Constructions for Clumps Statistics

P. Nicodème

CNRS - LIX École polytechnique, Palaiseau

Joint work with F. Bassino, J. Clément and J. Fayolle

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27	17	1000000000
28		00111111011
29	18	11000011001
30		0111001000

What is going on?

- Probability of appearance at a given position

$$\mathbf{P}(100) = \frac{1}{8}$$

$$\mathbf{P}(111) = \frac{1}{8}$$

What is going on?

- Probability of appearance at a given position

$$\mathbf{P}(100) = \frac{1}{8}$$

$$\mathbf{P}(111) = \frac{1}{8}$$

- BUT the 111 occur often by CLUMPS

$$\begin{array}{ll} \cdots 0111110 & \qquad\qquad \cdots 011110 \\ \quad\ \ 111 & \qquad\qquad\quad\ \ 111\cdots \\ \quad\ \ \ \ 111\cdots & \end{array}$$

- while the 100 NEVER OVERLAP

What is going on?

- Probability of appearance at a given position

$$\mathbf{P}(100) = \frac{1}{8}$$

$$\mathbf{P}(111) = \frac{1}{8}$$

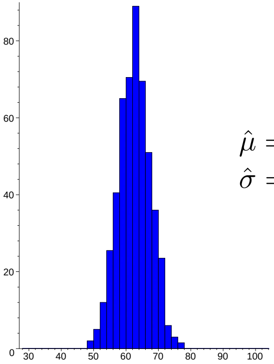
- BUT the 111 occur often by CLUMPS

$$\begin{array}{ll} \cdots 0111110 & \qquad \cdots 011110 \\ \quad 111 & \qquad \quad 111 \cdots \\ \quad\; 111 \cdots & \end{array}$$

- while the 100 NEVER OVERLAP

Expected waiting time: $111 - 14$ $\qquad 100 - 7$

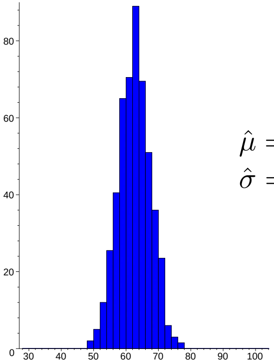
Number of occurrences 1000 texts of size 500



$$\hat{\mu} = 62.38$$
$$\hat{\sigma} = 4.90$$

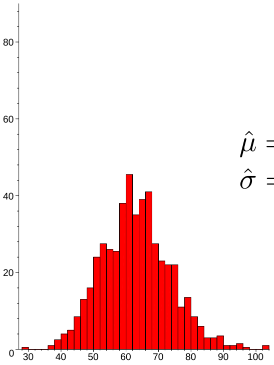
100

Number of occurrences 1000 texts of size 500



$$\hat{\mu} = 62.38$$
$$\hat{\sigma} = 4.90$$

100



$$\hat{\mu} = 62.07$$
$$\hat{\sigma} = 10.79$$

111

Aim of our work

- a **combinatorial** analysis for counting clumps of reduced sets of words
- an **algorithmic** contruction by **automata** that solves the counting problem in the general case and implies a normal limit law
- future extensions to **tandem repeats**?

$\ldots$GGGGATCGA|GATCGA|GATCGA|GATCGAGGG$\ldots$

Counting clumps

$w = aa \qquad T = bbbbaaaabbbbaaa$

- word counting with overlaps: 5 matches $(aa|a|a|, aa|a|)$
- clump counting: 2 matches $(aaaa|, aaa|)$

Definition of a clump of a word w

Set of **maximal contiguous** positions such that:

- each position is inside an occurrence of w
- if a position is at the beginning or at the end of an occurrence of w, it is covered by two occurrences of w, with exception of the first and last positions of the clump

Probability of start at a position i

(A) word $w = 1^k$ $\qquad$ $p = \mathbf{P}(1) = 1 - \mathbf{P}(0) = 1 - q$

$$\ldots \overbrace{\boxed{1}\boxed{1}\ldots\boxed{1}}^{k} \ldots$$

$\mathbf{P}(\mathsf{start}) = p^k$

Probability of start at a position i

(A) word $w = 1^k$ $\qquad$ $p = \mathbf{P}(1) = 1 - \mathbf{P}(0) = 1 - q$

$$\dots \overbrace{\boxed{1}\boxed{1}\dots\boxed{1}}^{k} \dots$$

$\mathbf{P}(\text{start}) = p^k$

(B) clump $\Gamma = 1^k.1^\star$

$$\begin{array}{ll}
\dots & \overbrace{\boxed{1}\boxed{1}\dots\boxed{1}}^{k}\boxed{0}\dots \\
\dots & \boxed{1}\boxed{1}\dots\boxed{1}\boxed{1}\boxed{0}\dots \\
\dots & \boxed{1}\boxed{1}\dots\boxed{1}\boxed{1}\boxed{1}\boxed{0}\dots \\
\dots & \dots
\end{array}$$

$$\mathbf{P}(\text{start}) = p^k \times q \times (1 + p + p^2 + \dots) = p^k \times \frac{q}{1-p} = p^k$$

