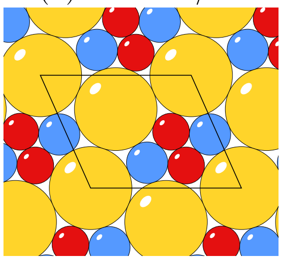
74 (S) 11rs / 1s1s1ssss



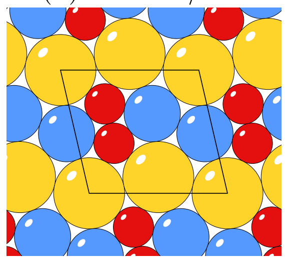
75 (E) 11rs / 1s1sss



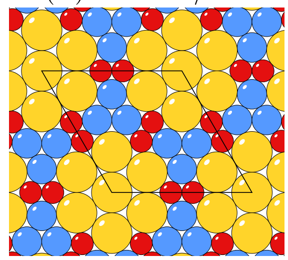
76 (L) 11rsr / 111ss



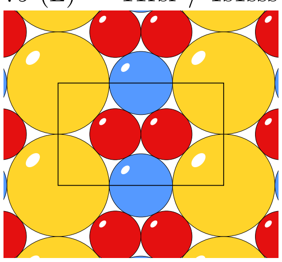
77 (L) 11rsr / 11r1ss



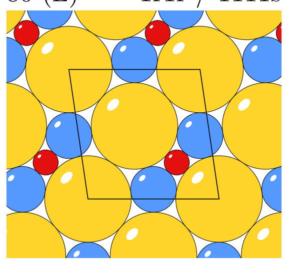
78 (H) 11rsr / 1rr1ss



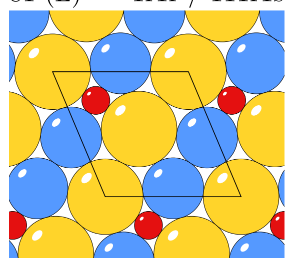
79 (L) 11rsr / 1s1sss



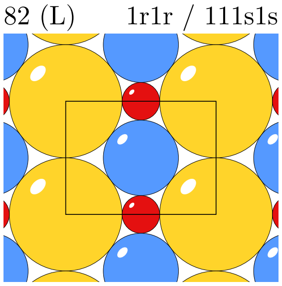
80 (L) 1r1r / 1111s



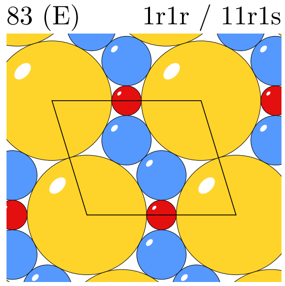
81 (L) 1r1r / 111r1s



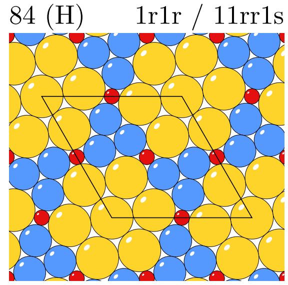
82 (L) 1r1r / 111s1s



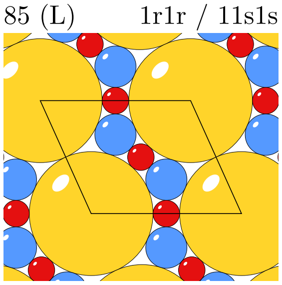
83 (E) 1r1r / 11r1s



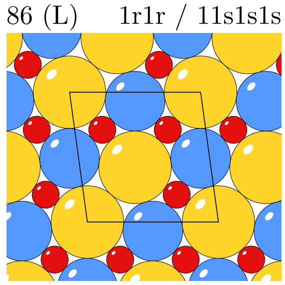
84 (H) 1r1r / 11rr1s



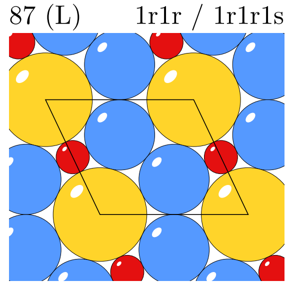
85 (L) 1r1r / 11s1s



86 (L) 1r1r / 11s1s1s



87 (L) 1r1r / 1r1r1s



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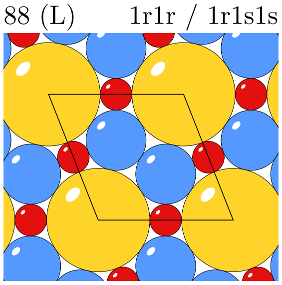
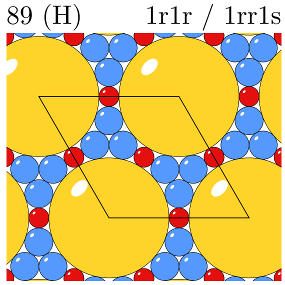
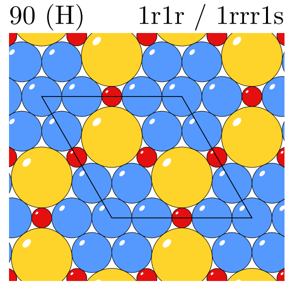


