Light Logics and Implicit Computational Complexity

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Implicit computational complexity

- From clocks to certificates:

```
def add5(x):
   return x+5

def dotwrite(ast):
   nodename = getNodename()
   label=symbol.sym_name.get(int(ast[0]),ast[0])
   print '    %s [label="%s' % (nodename, label),
   if isinstance(ast[1], str):
      if ast[1].strip():
         print '= %s"];' % ast[1]
      else:
         print '"]'
    else:
       print '"];'
       children = []
       for  in n, childenumerate(ast[1:]):
          children.append(dotwrite(child))
       print ,'    %s -> {' % nodename
       for  in :namechildren
          print '%s' % name,
```

vs.


```
def add5(x):
   return x+5

def dotwrite(ast):
   nodename = getNodename()
   label=symbol.sym_name.get(int(ast[0]),ast[0])
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```
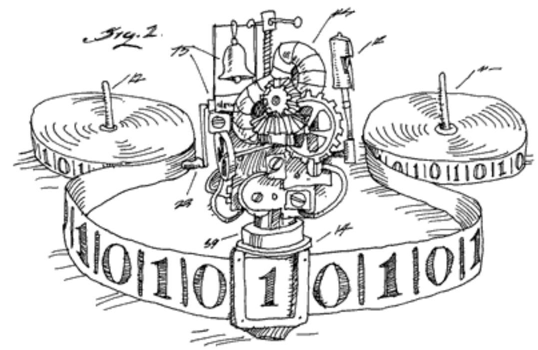


- Ultimate goal: understanding why a program has a given complexity.

- *E.g.*: What does a polytime program look like?

An analogy: termination



- What does a terminating program look like?

- Subsumes an undecidable problem, OK, but it doesn't mean we can't:

  1. non-trivially characterize termination ($e.g.$ intersection types);
  2. find *decidable* criteria isolating an interesting subset of terminating programs ($e.g.$ simple types, `ML` polymorphism);
  3. find programming languages whose programs *intrinsically* terminate and which nevertheless have reasonable expressive power ($e.g.$ primitive recursive functions).

The polytime side of the analogy

Some of the things we will see:

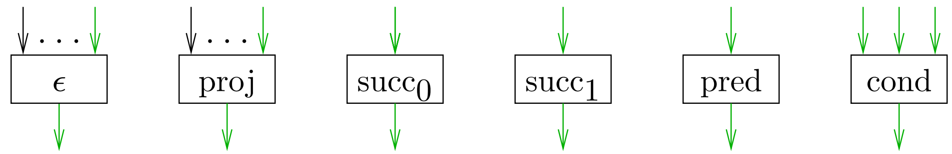
1. $d\ell\mathrm{PCF}$ by Dal Lago and Gaboardi (2011).

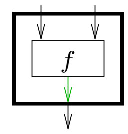
2. **DLAL** by Baillot and Terui (2004), STA (Gaboardi and Ronchi 2007), quasi-interpretations (Bonfante, Marion, Moyen 2007), . . .

3. λ-calculi based on light logics (Girard 1998, Lafont 2003), ramification and predicative recursion (Leivant 1991, Bellantoni and Cook 1992, Leivant and Marion 1993, Bellantoni, Niggl, Schwichtenberg 2000, . . .)

Example: Leivant via Bellantoni-Cook

- Idea: strings are both *data* (safe) and *recursion templates* (normal).

- Basic functions: 

- Closed under (well-sorted) composition, *lifting* 

and *predicative recursion on notation*:

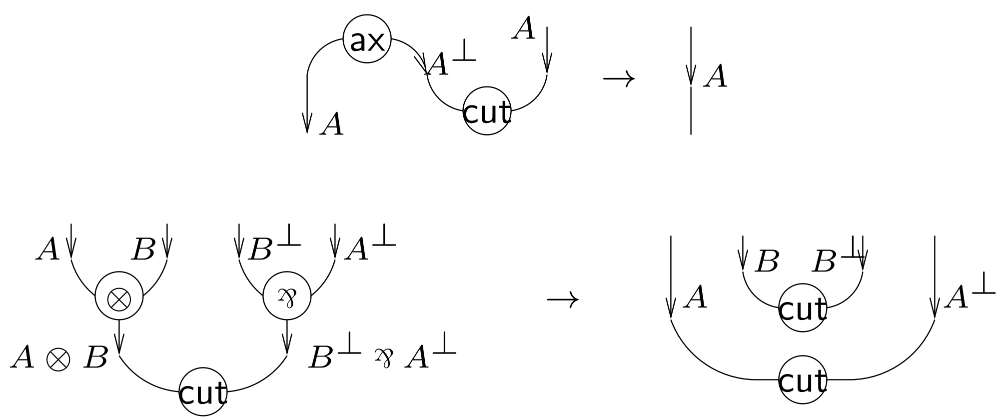
$$\mathrm{rec}[f_0, f_1, g](\epsilon, \vec{x}; \vec{a}) = g(\vec{x}; \vec{a})$$

$$\mathrm{rec}[f_0, f_1, g](zi, \vec{x}; \vec{a}) = f_i(z, \vec{x}; \vec{a}, \mathrm{rec}[f_0, f_1, g](z, \vec{x}; \vec{a}))$$

- Polytime functions: normal inputs to safe output.