Probability of start at a position i

(A) word $w = 1^k$ $\qquad$ $p = \mathbf{P}(1) = 1 - \mathbf{P}(0) = 1 - q$

$$\dots \overbrace{\boxed{1}\boxed{1}\dots\boxed{1}}^{k} \dots$$

$\mathbf{P}(\text{start}) = p^k$

(B) clump $\Gamma = 1^k.1^\star$

$$\begin{array}{ll}
\dots & \overbrace{\boxed{1}\boxed{1}\dots\boxed{1}}^{k}\boxed{0}\dots \\
\dots & \boxed{1}\boxed{1}\dots\boxed{1}\boxed{1}\boxed{0}\dots \\
\dots & \boxed{1}\boxed{1}\dots\boxed{1}\boxed{1}\boxed{1}\boxed{0}\dots \\
\dots & \dots
\end{array}$$

$$\mathbf{P}(\text{start}) = p^k \times q \times (1 + p + p^2 + \dots) = p^k \times \frac{q}{1-p} = p^k$$

FALSE

Probability of start at a position i

(A) word $w = 1^k$

$$\dots \overbrace{\boxed{1}\boxed{1}\dots\boxed{1}}^{k}\dots$$

$\mathbf{P}(\text{start}) = p^k$

(B) clump $\Gamma = 1^k.1^\star$

$\dots 1|\overbrace{1\dots1}^{k}\dots$ no clump **beginning** at position i

$\dots 0|\overbrace{1\dots1}^{k}\dots$ a clump **begins** at position i

$\mathbf{P}(\text{start}) = p^k \times q = p^k \times (1-p)$

Probabilistic approach

Prum, Reinert, Schbath, Pape, ...

$w = aaa$

$\mathbf{P}(a)$ small $\rightsquigarrow$ $\mathbf{P}$(start-of-a-clump-at-a-position) small

$\rightsquigarrow$

Poisson law for the number of clumps

Geometric law for the number of words in a clump

This extends to the general case.

Chen-Stein Poisson approximation provides bounds for the total variation distance to the composed law.

Combinatorial approach

Jacquet-Szpankowski

I- Combinatorial approach

1 word - Bernoulli model

$\mathcal{A}$ alphabet, w considered word

Polynomial of autocorrelation

$\mathcal{C}_w = \{\ h, \quad w.h = u.w \quad \text{and} \quad |u| < |w|\ \}$

$w = ababa$

$$\begin{array}{l} ababa \\ ababa| \\ \quad ababa \\ \qquad ababa \end{array}$$

$\mathcal{C}_{ababa} = \{\epsilon, ba, baba\}$

$$C_{ababa}(z) = \sum_{v \in \mathcal{C}_{ababa}} \mathbf{P}(v) z^{|v|} = 1 + \pi_a \pi_b z^2 + \pi_a^2 \pi_b^2 z^4$$

Régnier and Szpankowski decomposition

First $\mathcal{R} = \{\ t = u.w \quad \text{and} \quad \nexists r, s, \quad t = r.w.s\ \}$
$aaaaa\,ababa \in \mathcal{R}, \quad bbbbb\,abababa \notin \mathcal{R}$

Régnier and Szpankowski decomposition

First $\mathcal{R} = \{\ t = u.w$ and $\nexists r, s, \quad t = r.w.s\ \}$
$aaaaaababa \in \mathcal{R}, \quad bbbbbabababa \notin \mathcal{R}$
Minimal $\mathcal{M} \ = \{\ t, \quad w.t = u.w$ and $\nexists r, s, \quad w.t = r.w.s\ \}$

$ababa_{aaaaababa} \in \mathcal{M} \quad ababa_{babbbbbbbababa} \notin \mathcal{M} \quad ababa_{ba} \in \mathcal{M}$

Régnier and Szpankowski decomposition

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$ababa_{aaaaababa} \in \mathcal{M} \quad ababa_{babbbbbbbbababa} \notin \mathcal{M} \quad ababa_{ba} \in \mathcal{M}$

Ultimate $\mathcal{U} = \{t, \quad \nexists r, s, \quad w.t = r.w.s\}$

$ababa_{aabbbabbbbbbb} \in \mathcal{U} \qquad ababa_{babbbbbbbbbb} \notin \mathcal{U}$