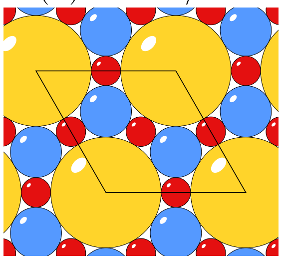
89 (H) 1r1r / 1rr1s



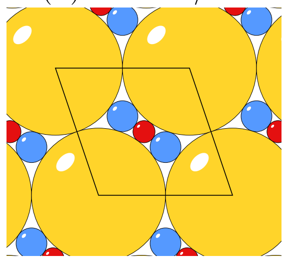
90 (H) 1r1r / 1rrr1s



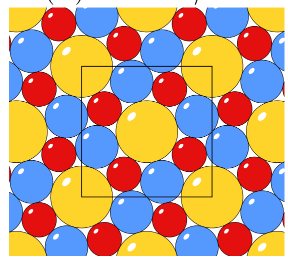
91 (H) 1r1r / 1s1s1s



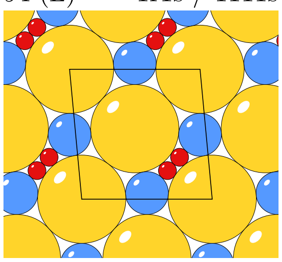
92 (E) 1r1r / 1srsrs



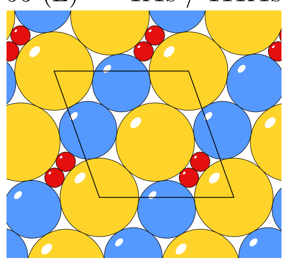
93 (H) 1r1rr / 1s1srs



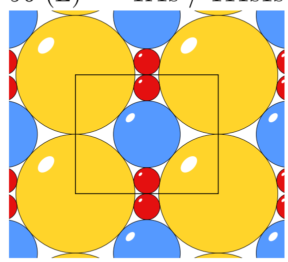
94 (L) 1r1s / 1111s



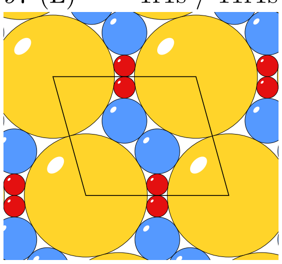
95 (L) 1r1s / 111r1s



96 (L) 1r1s / 111s1s



97 (L) 1r1s / 11r1s



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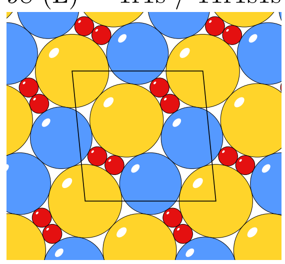

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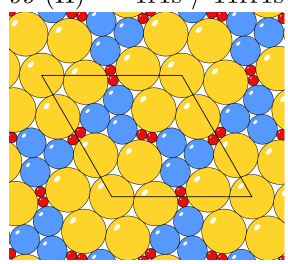

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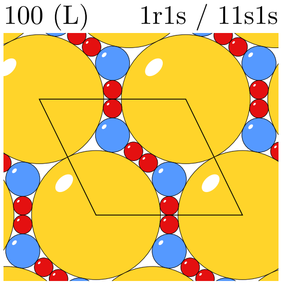

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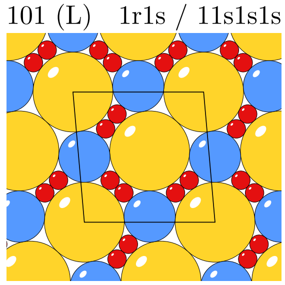
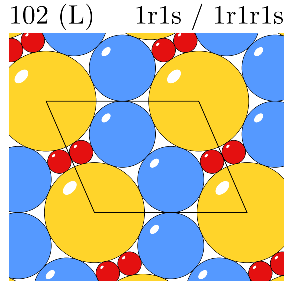


102 (L) 1r1s / 1r1r1s



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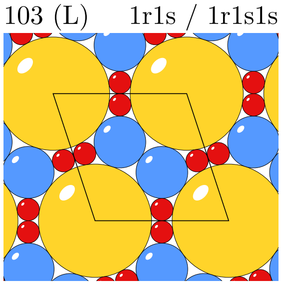

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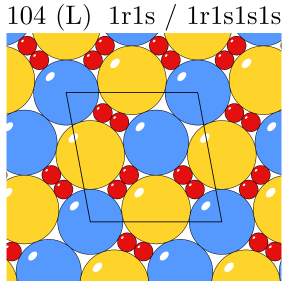

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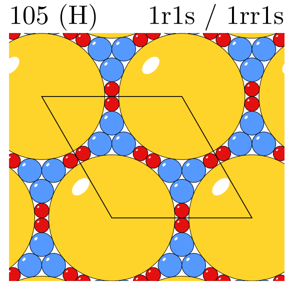

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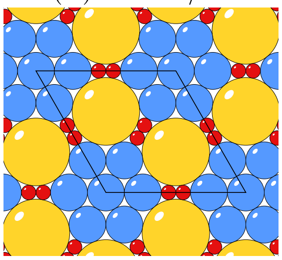