A linear (and trivial) example: the affine λ-calculus

- Remember cut-elimination in multiplicative linear logic:



- Number of steps bounded by the size of the initial proof net.

- Affine λ-calculus: $t, u ::= x \mid \lambda x.t \mid tu \text{ s.t. } \mathrm{fv}(t) \cap \mathrm{fv}(u) = \emptyset$.

A parenthesis: the complexity/ies of $\mathbf{MLL}$

- Cut-elimination ($i.e.$, given two $\mathbf{MLL}$ proof nets, do they have the same cut-free form?) is $\mathbf{P}$-complete (Mairson and Terui 2003). We have basically seen that it is in $\mathbf{P}$; hardness is shown by encoding Boolean circuits in $\mathbf{MLL}$ proof nets.

- Interestingly, $\mathbf{MLL}$ cut-elimination *with atomic axioms* is in $\mathbf{L}$. The algorithm uses the geometry of interaction! What's hiding behind η?

- Correctness ($i.e.$, is an $\mathbf{MLL}$ proof structure a proof net?) is $\mathbf{NL}$-complete (de Naurois and Mogbil 2009). There is a correctness criterion the verification of which subsumes reachability.

- Provability ($i.e.$, is the $\mathbf{MLL}$ formula A provable?) is $\mathbf{NP}$-complete. Can you see why it is in $\mathbf{NP}$?

Naive set theory

- Terms: $t, u ::= x \mid \{x \mid A\}$
- Formulas: $A, B ::= t \in u \mid t \notin u \mid A \wedge B \mid A \vee B \mid \forall x.A \mid \exists x.A$
- One-sided classical sequent calculus $(\mathbf{LK})$, plus

$$\frac{\vdash \Gamma, A[t/x]}{\vdash \Gamma, t \in \{x \mid A\}} \in \qquad\qquad \frac{\vdash \Gamma, \neg A[t/x]}{\vdash \Gamma, t \notin \{x \mid A\}} \notin$$

- Standard cut-elimination rules, plus the obvious one for membership.

Russel's antinomy

- Define:

$$M := x \notin x,$$
$$r := \{x \mid M\},$$
$$R := \neg M[r/x] = r \in r$$

- We have

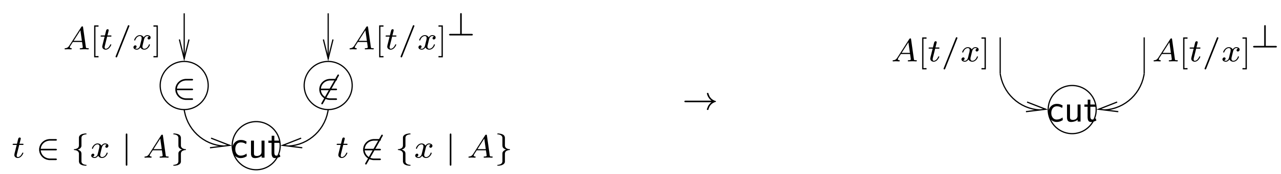
$$\dfrac{\dfrac{\dfrac{}{\vdash \neg R, R}}{\vdash \neg R, \neg R}\,\notin}{\vdash \neg R} \qquad \dfrac{\dfrac{\dfrac{}{\vdash \neg R, R}}{\vdash R, R}\,\in}{\vdash R}$$
$$\overline{\qquad\qquad\vdash\qquad\qquad}$$

- As a consequence, cut-elimination does not terminate.

Naive set theory in MLL

- Terms: $t, u ::= x \mid \{x \mid A\}$
- Formulas: $A, B ::= t \in u \mid t \notin u \mid A \otimes B \mid A \parr B \mid \forall x.A \mid \exists x.A$
- Usual multiplicative proof nets (without units), plus

- Usual multiplicative cut-elimination rules, plus



No contraction, no contradiction

- Define M, r and R as before. It is still true that R is equivalent to $R^\perp$ ($i.e.$, $(R \multimap R^\perp) \otimes (R^\perp \multimap R)$ is derivable).

- However, the empty sequent is no longer derivable!

Why?

No contraction, no contradiction

- Define M, r and R as before. It is still true that R is equivalent to $R^\perp$ ($i.e.$, $(R \multimap R^\perp) \otimes (R^\perp \multimap R)$ is derivable).

- However, the empty sequent is no longer derivable!