Régnier and Szpankowski decomposition

First $\mathcal{R} = \{\ t = u.w \quad \text{and} \quad \nexists r, s, \quad t = r.w.s\ \}$
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$ababa_{aabbbabbbbbb} \in \mathcal{U} \qquad ababa_{babbbbbbbbb} \notin \mathcal{U}$

Not $\mathcal{N} = \overline{\mathcal{A}^{\star}.w.\mathcal{A}^{\star}} = \{t, \ \nexists r, s, \quad t = r.w.s\}$

Régnier and Szpankowski decomposition

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Not $\mathcal{N} = \overline{\mathcal{A}^\star.w.\mathcal{A}^\star} = \{t, \ \nexists r, s, \quad t = r.w.s\}$

$$\mathcal{A}^\star = \mathcal{N} + \mathcal{R}.\left(\mathcal{M}\right)^\star.\mathcal{U} \quad \Rightarrow \quad \mathcal{L}_x = \mathcal{N} + \mathcal{R}x.\left(\mathcal{M}x\right)^\star.\mathcal{U}$$

Languages $\Rightarrow$ generating functions

$$
\begin{array}{llll}
\text{(I)} & \Sigma^\star = \mathcal{U} + \mathcal{M}\Sigma^\star & \Rightarrow & \dfrac{1}{1-z} = U(z) + \dfrac{M(z)}{1-z} \\
\text{(II)} & \Sigma^\star w = \mathcal{R}.\mathcal{C} + \mathcal{R}.\Sigma^\star.w & \Rightarrow & \dfrac{\pi_w z^m}{1-z} = R(z)\left(C(z) + \dfrac{\pi_w z^m}{1-z}\right) \\
\text{(III)} & \mathcal{M}^+ = \Sigma^\star.w + \mathcal{C} - \epsilon & \Rightarrow & \dfrac{M(z)}{1-M(z)} = \dfrac{\pi_w z^m}{1-z} + C(z) - 1 \\
\text{(IV)} & \mathcal{N}.\Sigma = \mathcal{R} + \mathcal{N} - \epsilon & \Rightarrow & zN(z) = R(z) + N(z) - 1
\end{array}
$$

Add the mark x after each match $\Rightarrow$ marked language $\mathcal{L}$

$$\mathcal{L} = \mathcal{N} + \mathcal{R}.x(\mathcal{M}.x)^\star\mathcal{U}$$

Translation into generating function

$$L(z,x) = N(z) + \frac{xR(z)U(z)}{1 - xM(z)}$$

Solving the system

$$R(z) = \frac{\pi_w z^{|w|}}{\pi_w z^{|w|} + (1-z)C(z)} \qquad U(z) = \frac{1}{\pi_w z^{|w|} + (1-z)C(z)}$$

$$N(z) = \frac{C(z)}{\pi_w z^{|w|} + (1-z)C(z)} \qquad M(z) = 1 + \frac{z-1}{\pi_w z^{|w|} + (1-z)C(z)}$$

$$L(z,x) = \frac{1}{1 - z + \pi_w z^{|w|} \dfrac{1-x}{x + (1-x)C(z)}}$$

Language equations for the clumps

Notation:

if $\mathcal{L} = \mathcal{W} \cdot w$ we write $\mathcal{L}^{-} = \mathcal{W}$

Combinatorial decomposition

$$\begin{aligned}
\mathcal{A}^{\star} &= \mathcal{N} + \mathcal{R}^{-} w\, \mathcal{C}^{\star} \Big((\mathcal{M} - \mathcal{K})^{-} w\, \mathcal{C}^{\star} \Big)^{\star} \mathcal{U} \\
&= \mathcal{N} + \mathcal{R}^{-} w\, \mathcal{K}^{\star} \Big((\mathcal{M} - \mathcal{K})^{-} w\, \mathcal{K}^{\star} \Big)^{\star} \mathcal{U} \\
&= \mathcal{N} + \mathcal{R}^{-} \mathbf{\Gamma} \Big((\mathcal{M} - \mathcal{K})^{-} \mathbf{\Gamma} \Big)^{\star} \mathcal{U}
\end{aligned}$$

Clump: $\mathbf{\Gamma} = w\, \mathcal{C}^{\star} = w\, \mathcal{K}^{\star}$

Some combinatorial properties

$w = aaaaa$
$\mathcal{C} - \{\epsilon\} = \{a, aa, aaa, aaaa\}$
$\mathcal{K} = \{a\}$
$\mathcal{M} = \{a, b(b + ab + aab + aaab + aaaab)^{\star}aaaaa\}$