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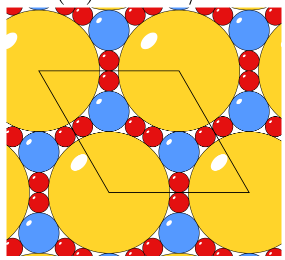

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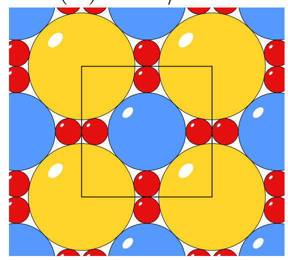
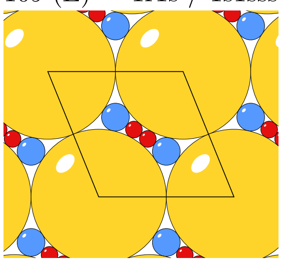
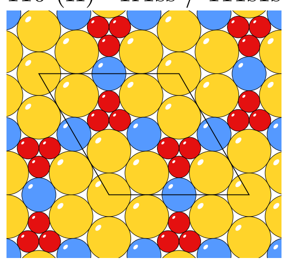


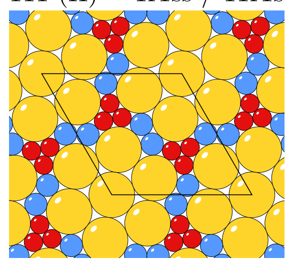
109 (E) 1r1s / 1s1sss



110 (H) 1r1ss / 111s1s



111 (H) 1r1ss / 11r1s



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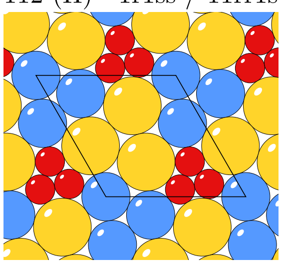

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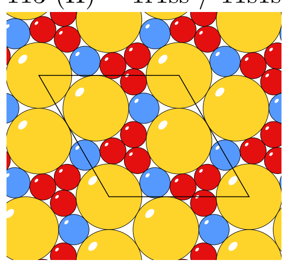

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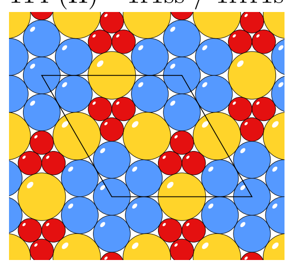

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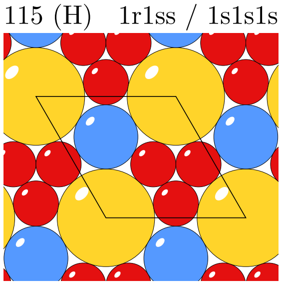

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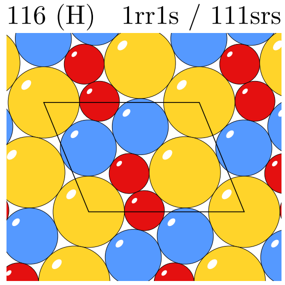

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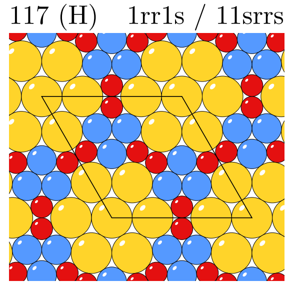
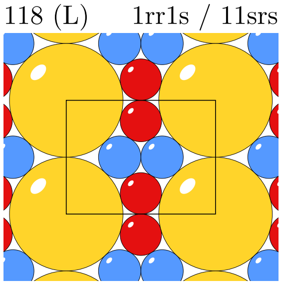


118 (L) 1rr1s / 11srs



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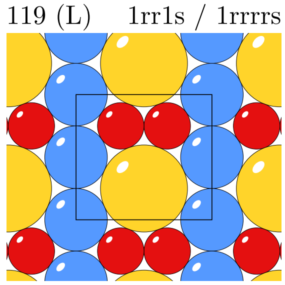

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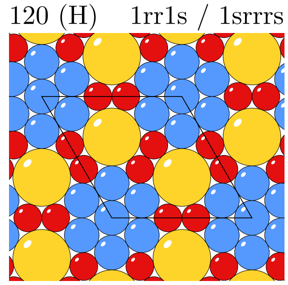

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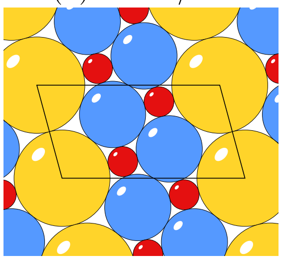

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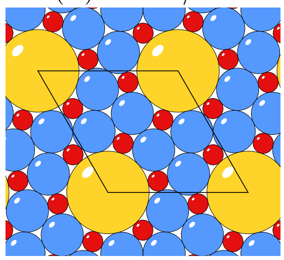

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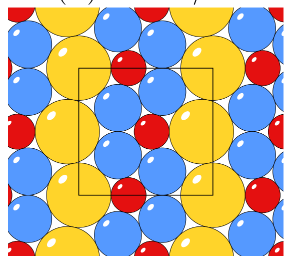

