Research Internship at Master 2 level — year 2012/'13

The "Red Stone Model": a cellular automaton with conservation rules

Andrea Sportiello

LIPN, Université Paris-Nord, CNRS UMR7030
99, av. J.-B. Clément, 93430 Villetaneuse, France
web page: `http://lipn.univ-paris13.fr/~sportiello/`

e-mail: `Andrea.Sportiello@lipn.fr`
tel.: +33 (0)1 49 40 40 84
skype: `asportiello`

Context

The aim of this internship is the study of a new cellular automaton, that we call the *Red Stone Model*. Cellular automata, even in one dimension, show a richness of behaviours [1], and are, under various aspects, untractable in general. This should not come as a surprise: these systems can be seen as models in Non-Equilibrium Statistical Mechanics, for which, contrarily to Boltzmann approach to Equilibrium, we lack a simple and solid formalism.

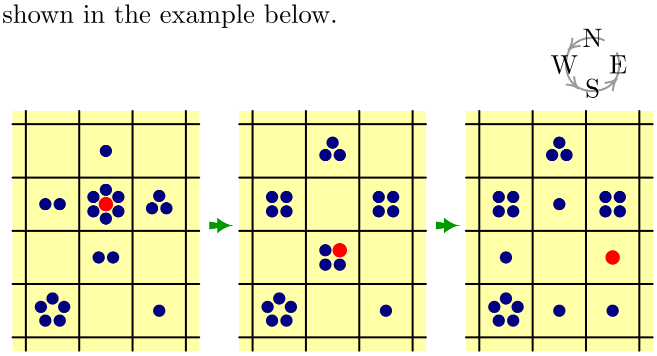
However, some rare examples have special 'solvability' properties, often relying on a "hidden mathematical structure", algebraic or combinatorial. One famous case is the *Abelian Sandpile Model* (ASM) [2], that, after the work of Dhar (and many others), has shown connections with the Kirchhoff Matrix-Tree Theorem, and the Tutte polynomial of the associated graph. Another case is the *Asymmetric Simple Exclusion Process* (ASEP) [3], that can be solved through a matrix ansatz (due to Derrida), and shows connections with integrable spin chains [6], and the Askey-Wilson polynomials [7]. Finally, we mention the *Box-Ball system* of Takahashi and Satsuma, which shows 'solitons' whose nature is related to Kashiwara crystal bases and Tropical Geometry [4].

Our scientific production within this topic up to now is concerned with the Abelian Sandpile Model, for which, together with Sergio Caracciolo and Guglielmo Paoletti, we have contributed to establish the emergence of one- and two-dimensional periodic patterns under deterministic protocols, and classified them through a hierarchical construction crucially based on certain properties of the modular group. An extensive account of this can be found in [5] (see also [9, 10, 11]).

This is the context in which the internship is situated… but the candidate is *not* required to know or understand all of it! (not even at the end of the period!) Our ambitions will be simpler and more concrete. Let us start by setting up our model…

Definitions

We start with an example. Consider a finite portion Λ of the square grid, e.g. compactified on a torus, $\Lambda = \mathbb{Z}^2 \big/_{\vec{x} \sim \vec{x} + a\vec{v}_1 + b\vec{v}_2}$. The cells of the grid are imagined as 'boxes' that can contain 'marbles' (or 'stones'). A configuration of marbles is determined by a pair $(\boldsymbol{z}, \vec{x}_r) \in \mathbb{N}^\Lambda \times \Lambda$, with all $z(x) \geq 0$ and $z(\vec{x}_r) \geq 1$. We have $z(\vec{x})$ black marbles at site $\vec{x} \neq \vec{x}_r$, while we have one red and $z(\vec{x}_r) - 1$ black marbles at $\vec{x}_r$. The dynamics is very simple: the red stone triggers the deployment (*toppling*) of its cell, marbles being given to the neighbours, N-W-S-E in cyclic order, the red one being given the last, so that it triggers the toppling on this neighbour, and so on, as shown in the example below.