Properties

- $\mathcal{K} \subset \mathcal{M}$
- $\mathcal{M} - \mathcal{K} = \mathcal{L}w$
- $\mathcal{K}^{\star} = \mathcal{C}^{\star}$ and $\mathcal{K}^{\star}$ is **unambiguous**

A Prefix Code generating the clumps

Lemma.
Let $\mathcal{C}_\circ = \mathcal{C} - \{\epsilon\}$ be the strict autocorrelation set of a word w

- the Prefix code $\mathcal{K} = \mathcal{C}_\circ - \mathcal{C}_\circ \mathcal{A}^+$ generates **unambiguously** $\mathcal{C}^+ - \{\epsilon\}$, which implies that $\mathcal{K}^\star = \mathcal{C}_\circ{}^\star$
- $\mathcal{K}^\star$ is **unambiguous**

Generating functions

$$F(z,\bullet,\bullet,\bullet) = \mathcal{N}(z) + \frac{\mathcal{R}(z)}{\pi_w z^{|w|}} \mathbf{\Gamma}(\bullet z,\bullet,\bullet) \frac{1}{1 - \frac{\mathcal{M}(z) - \mathcal{K}(z)}{\pi_w z^{|w|}} \times \mathbf{\Gamma}(\bullet z,\bullet,\bullet)} \mathcal{U}(z)$$

- number of w and number of clumps:

  $$\mathbf{\Gamma}(z,u,x) = ux\pi_w z^{|w|} \frac{1}{1 - x\mathcal{K}(z)}$$

- number of clumps and total number of positions inside clumps:

  $$\mathbf{\Gamma}(tz,u) = u\pi_w (tz)^{|w|} \frac{1}{1 - \mathcal{K}(tz)}$$

- number of w and total number of positions inside clumps:

  $$\mathbf{\Gamma}(tz,x) = x\pi_w (tz)^{|w|} \frac{1}{1 - x\mathcal{K}(tz)}$$

- number of "stuttering" w and total number of positions inside clumps:

  $$\mathbf{\Gamma}(tz,x) = \frac{x}{1-x}\pi_w (tz)^{|w|} \frac{1}{1 - \frac{x}{1-x}\mathcal{K}(tz)}$$

One word - Expectation - Variance

clumps

$$\mathbf{E}(O_n^{\mathfrak{K}}) = (\mathbf{n} - |w| + 1)\pi_w(1 - \mathcal{K}(1)) - \pi_w \mathcal{K}'(1)$$

$$\mathbf{Var}(O_n^{\mathfrak{K}}) = \mathbf{n} \times (1 - \mathcal{K}(1))^2 \mathbf{V}_w - \mathbf{n} \times \pi_w(1 - \mathcal{K}(1))(\mathcal{K}(1) - 2\pi_w \mathcal{K}'(1))$$

one word

$$\mathbf{E}(O_n^w) = (\mathbf{n} - |w| + 1)\pi_w, \qquad \mathbf{Var}(O_n^w) = \mathbf{n} \times \mathbf{V}_w + O(1).$$

$$\mathbf{V}_w = \pi_w\big(2\mathcal{C}(1) - 1 - (2|w| - 1)\pi_w\big)$$

Reduced compound patterns

$W = \{w_1, w_2\}$ and w_1 (resp. w_2) is **not factor** of w_2 (resp. w_1)

Correlation of words

$$\mathbb{C}(z) = (\mathcal{C}_{i,j}(z)) \qquad \mathcal{C}_{i,j} = \{h,\ w_i.h = u.w_j\}$$

Example $W = \{aab, abaa\}$

$$\mathbb{C}(z) = \begin{pmatrix} 1 & \pi_a^2 z^2 \\ \pi_b z & 1 + \pi_a^2 \pi_b z^3 \end{pmatrix}$$

Languages **Right**, **Minimal**, **Ultimate** $\mathcal{R}_i, \mathcal{M}_{i,j}, \mathcal{U}_i$

The set of language equations

$$\bigcup_{k\geq 1}\left(\mathbb{M}^k\right)_{i,i} = \mathcal{A}^\star \cdot w_i + \mathcal{C}_{ii} - \epsilon, \qquad 1 \leq i \leq 2,$$

$$\bigcup_{k\geq 1}\left(\mathbb{M}^k\right)_{i,j} = \mathcal{A}^\star \cdot w_j + \mathcal{C}_{ij}, \qquad 1 \leq i \neq j \leq 2,$$

$$\mathcal{U}_i \cdot \mathcal{A} = \bigcup_j \mathcal{M}_{ij} + \mathcal{U}_i - \epsilon, \qquad 1 \leq i \leq 2,$$

$$\mathcal{A} \cdot \mathcal{R}_j - (\mathcal{R}_j - w_j) = \bigcup_i w_i \mathcal{M}_{ij}, \qquad 1 \leq j \leq 2,$$

$$\mathcal{N} \cdot w_j = \mathcal{R}_j + \bigcup_i \mathcal{R}_i \left(\mathcal{C}_{ij} - \delta_{ij}\epsilon\right), \qquad 1 \leq i, j \leq 2$$

$$F(z,x,y) = \mathbb{R}(z,x,y)\left(\mathbb{I} - \mathbb{M}(z,x,y)\right)^{-1}\mathbb{U}(z)$$