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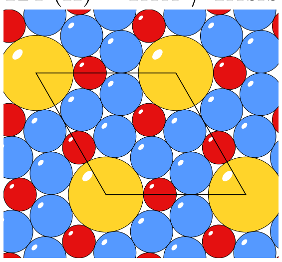

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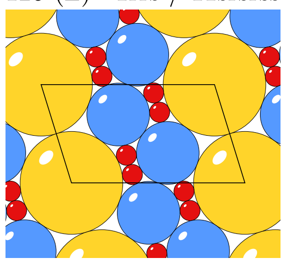

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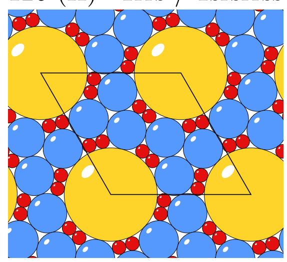
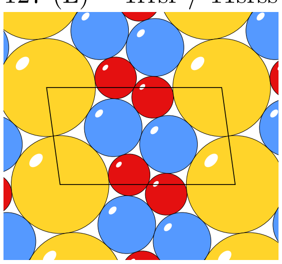
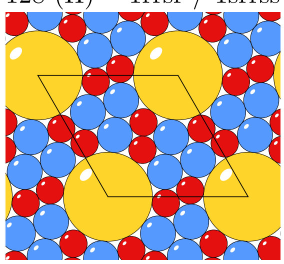


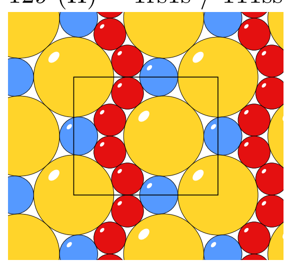
127 (L) 1rrsr / 11srss



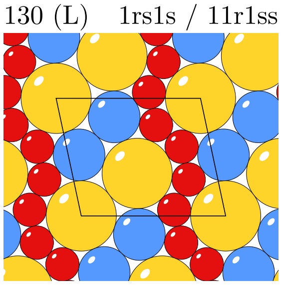
128 (H) 1rrsr / 1srrss



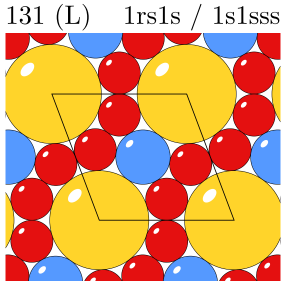
129 (H) 1rs1s / 111ss



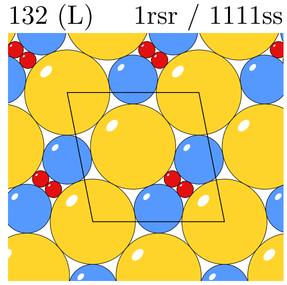
130 (L) 1rs1s / 11r1ss



131 (L) 1rs1s / 1s1sss



132 (L) 1rsr / 1111ss



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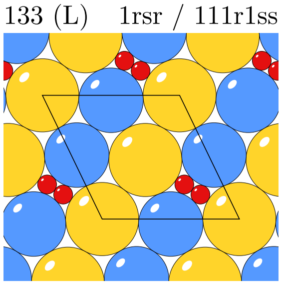
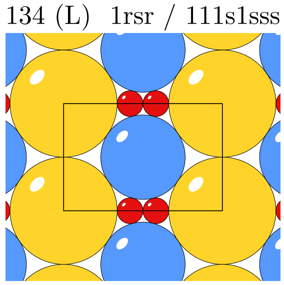
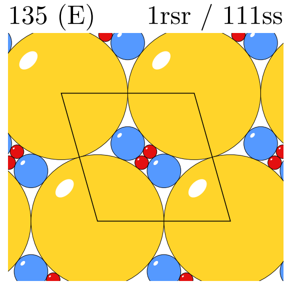


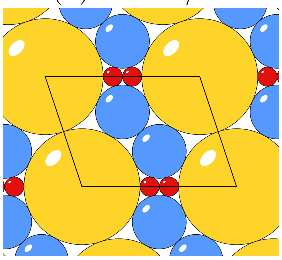
134 (L) 1rsr / 111s1sss



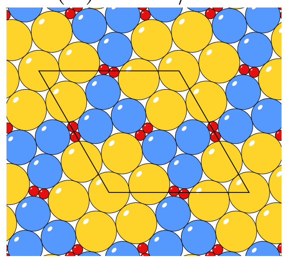
135 (E) 1rsr / 111ss



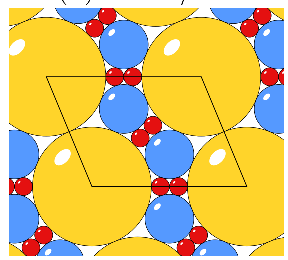
136 (L) 1rsr / 11r1ss



137 (H) 1rsr / 11rr1ss



138 (L) 1rsr / 11s1sss



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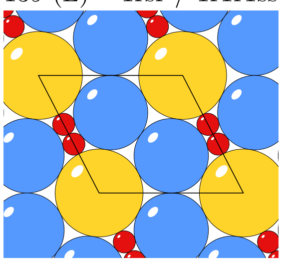