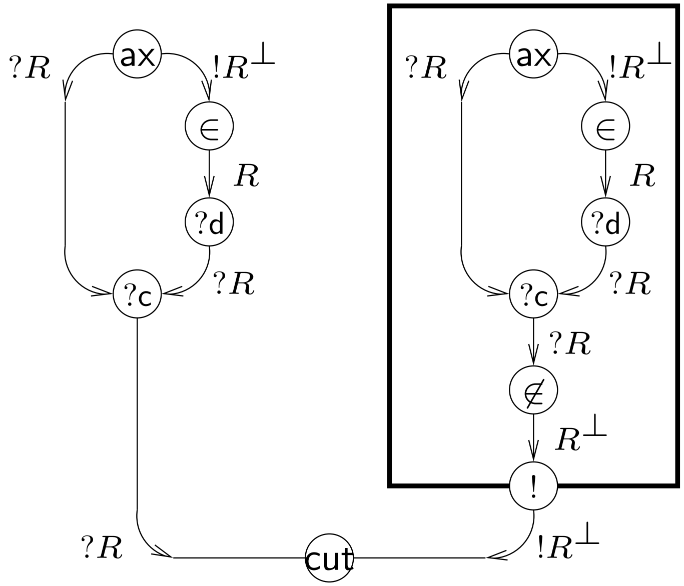
- Because cut-elimination holds by the usual argument:

size-decrease $+$ preservation of correctness

- Girard's insight: the key is *untyped* cut-elimination, $i.e.$, a cut-elimination proof not relying on formulas.

Russel's antinomy in MELL

- Remember the translation of classical negation in linear logic: R is equivalent to $!R^{\perp}$.



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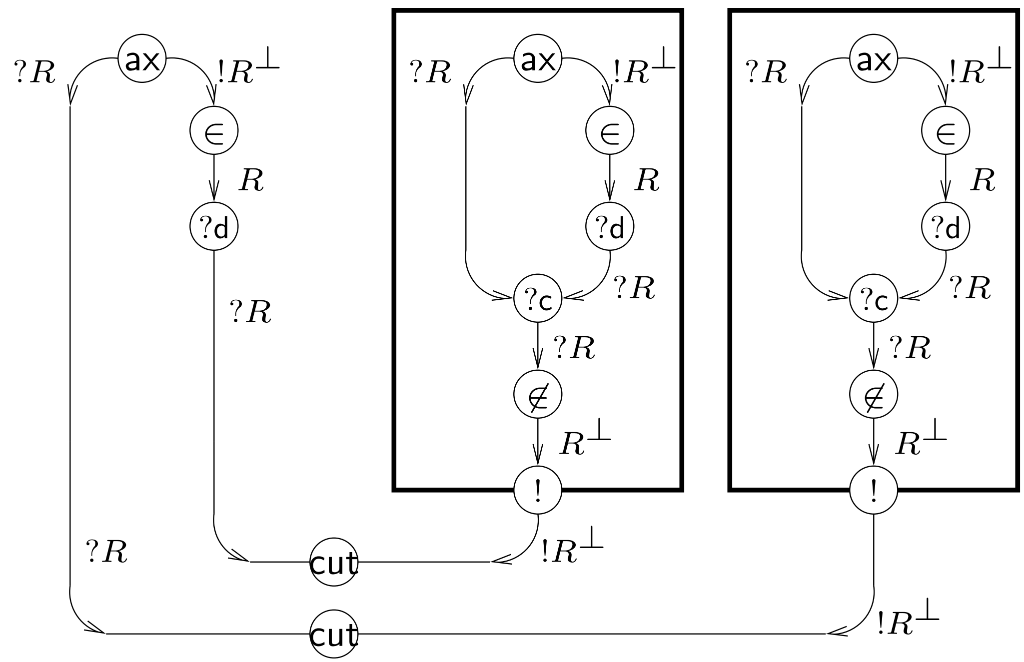