This dynamics induces a map on the set $X(\Lambda, n)$, the configurations on Λ with n marbles (which are a finite number). Thus $X(\Lambda, n)$ is partitioned into a collection of periodic orbits, consisting of *recurrent* configurations, on which arborescences of *transient* configurations are rooted. Let's call X_R and X_T the subsets of X corresponding to recurrent and transient configurations, respectively.

A useful special property of this model is that the red stone itself performs a walk on Λ, while the whole configuration performs a walk on $X(\Lambda, n)$, a fact that is of some relevance when characterising the orbits (especially in a low-density regime).

The model is clearly generalisable to arbitrary adjacency structures. We can replace the lattice Λ, and the "N-W-S-E prescription", with a collection of m cells, and, for each $i \in [m]$, a finite sequence $\vec{\phi}_i = (\phi_1^{(i)}, \phi_2^{(i)}, \dots, \phi_{d(i)}^{(i)})$, with $\phi_j^{(i)} \in [m]$. When the red stone is in i, the toppling gives one stone to $\phi_1^{(i)}$, one to $\phi_2^{(i)}$, and so on, cyclically, up to full deployment, and giving the red stone as last one. The equivalent of Λ is a digraph with certain labels on the edges, with oriented edges $(i, \phi_j^{(i)})$ (having a label j). We will denote Φ the collection of $\vec{\phi}_i$'s, and $X(\Phi, n)$ the analogue of $X(\Lambda, n)$ above.

Goals

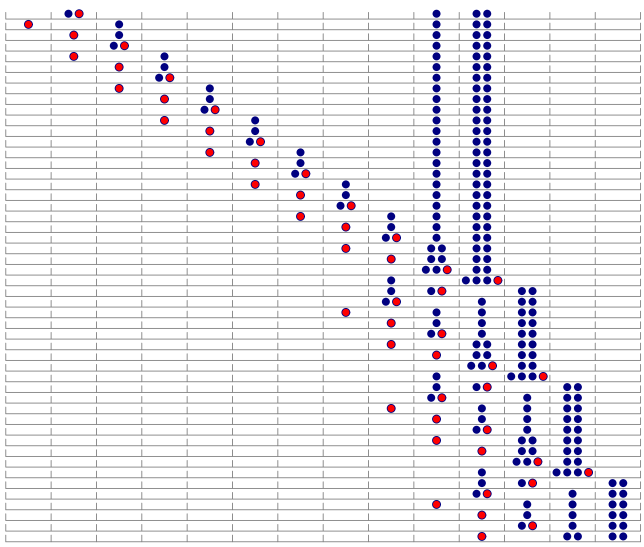
We will be interested in various kinds of questions, that can be roughly characterised as:

generic properties: facts that hold true for *all* configurations, in a general system $X(\Phi, n)$. Example: an algorithm to determine if a configuration in $X(\Phi, n)$ is transient or recurrent.

statistical properties: facts that hold true 'almost surely', for large regular systems, e.g. on portions of the hypercubic lattice dimension D. Example: the average velocity of the red stone, as a function of the density.

special configurations: determine configurations with peculiar or extremal properties. Example: longest or shortest orbit in $X(\Phi, n)$, or orbits such that the trajectory of the red stone has some 'nice property', such as being a Hamiltonian cycle.

specialties at $D = 1$: when the system is on a linear strip, the dynamics may induce "solitons"[1]. It is of separate interest to investigate the conditions leading to this feature, and study the resulting scattering theory. For examples, the model in $D = 1$ shows waves of velocity $1/(2^k - 1)$, involving 1 red and $2^{k-1} - 1$ black stones (see figure below, showing the time evolution of a 1-dimensional automaton, to be compared with those in [11], showing a single time slice of a 2-dimensional ASM).