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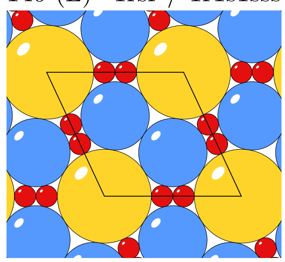

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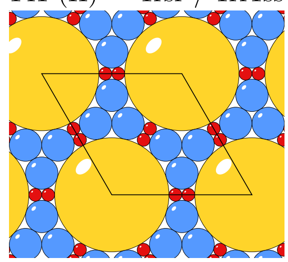

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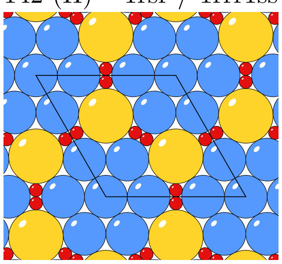

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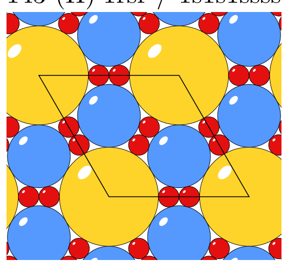
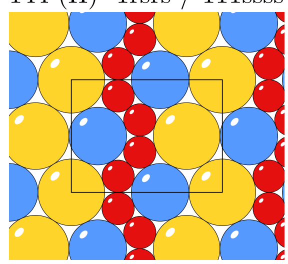


144 (H) 1rsrs / 111ssss



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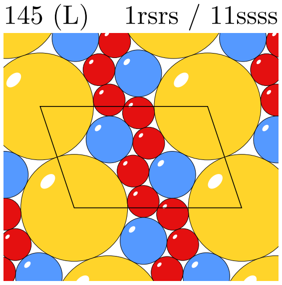

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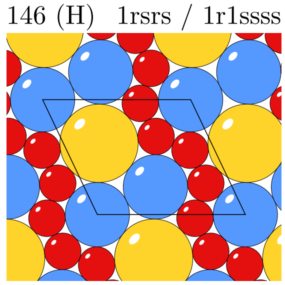
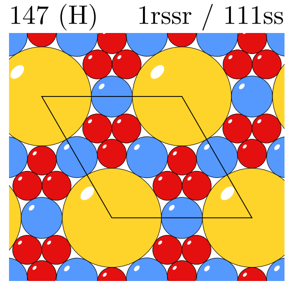
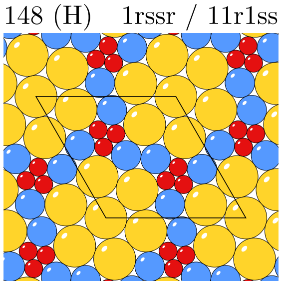