Putting up equations for clumps of two words

Minimal Correlation Language: $\mathcal{K}_{ij} = \mathcal{C}_{ij} - \mathcal{C}_{ij}\mathcal{A}^{+}$
Lemma: $\mathcal{M}_{ij} - \mathcal{K}_{ij} = \mathcal{L}w_j$

$$\mathbb{K} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{pmatrix}$$

$$\boldsymbol{\Xi} = \mathbb{K}^{\star} \qquad \mathbb{G} = \begin{pmatrix} w_1 \boldsymbol{\Xi}_{11} & w_1 \boldsymbol{\Xi}_{12} \\ w_2 \boldsymbol{\Xi}_{21} & w_2 \boldsymbol{\Xi}_{22} \end{pmatrix}$$

$$\mathcal{A}^{\star} = \mathcal{N} + (\mathcal{R}_1^{-}, \mathcal{R}_2^{-})\, \mathbb{G} \left((\mathbb{M} - \mathbb{K})^{-} \mathbb{G} \right)^{\star} \begin{pmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{pmatrix}$$

II- Automaton approach

Clumps of the set of words $\mathcal{U} = \{aabaa, baab\}$

$\mathcal{E}_{w_1,w_2}$ extension set from w_1 to w_2

$$\begin{aligned} \mathcal{E}_{aabaa,aabaa} &= \{baa, abaa\} & \mathcal{E}_{baab,baab} &= \{aab\} \\ \mathcal{E}_{aabaa,baab} &= \{b\} & \mathcal{E}_{baab,aabaa} &= \{aa\} \end{aligned}$$

Clumps of the set of words $\mathcal{U} = \{aabaa, baab\}$

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Algorithm

1. Build the set of strings

Clumps of the set of words $\mathcal{U} = \{aabaa, baab\}$

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Algorithm

1. Build the set of strings

$$\begin{array}{rllll} X = & & \{aabaa.(\epsilon + \mathcal{E}_{aabaa,aabaa})\} & \cup & \{aabaa.\mathcal{E}_{aabaa,baab}\} \\ & \cup & \{baab.(\epsilon + \mathcal{E}_{baab,baab})\} & \cup & \{baab.\mathcal{E}_{baab,aabaa}\} \end{array}$$

Clumps of the set of words $\mathcal{U} = \{aabaa, baab\}$

$\mathcal{E}_{w_1,w_2}$ extension set from w_1 to w_2

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2. Build a trie $\mathcal{T}$ on X

Clumps of the set of words $\mathcal{U} = \{aabaa, baab\}$

$\mathcal{E}_{w_1,w_2}$ extension set from w_1 to w_2

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Algorithm

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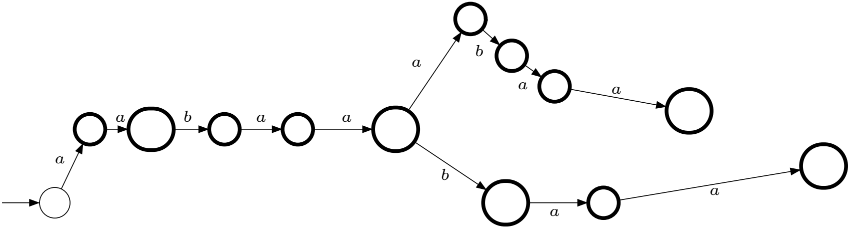
2. Build a trie $\mathcal{T}$ on X

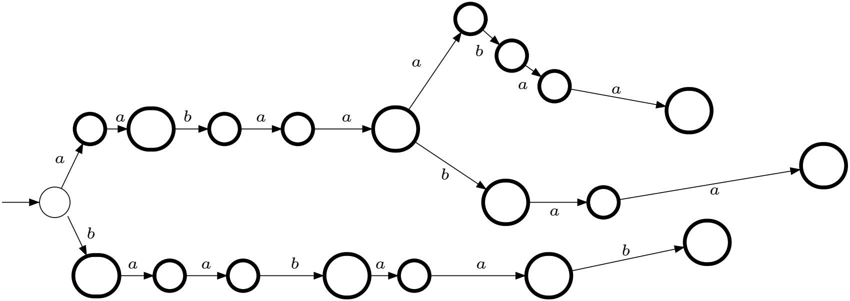
3. Build a Aho-Corasick like automaton upon $\mathcal{T}$. For each node ν of $\mathcal{T}$ with "access word" v, use the transition function δ