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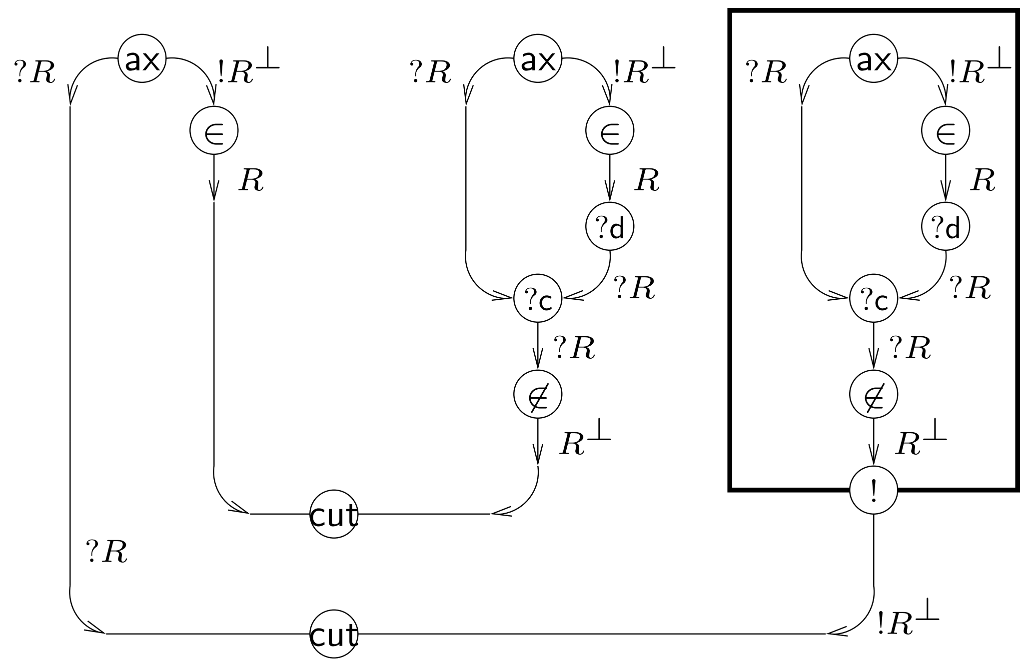
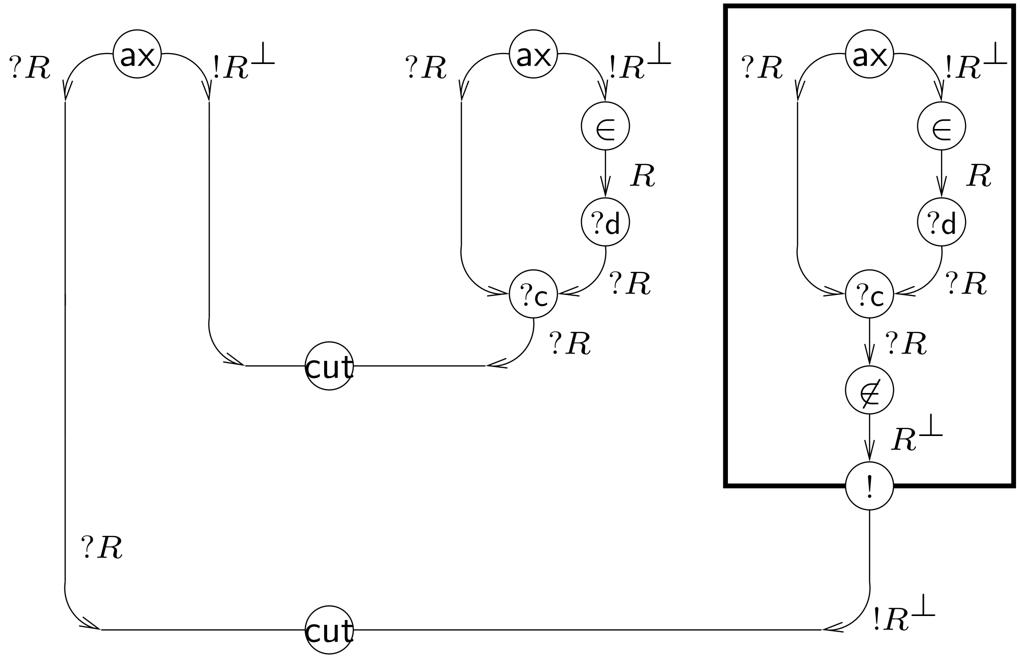


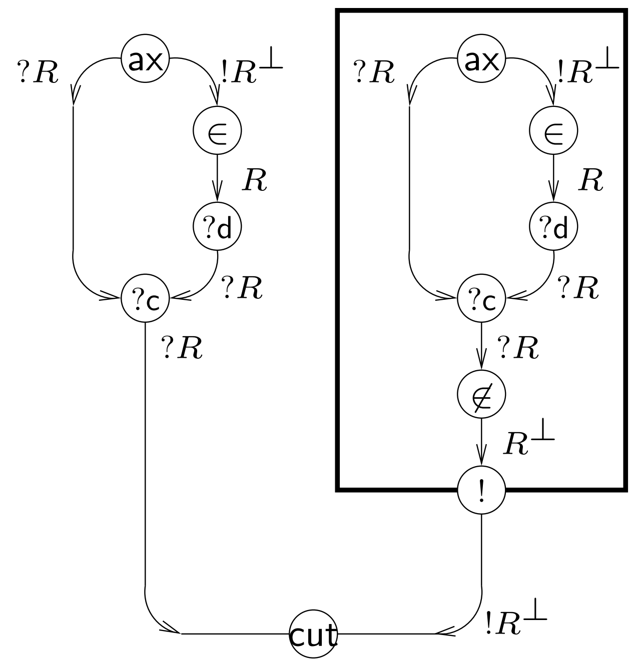
Russel's antinomy in $\mathbf{MELL}$

- Remember the translation of classical negation in linear logic: R is equivalent to $!R^\perp$.



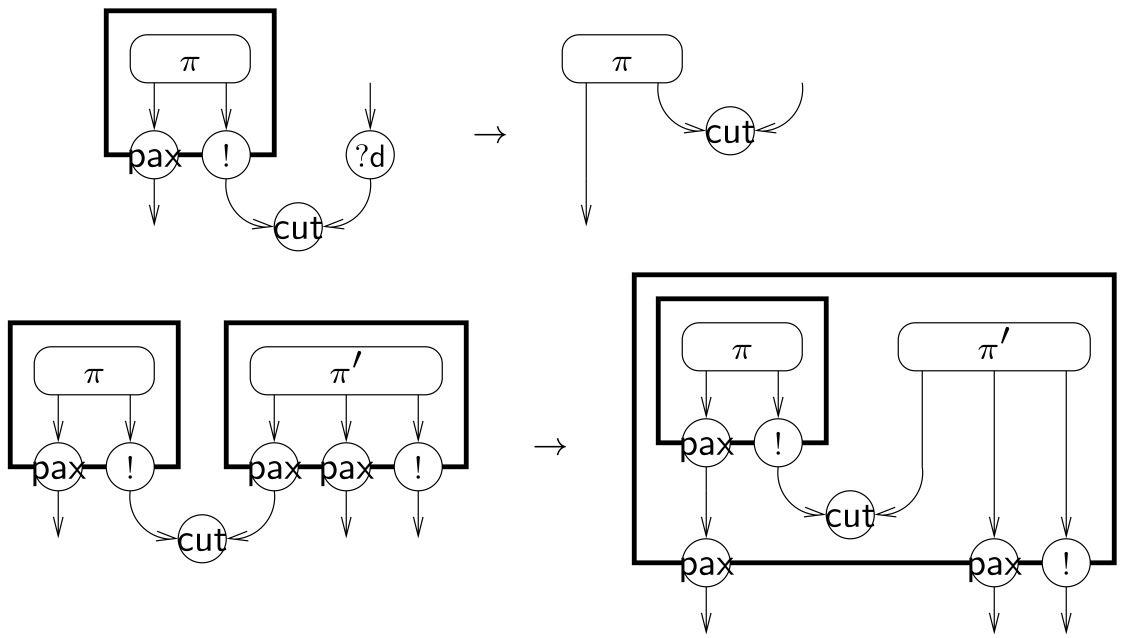
Russel's antinomy in MELL

- Remember the translation of classical negation in linear logic: R is equivalent to $!R^{\perp}$.