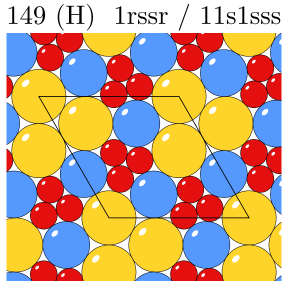
147 (H) 1rssr / 111ss



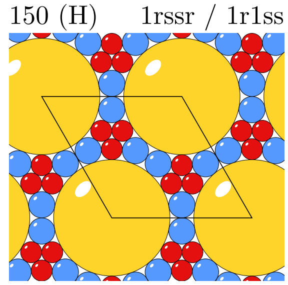
148 (H) 1rssr / 11r1ss



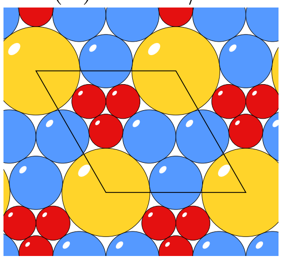
149 (H) 1rssr / 11s1sss



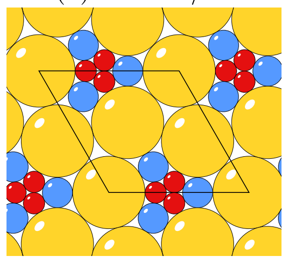
150 (H) 1rssr / 1r1ss



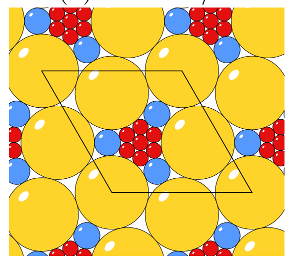
151 (H) 1rssr / 1rr1ss



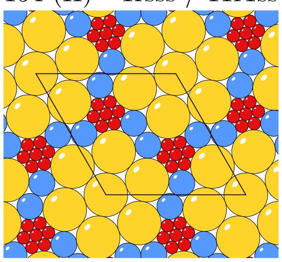
152 (S) 1rssr / 1s1sss



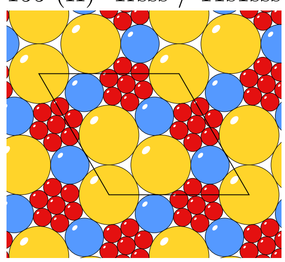
153 (S) 1rsss / 111ss



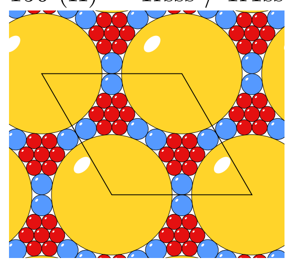
154 (H) 1rsss / 11r1ss



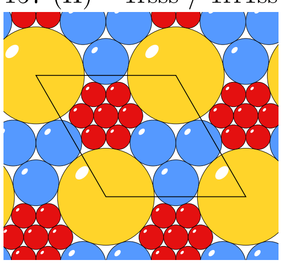
155 (H) 1rsss / 11s1sss



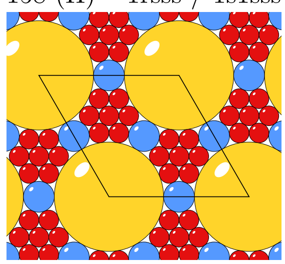
156 (H) 1rsss / 1r1ss



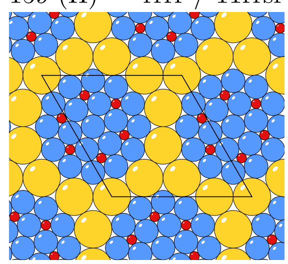
157 (H) 1rsss / 1rr1ss



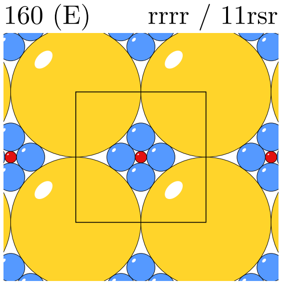
158 (H) 1rsss / 1s1sss



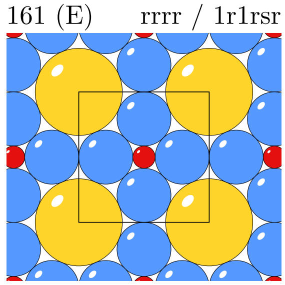
159 (H) rrrr / 11rrsr



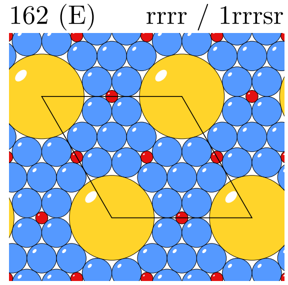
160 (E) rrrr / 11rsr



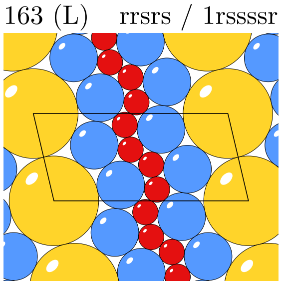
161 (E) rrrr / 1r1rsr



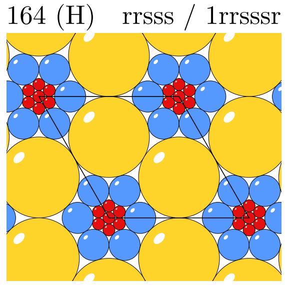
162 (E) rrrr / 1rrrsr



163 (L) rrsrs / 1rssssr



164 (H) rrsss / 1rrsssr



B Classification

Appendix A gives an example of compact packing for each of the 164 pairs (r, s) which allows a compact packing by discs of size $s < r < 1$. However, many pairs allow not only one but a whole set of compact packings, namely a *tiling space* in the terminology of [22] (which extends symbolic dynamical systems to tilings). In order to give an idea of the variety of possible packings, we assign to each case a *type* (letter **H**, **L**, **S** or **E** in brackets in App. A, Fig. 1 and Fig. 2). We distinguish four basic types with possible refinements in the following sense. A packing set Y is said to be a *refinement* of a paking set X if there is a "local recoding" (a surjective continuous map which commutes with isometries) which maps Y onto X. Roughly, the local recoding simply removes the flourish. In the terminology of dynamical system [22], Y is said to *factor* on X and the local recoding is called a *factor map*. The case c9, for example, is a refinement of the hexagonal compact packing with one size of discs: the local recoding removes the small discs between large discs. The same holds for c5, with the local recoding replacing the clusters of 7 small discs by a large discs. Two sets which are mutual refinements are said to be *conjugated*: they are pretty much the same (*e.g.*, c8 and c9 or c4 and 160). This allows to focus more on the very structure of packing sets.

Periodic packings (H). This is the simplest type: the disc sizes allow only finitely many compact packings with two independent periodic directions. This includes the hexagonal compact packing with one size of discs (whence the letter H), the compact packing with two sizes of discs labelled c6 in Fig. 1 and 52 cases with three sizes of discs. One checks that 10 out of the 17 wallpaper groups appear as symmetry groups of these periodic compact packings (Tab. 5). Refinements include the cases c5, c8, c9 and 22 cases with three sizes of discs.

p6m	p6	p31m	pmg	p3m1	cmm	p3	p4g	p4m	pgg
78	49	113	47	115	116	112	93	108	53

Table 5: Examples of periodic compact packings with three sizes of discs (numbers refer to App. A) for each possible symmetry group.

Laminated packings (L). The disc sizes allow only compact packings with exactly one periodic direction (and maybe finitely many degenerated cases). This includes c1 and c3, already described in [16], as well as 54 cases with three sizes of discs (7 of which are refinements). Fig. 18 gives a typical example.