   $\delta(\nu, \ell) =$ node accessed by the **longuest prefix** in X that is **suffix** of $v.\ell$

$X = \{aabaa, aabaabaa, aabaaabaa, aabaab, \qquad\qquad\qquad \}$

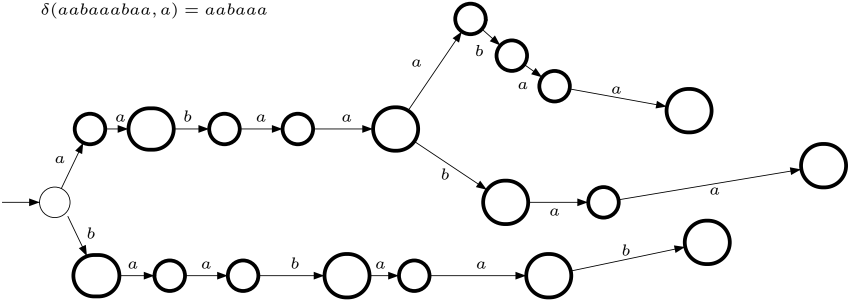


$X = \{aabaa, aabaabaa, aabaaabaa, aabaab, baab, baabaab\}$



$$X = \{aabaa, aabaabaa, aabaaabaa, aabaab, baab, baabaab\}$$

$$\delta(aabaaabaa, a) = aabaaa$$



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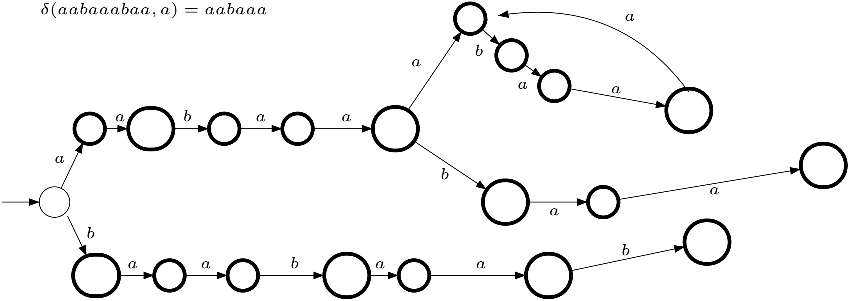
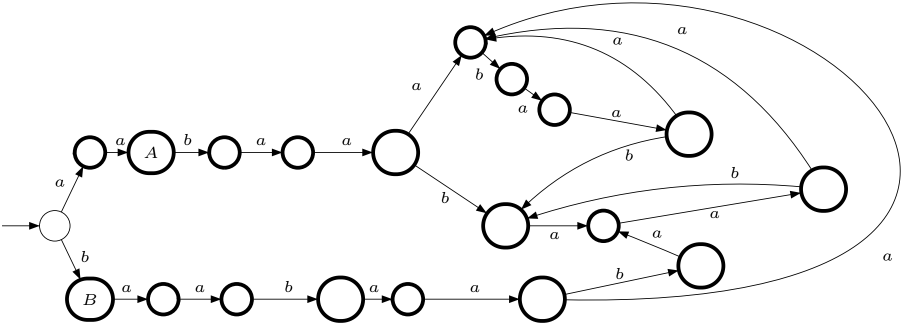
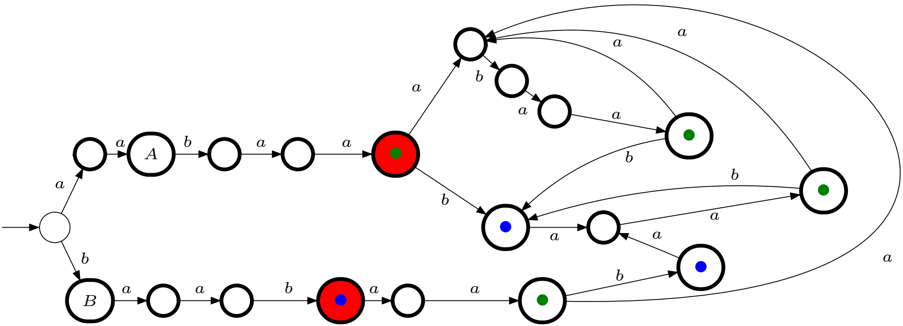