Opening boxes, boxing boxes

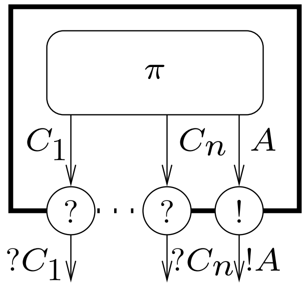
- Remember the *depth* of a proof net: it is the maximum number of boxes nested one into the other. It is altered by two cut-elimination steps:



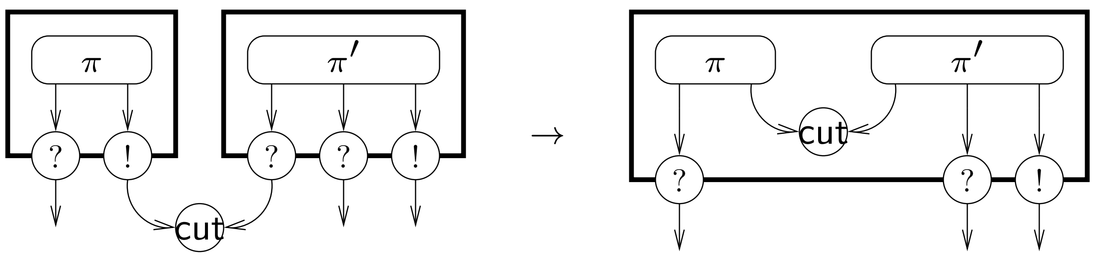
- Depth-changing is needed in Russel's antinomy!

ELL: functorial boxes

- We eliminate $?$d links, and replace boxes with *functorial boxes*:



- Cut-elimination does not alter the depth:



Untyped cut-elimination

- We consider 3 cut-elimination steps: axiom, multiplicative, exponential (contraction/weakening + functorial box).

- Let $|\pi|_i$ be the size of the proof net π at depth $i \geq 0$, and let $|\pi|_i = 0$ for all $i < 0$. If π has depth d, we define

$$\begin{array}{rccc} \alpha_\pi : & \mathbb{N} & \to & \mathbb{N} \\ & n & \mapsto & |\pi|_{d-n} \end{array}$$

- We see α_π as an ordinal $< \omega^\omega$ and verify that

$$\pi \to \pi' \qquad \text{implies} \qquad \alpha_\pi > \alpha_{\pi'}.$$

- Correctness is preserved, so we have (untyped) cut-elimination!

Quantifying the runtime

- When we operate at depth i, nothing happens at depth $j < i$. So, if π has depth d and normal form π', we may go "depth by depth":

$$\pi = \pi_0 \to^* \pi_1 \to^* \pi_2 \to^* \cdots \to^* \pi_n \to^* \pi_{n+1} = \pi', \qquad n \leq d$$

  - The length of $\pi_i \to^* \pi_{i+1}$ is bounded by $|\pi_i|$ (the size of π_i);
  - cut-elimination steps at most square the size of proof nets, so

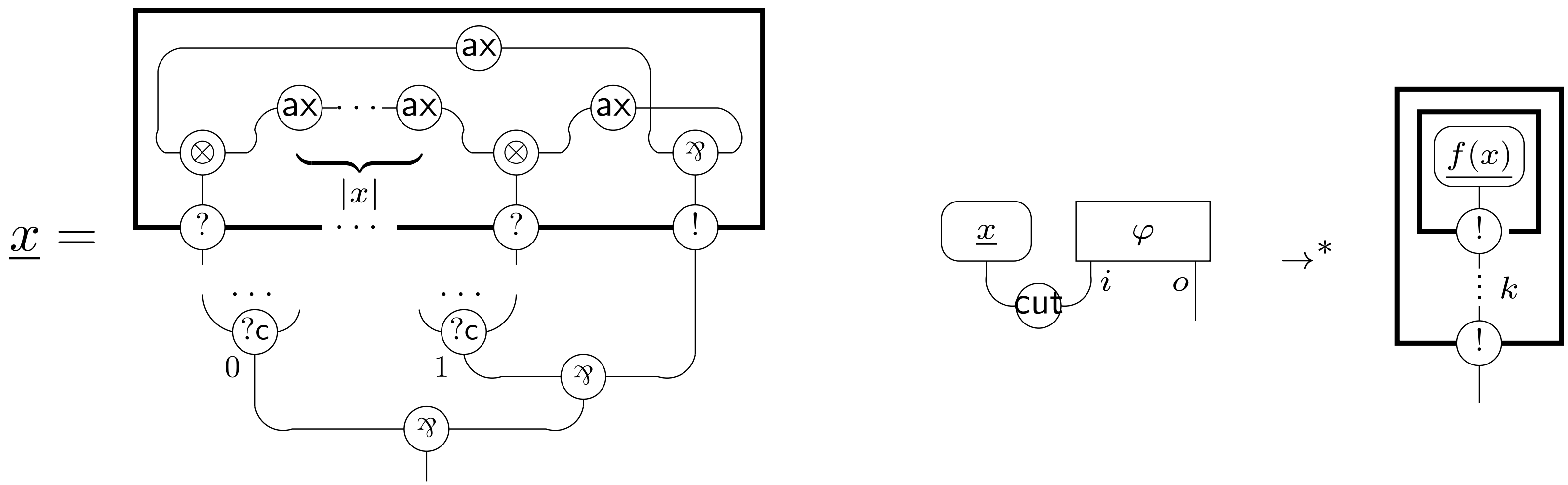
$$|\pi_{i+1}| \leq |\pi_i|^{2^{|\pi_i|}} \leq 2^{2^{2^{|\pi_i|}}} = 2_3^{|\pi_i|}$$