Shield packings (S). The disc sizes allow compact packings which can be seen as tilings by an equilateral triangle and a *shield*, that is, a convex hexagon with two different angles (one obtuse and one acute) which alternate. This case includes c2, as well as 13 cases with three sizes of discs (7 of which are refinements). Fig. 19 describe these packings ([16] gives only two examples).

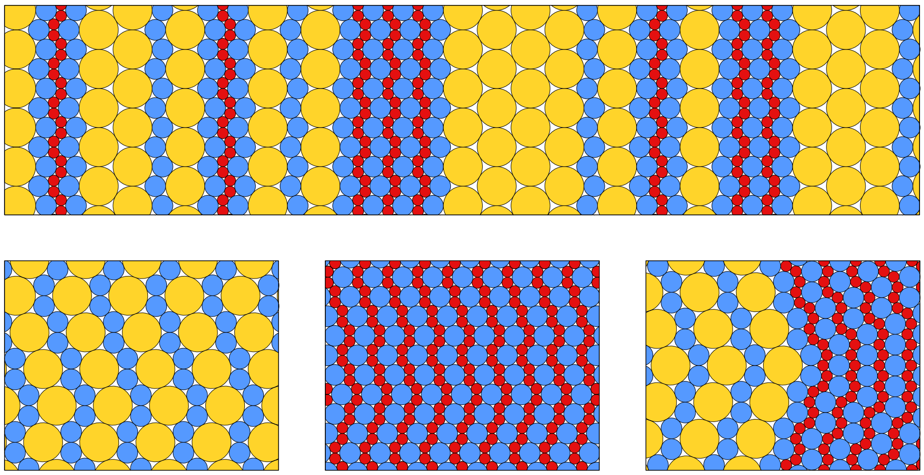


Figure 18: The typical packings of case 163 alternate lines of large, medium and small discs, such that there is always a line of medium discs between a line of large discs and a line of small discs, and only large discs can form two consecutive lines (top). There are also the laminated packings of case c3, where the lines of large and medium discs can be bended (bottom-left and bottom-center) and a single (up to isometry) degenerated case (bottom-right).

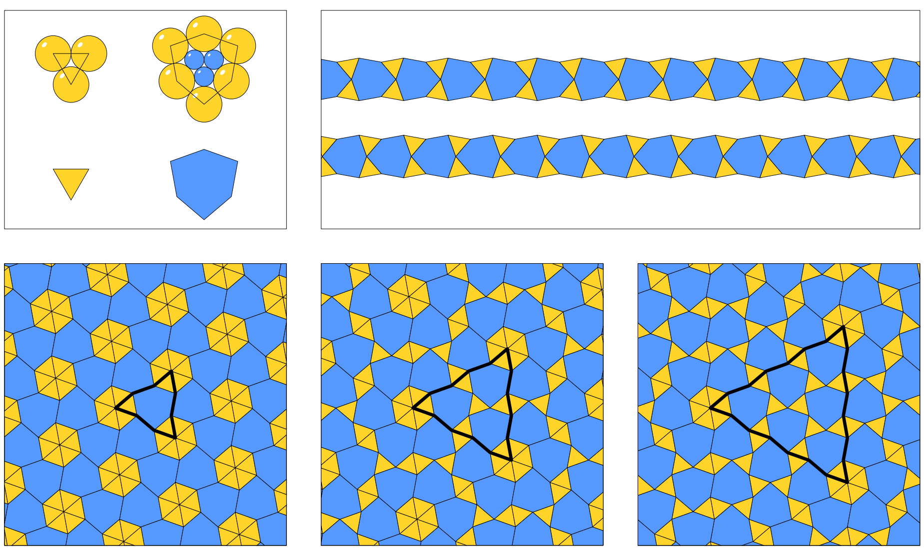


Figure 19: The compact packings which can be seen as tilings by a triangle and a shield (top-left) can form laminated packings (top-right: the two stripes can freely alternate) and a family of periodic packings looking like triangular grids of arbitrarily large size (bottom, the three first grids).

Positive entropy packings (E). The set of discs of a packing which intersect a ball of radius r forms what is called an *r-pattern*. A packing set is said to have *zero entropy* if the number of different r-patterns (up to an isometry) grows subexponentially with r^2 (the volume of the ball). The notion of entropy comes from dynamical systems, where it is used to measure the "complexity" of a system (in particular, to distinguish non-conjugated systems). In pratice for our classification, zero-entropy means that the set of possible packings is rather easy to describe. Periodic, laminated or shield packings do have zero entropy. Not their refinements, because flourish can be added or not independently at each position, but this does not affect the very structure of packing sets which are still easy to describe. The 23 remaining cases, however, do not have zero entropy nor are refinements of zero entropy cases. They are thus somehow more complicated to describe.

Actually, most of them can be seen as tilings by a square and a regular triangle, known in statistical mechanics as *square-triangle tilings* (Fig. 20). This includes c4 and (up to a refinement) 8 cases with three sizes of discs (numbered 1, 11, 16, 27, 61, 160, 161 and 162 in App. A). This also includes, up to a shear of the square into a rhombus which does not modify the combinatorics, c7 and (up to a refinement) 11 cases with three sizes of discs (numbered 2, 9, 12, 21, 30, 68, 71, 75, 92, 109, 135).