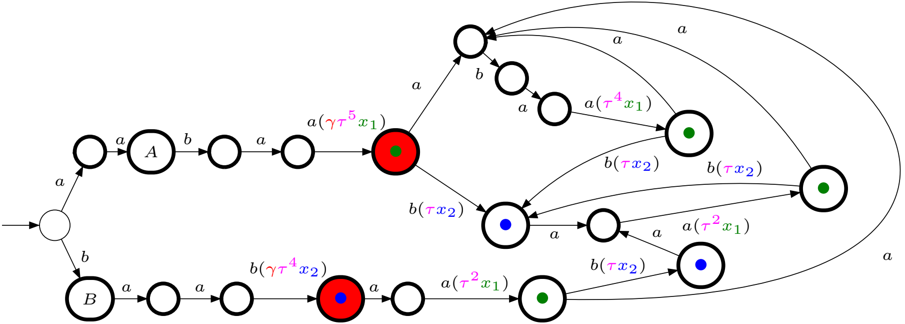


An automaton for $\mathcal{V} = \{v_1 = aabaa, v_2 = baab\}$. All transitions labeled by a and b ending respectively on state A and B are omitted.



An automaton for $\mathcal{V} = \{v_1 = aabaa, v_2 = baab\}$. All transitions labeled by a and b ending respectively on state A and B are omitted.

- ●, ● → the corresponding prefix (or state) ends with some occurrence of $aabaa$, $baab$.
- **red states** → states where we have entered a **new clump**



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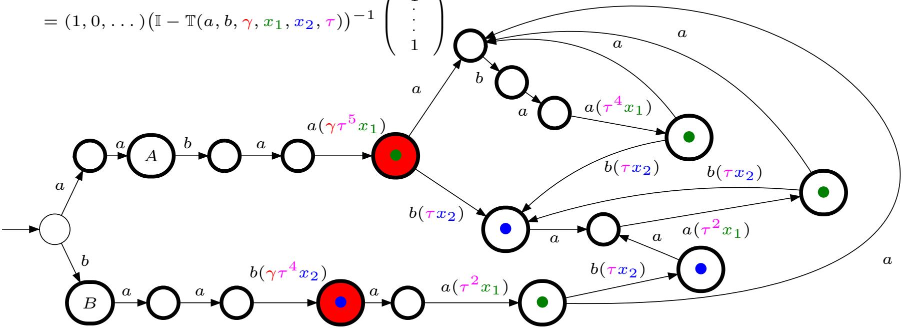
- ●, ● $\longrightarrow$ the corresponding prefix (or state) ends with some occurrence of $aabaa, baab$.
- **red states** $\longrightarrow$ states where we have entered a **new clump**

Formal weights on transitions

- $\gamma \longrightarrow$ the number of clumps
- $\tau \longrightarrow$ total length of clumps
- $x_1, x_2 \longrightarrow$ occurrences of $aabaa, baab$

$$F(a,b,\gamma,x_1,x_2,\tau) = (1,0,\dots)(\mathbb{I} - \mathbb{T}(a,b,\gamma,x_1,x_2,\tau))^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$



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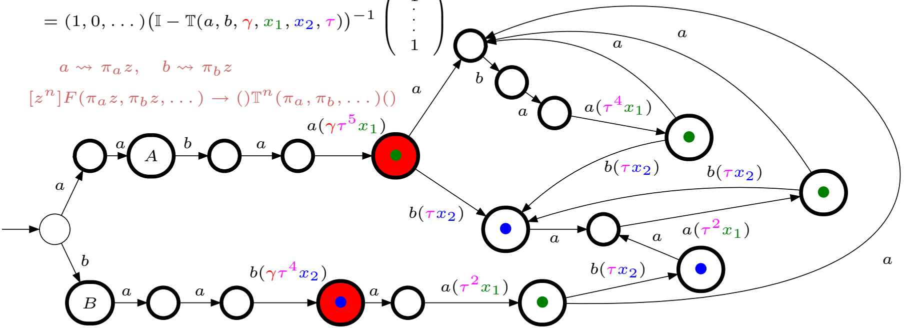
$$a \rightsquigarrow \pi_a z, \quad b \rightsquigarrow \pi_b z$$

$$[z^n]F(\pi_a z, \pi_b z, \dots) \to ()\mathbb{T}^n(\pi_a, \pi_b, \dots)()$$



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- ●, ● $\to$ the corresponding prefix (or state) ends with some occurrence of $aabaa$, $baab$.
- **red states** $\to$ states where we have entered a **new clump**

Formal weights on transitions

- $\gamma \to$ the number of clumps
- $\tau \to$ total length of clumps
- $x_1, x_2 \to$ occurrences of $aabaa$, $baab$