- Therefore, the total runtime is bounded by

$$\sum_{i=0}^{n} |\pi_i| \leq \sum_{i=0}^{n} 2_{3i}^{|\pi|} \leq (n+1) 2_{3n}^{|\pi|} \leq (d+1) 2_{3d}^{|\pi|}$$

Representing functions on strings

- A function $f : \{0,1\}^* \to \{0,1\}^*$ is *representable* in $\mathbf{ELL}$ if there are $k \geq 0$ and a proof net φ with two conclusions, i and o, such that



- In fact, we are using Church strings: $\lambda f_0.\lambda f_1.\lambda z.f_{i_1}(\ldots f_{i_n} z \ldots)$.

A characterization of elementary functions

- Let π be the proof net obtained by cutting φ with $\underline{x}$ on i.
  - $|\pi| = \Theta(|x|)$;
  - the depth of π does not depend on x.

- Cut-elimination on Turing machines has only a polynomial slowdown. Hence, all functions representable in $\mathbf{ELL}$ are elementary.

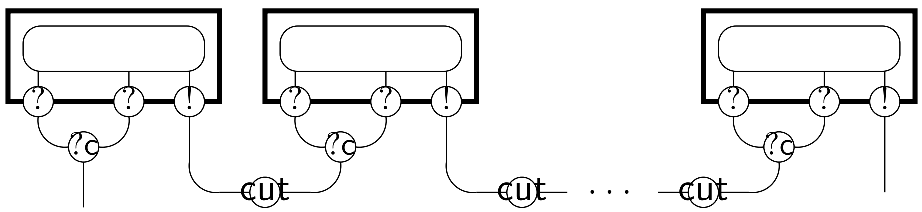
- Conversely, one may show that every elementary function may be represented in $\mathbf{ELL}$. Furthermore, we may restrict to intuitionistic second-order typable proof nets, of type $\mathbf{S} \vdash !^k\mathbf{S}$ for some $k \geq 0$, where

$$\mathbf{S} := \forall X.!(X \multimap X) \multimap !(X \multimap X) \multimap !(X \multimap X),$$

  which is a decoration of the system F type of Church binary strings.

LLL: forbidding exponential chains

- The exponential blow-up in the normalization of **ELL** is essentially due to configurations such as the following:



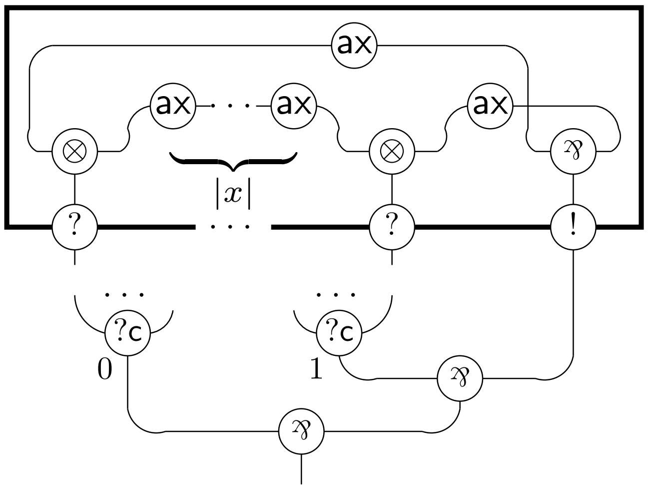
- **LLL** is defined by restricting to boxes with at most one auxiliary door.