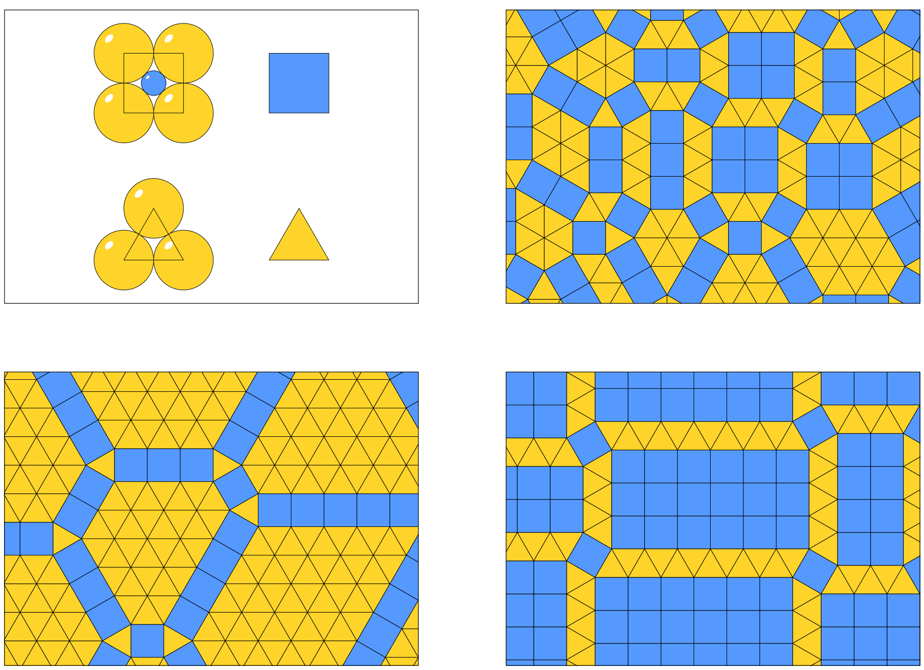


Figure 20: The compact packings which can be seen as tilings by a square and a regular triangle (top-left) can form a wide range of different packings, more or less random and with various proportions of tiles.

Besides square-triangle tilings, there are three cases which are a sort of mix of two different compact packings with two sizes of discs, namely those numbered 6, 7 and 8 which respectively mix c3/c4, c4/c7 and c3/c7.

Last but not least, the case 83. It is unique in the sense that it is neither a refinement of another case nor conversely. The compact packings can form rather complicated curves which alternate a small and two medium discs (Fig. 21, left). These packings can be seen as the tilings by a square, a regular triangle and an irregular one (Fig. 21, center and right). The edges of the irregular triangle have length 2, $2+2s$ and $2\sqrt{1+2r}$, where $r=\sqrt{2}-1$ and $s\simeq 0.249$ is root of X^4+4X-1. The smallest angle is $\frac{\pi}{4}$ and the largest one is $\arccos(1-\frac{1}{\sqrt{2}})$.

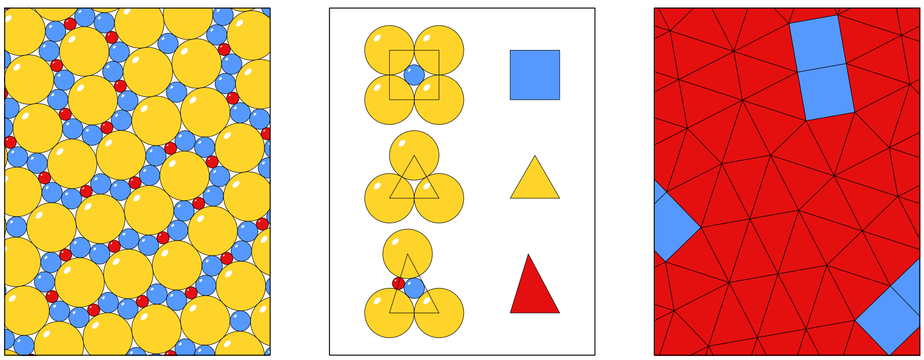


Figure 21: Case 83: a compact packing and the corresponding tiling.

C Code

Computations and case checking have been done with Python and SageMath. The full commented code is provided in supplementary materials. It is organized in five programs here briefly described (the numbers in brackets give the number of code lines, comments included):

• `coronas.sage` (191) contains functions to convert vector angles to sequence and conversely, to find all the possible small and medium coronas (Sections 4 and 6), to find the coronas compatible with interval values of r and s.

• `equations.sage` (126) contains functions to compute the polynomial associated with a corona (Section 5) and to check exactly whether given algebraic values of r and s are compatible with a given corona.

• `two_phases.sage` (28) deals with the large separated packings (Section 7).

• `two_small_coronas.sage` (127) deals with the packings with two different s-coronas (Section 8). It implements the hidden variable method, then apply interval arithmetic and exact filtering.

• `one_small_coronas.sage` (254) deals with the packings with only one s-coronas (Section 9). It implements the (pre)cover condition and the hidden variable method, then apply interval arithmetic and exact filtering.