[1]From wikipedia: *a soliton is a [..] wave that maintains its shape while it travels at constant speed. [..] Drazin & Johnson [..] ascribe three properties to solitons: (i) They are of permanent form; (ii) They are localised within a region; (iii) They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.*

Roadmap

A more precise list of issues that can be addressed, within the internship or on a longer time-scale (e.g. on a potential PhD continuation), are as follows. The number of ⋆ symbols denotes the level of difficulty (up to my present perspective).

General graphs:

⋆ 1) Understand low- and high-density regimes.

⋆⋆⋆ 2) Classify the invariants (the ASM analogue of this being the theory developed in [8]).

⋆⋆⋆ 3) Construct an efficient (linear-time?) algorithm to recognize if a configuration is transient or recurrent (the ASM analogue of this being the *Burning Test* [2]).

⋆⋆⋆ 4) The cardinality of $X(\Phi, n)$ is trivial (it is just $m\binom{m+n-2}{n-1}$). Can we determine $|X_R(\Phi, n)|$? Do we have an analogue of the ASM bijection with spanning trees?

Regular lattices in dimension D:

⋆⋆ 1) Determine the average velocity as a function of the density, when we start from a random configuration of given density.

⋆⋆⋆ 2) The evolution of a periodic initial configuration may remain (quasi-)periodic for a good while ($t \sim \alpha V$), with *varying* periodic patterns, and then suddenly break into an apparently random configuration (cf. figure on next page). Understand and formalise such a behaviour, and determine if a maximal α exists, or α is unbounded.

⋆⋆ 3) W.r.t. the previous question, I conjecture that we either enter an orbit of length $\mathcal{O}(V)$, or we reach asymptotically the same average velocity as in the random case. Is this true?

1-dimensional strips:

⋆ 1) Discuss conservation of momentum in $D = 1$.

⋆⋆ 2) Study which systems have a form of conservation of momentum, and what can be deduced from this.

⋆⋆ 3) Study 'solitons' in systems on a 1-dimensional strip, which have conservation of momentum. Determine their 'spectrum', and the 'scattering matrix' between the soliton and a fixed obstacle.

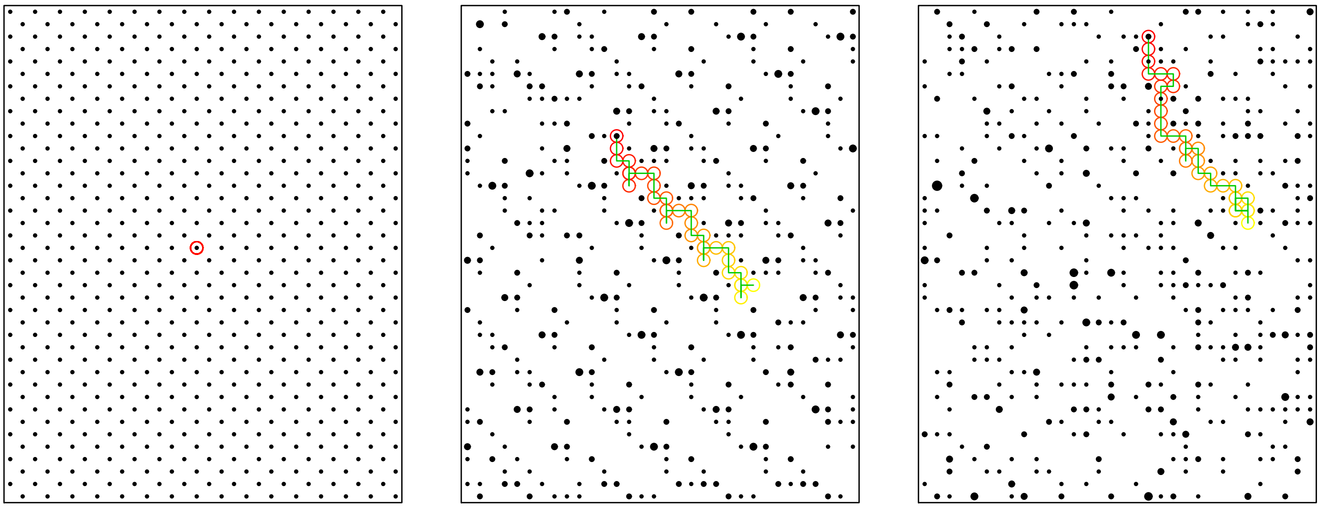
⋆⋆⋆⋆ 4) Introduce more red stones. Can we extend the dynamics, and design a system, such that the resulting scattering matrix satisfies the Yang–Baxter relation?