- The total arity of contractions at depth i *does not increase* during cut-elimination at depth i. Therefore, $|\pi_{i+1}| \leq |\pi_i|^2$, and we get

$$\text{runtime} \leq \sum_{i=0}^{n} |\pi_i| \leq \sum_{i=0}^{n} |\pi|^{2^i} \leq (n+1)|\pi|^{2^n} \leq (d+1)|\pi|^{2^d}$$

A problem of expressiveness

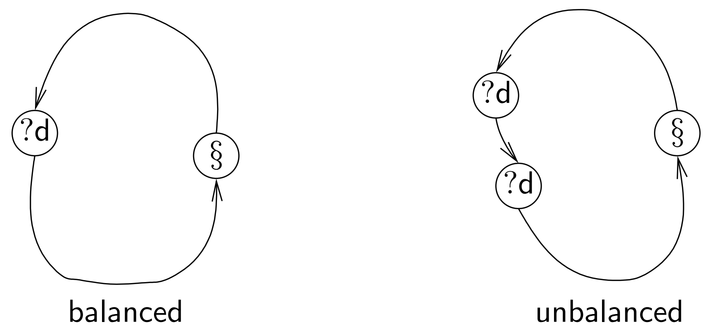
- Recall the representation of binary strings in $\mathbf{ELL}$:



- In $\mathbf{LLL}$, this only works for strings of length at most $1\ldots$

The paragraph

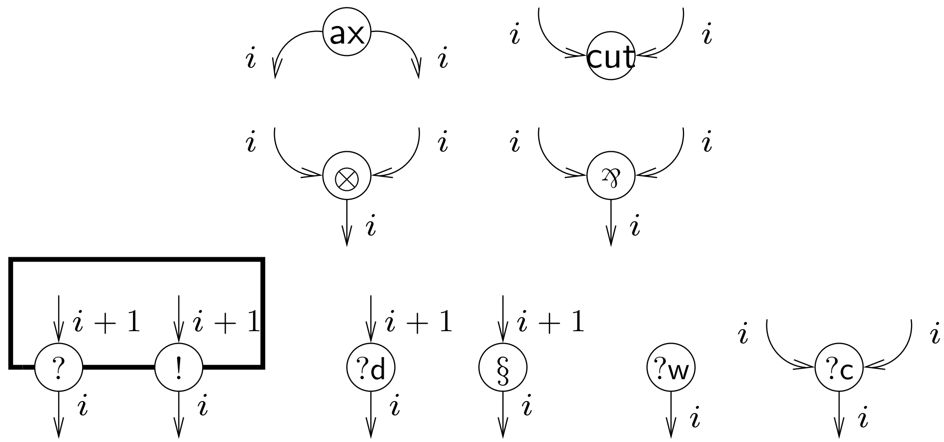
- We re-introduce $?\mathrm{d}$ links, plus a new unary link §.
- We define *balanced cycles* (ignoring switches!!!) by counting the number of $?\mathrm{d}$ and § links crossed going "up" and "down":



- A proof net with § links is *balanced* if all of its cycles are balanced (cycles are allowed to jump between conclusions).

Levels

- A proof net is balanced iff there exists a labelling of its links in $\mathbb{N}$ s.t.



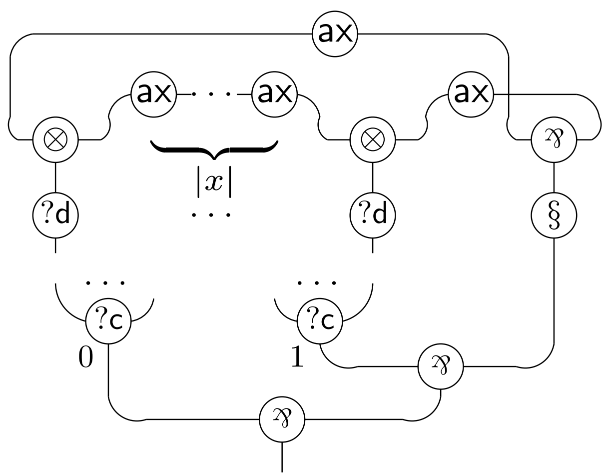
and all conclusions have the same label (Baillot and M. 2010, Boudes, M. and Tortora de Falco 2013).

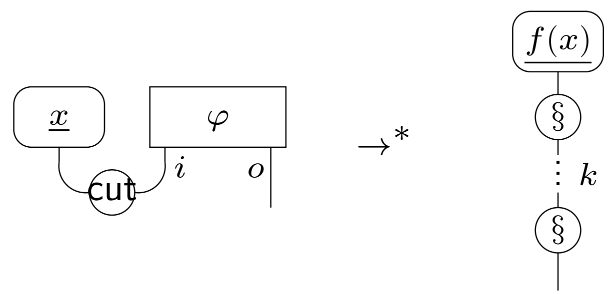
- This integer is the *level* of a link. It behaves very much like the depth.

Representing functions in $\mathbf{LLL}$

- A function $f : \{0,1\}^* \to \{0,1\}^*$ is *representable* in $\mathbf{LLL}$ if there are $k \geq 0$ and a proof net φ with two conclusions, i and o, such that

$\underline{x} =$





- We are using again Church strings, but with a different decoration.

A characterization of polytime functions

- Let π be the proof net obtained by cutting φ with $\underline{x}$ on i.