Asymptotic Limit Laws

- **One word**
  - $O_n^{\mathfrak{K}} = O(1)$ Poisson law for the number of clumps
- **General non-reduced sets**
  - $O_n^{\mathfrak{K}} = \Theta(n)$ Normal limit law (number of clumps, size covered)

Proofs

- Poisson law: Rouché theorem, singularity analysis
- Normal law: automaton, Perron-Frobenius for $\mathbb{T}(\dots)$, singularity analysis, large powers theorem

Complexity of computing the prefix code(s)

- **one word**

$$|\mathcal{K}| \log(|\mathcal{K}|)$$

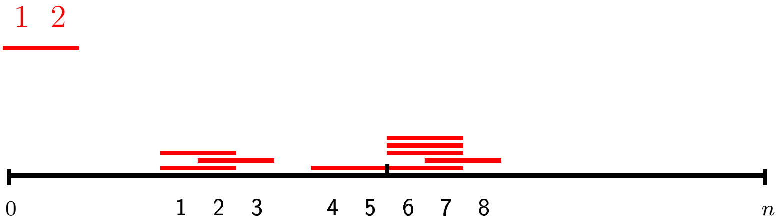
- **several words** (reduced case)

$$\left(\sum_{i,j} |\mathcal{K}_{i,j}|\right) \log \left(\sum_{i,j} |\mathcal{K}_{i,j}|\right)$$

Complexity of **insertion** of random keys in **tries**

Throwing flat dimers on an “integral” segment

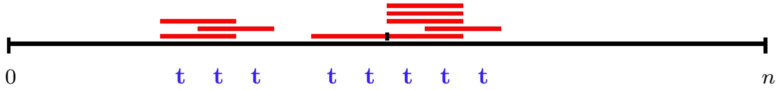
Throwing flat dimers on an “integral” segment



Size covered by the dimers?

Throwing flat dimers on an "integral" segment

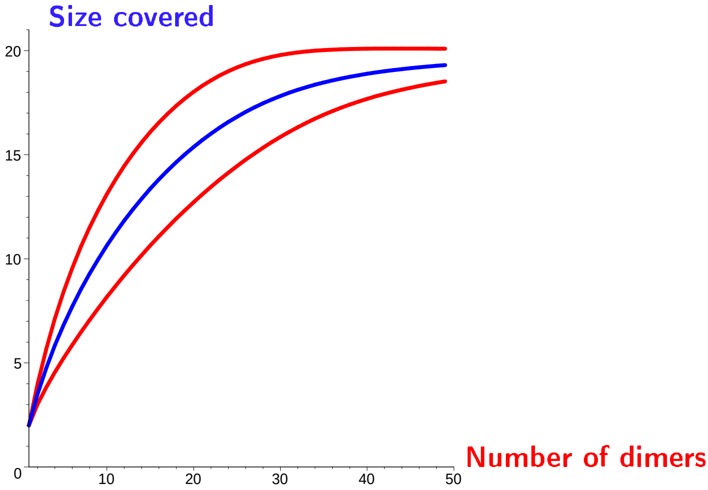
$a \quad a$ 　 **stuttering** occurrences of **aa** on $(\mathbf{a}+\mathbf{b})^{\mathbf{n}}$



$$\mathbf{P}(a) = \mathbf{P}(b) = \frac{1}{2} \qquad \mathbf{P}(aa) = \frac{1}{4}$$

stuttering by **substituting** $\quad \mathbf{x} \rightsquigarrow \dfrac{\mathbf{x}}{1-\frac{1}{4}\mathbf{x}} \quad$ **in** $\quad \Gamma(\mathbf{z}\mathbf{t}, \mathbf{x})$

Throwing flat dimers on an “integral” segment



$$\mathbf{n} = |\text{segment}| = \mathbf{20}$$

The Prefix Code $\mathcal{K} = \mathcal{C}_\circ - \mathcal{C}_\circ\mathcal{A}^+$ generates $\mathcal{C}_\circ{}^+$

w word, $\quad \mathcal{C}_\circ = \mathcal{C} - \epsilon, \qquad v \in \mathcal{C}_\circ \implies v = \kappa_1 \dots \kappa_r, \quad \kappa_i \in \mathcal{K}$

- $v \notin \mathcal{K} \implies v = \kappa_1 v'$ with $\kappa_1 \in \mathcal{K}$ and $v' \in \mathcal{A}^+$.
- $v, \kappa_1 \in \mathcal{C}_\circ \implies \exists p, p', s'$ such that $p = p's'$, $wv = pw$ and $w\kappa_1 = p'w$. Therefore $wv' = s'w$ and $v' \in \mathcal{C}_\circ$.
- iterate a finite number of time