Prerequisites

Mastering English is not a prerequisite! Knowing *some* english is good, for reading books and papers... but I *do* speak French.

Mathematics will be the main language here. There are no strong specific prerequisites, though, not even the texts mentioned in the bibliography. Some very basics Combinatorics and Probability (e.g., what is a Markov Chain) is useful. The notions of Physics (mostly Statistical Mechanics) you might have noticed lurking in the text are not a prerequisite: in case, I'll tell you what you need, when you need it.

Some knowledge of programming languages at your choise (say, among C++, Mathematica, Maple or Java) could be useful for numerical investigations.



Time evolution of a periodic configuration, at times $t = 0$, $2V$ and $10V$. Orange circles show the last few steps of the red stone trajectory.

Books, reviews, lecture notes

[1] S. Wolfram, *Cellular Automata And Complexity: Collected Papers,* Westview Press, 1994 http://www.stephenwolfram.com/publications/books/ca-reprint/

[2] D. Dhar, *The abelian sandpile and related models,* Physica A**263**, 4 (1999) http://arxiv.org/abs/cond-mat/9808047

[3] B. Derrida, *An exactly soluble non-equilibrium system: The asymmetric simple exclusion process,* Physics Reports **301** 65 (1998) http://www.lps.ens.fr/~derrida/PAPIERS/1998/altenberg-1998.pdf

[4] R. Inoue, A. Kuniba and T. Takagi, *Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry,* J. Phys. A **45** 073001 (2012) http://arxiv.org/abs/1109.5349

[5] G. Paoletti, *Deterministic Abelian Sandpile Models and Patterns,* PhD thesis at Università di Pisa, 2012 http://pcteserver.mi.infn.it/~caraccio/PhD/Paoletti.pdf

Research papers

[6] O. Golinelli and K. Mallick, *The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics,* J. Phys. A: Math. Gen. **39**, 12679-12705 (2006) http://arxiv.org/abs/cond-mat/0611701; *Family of Commuting Operators for the Totally Asymmetric Exclusion Process,* J. Phys. A: Math. Gen. **40**, 5795-5812 (2007) http://arxiv.org/abs/cond-mat/0612351

[7] S. Corteel and L.K. Williams, *Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials,* Duke Math. J. **159** 385-415 (2011) http://arxiv.org/abs/0910.1858

[8] D. Dhar, P. Ruelle, S. Sen and D.-N. Verma, *Algebraic aspects of Abelian sandpile models,* J. Phys. A**28**, 805-831 (1995) http://arxiv.org/abs/cond-mat/9408022

[9] S. Ostojic, *Patterns formed by addition of grains to only one site of an Abelian Sandpile,* Physica A**318**, 187-199 (2003).

[10] D. Dhar, T. Sadhu and S. Chandra, *Pattern formation in growing Sandpiles,* Europhys. Lett. **85**, 48002 (2009) http://arxiv.org/abs/0808.1732

[11] S. Caracciolo, G. Paoletti and A. Sportiello, *Conservation laws for strings in the Abelian Sandpile Model,* Europhys. Lett. **90**, 60003 (2010) http://arxiv.org/abs/1002.3974