  - $|\pi| = \Theta(|x|)$;
  - the level of π does not depend on x.

- Cut-elimination on Turing machines has only a polynomial slowdown. Hence, all functions representable in $\mathbf{LLL}$ are polytime.

- Conversely, one may show that every polytime function may be represented in $\mathbf{LLL}$. Furthermore, we may restrict to intuitionistic second-order typable proof nets, of type $\mathbf{S}' \vdash \S^k\mathbf{S}'$ for some $k \geq 0$, where

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  which is another decoration of the system F type of Church binary strings.

A word on the completeness proofs

- For $\mathbf{ELL}$, it is possible to use recursive-theoretic characterizations of elementary functions ($e.g.$ Danos and Joinet (2003) use Kalmar's: elementary functions contain constants, projections, addition, multiplication, equality test and are closed under composition, bounded sums and bounded products).

- For $\mathbf{LLL}$, it would be nice to use Bellantoni and Cook's characterization, but it doesn't work. So we do things "manually" (Girard 1998):
  - we show that $\mathbf{LLL}$ can encode one step of computation of arbitrary Turing machines;
  - we show that polynomials (on unary integers) are representable in $\mathbf{LLL}$;
  - so we have Turing machines with polynomial clocks, and we are done.

A word on representations

- Observe that the type of binary strings, both in $\mathbf{ELL}$ and $\mathbf{LLL}$, is *not* what you would obtain by applying Girard's (CbN) translation of intuitionistic logic into linear logic:

$$\forall X.!(!X \multimap X) \multimap !(!X \multimap X) \multimap !X \multimap X.$$

- In fact, let $\mathbf{PN}_\lambda$ denote the set of all $\mathbf{MELL}$ proof nets which are CbN translations of some λ-term, and let $\mathbf{PN}_{\mathbf{ELL}}$ be the set of $\mathbf{ELL}$ proof nets (embedded in $\mathbf{MELL}$ in the obvious way). Then

$$\mathbf{PN}_\lambda \cap \mathbf{PN}_{\mathbf{ELL}} = \emptyset.$$

- Moreover, there is no such thing as polarized $\mathbf{ELL}$, $\mathbf{LLL}$, etc.

Soft linear logic

- Replace the usual exponential rules of sequent $\mathbf{LL}$ calculus

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$$

with

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\ \text{functorial promotion} \qquad \frac{\vdash \Gamma, \overbrace{A, \ldots, A}^{n}}{\vdash \Gamma, ?A}\ \text{multiplexing}$$

- Untyped cut-elimination in $\mathcal{O}(s^d)$ steps (size s, depth d), with a marvelously simple proof. Entails polytime soundness.

- By contrast, proving polytime completeness is tricky. As a programming language, $\mathbf{SLL}$ is far from user friendly. . .

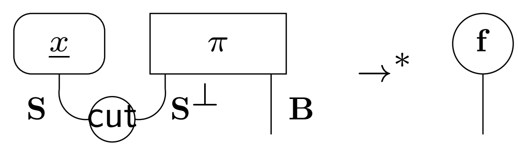
A word on diagonalization

- How do we separate primitive recursive sets from recursive sets? Diagonalization, of course:

$$\{x \in \{0,1\}^* \mid \exists P \text{ prim. rec. } x = \ulcorner P \urcorner \text{ and } P(x) = \text{false}\} \in \mathbf{R} \setminus \mathbf{PR}$$

- What happens if we diagonalize $\mathbf{P}$? The set

$$\Bigg\{ x \in \{0,1\}^* \mid \exists \pi \in \mathbf{SLL}.\ x = \ulcorner \pi \urcorner \text{ and}$$



$$\Bigg\}$$

cannot be in $\mathbf{P}$ by construction. Can you show an upper bound to its complexity? (Following the recursion-theory analogy, it should be in $\mathbf{NP} \cap \mathbf{coNP}$, but probably it is not even in $\mathbf{PSPACE}$. . .).

Dual light affine logic

- Recall how, in $\mathbf{LLL}$, we may actually restrict to intuitionistic, second-order typed proof nets, $i.e.$, λ-terms. The following type system is due to Baillot and Terui (2004), using Barber and Plotkin (1997):

$$\frac{}{\Theta; x : A \vdash x : A}$$

$$\frac{\Theta; \Gamma, x : A \vdash t : B}{\Theta; \Gamma \vdash \lambda x.t : A \multimap B} \qquad \frac{\Theta; \Gamma \vdash t : A \multimap B \quad \Theta; \Delta \vdash u : A}{\Theta; \Gamma, \Delta \vdash tu : B}$$

$$\frac{\Theta, z : A; \Gamma \vdash t : B}{\Theta; \Gamma \vdash \lambda x.t : A \Rightarrow B} \qquad \frac{\Theta; \Gamma \vdash t : A \Rightarrow B \quad ; x : C \vdash u : A}{\Theta \cup z : C; \Gamma \vdash tu : B}$$

$$\frac{; \Gamma, \Delta \vdash t : A}{\Gamma; \S\Delta \vdash t : \S A} \qquad \frac{\Theta; \Delta \vdash u : \S A \quad \Theta; \Gamma, x : \S A \vdash t : B}{\Theta; \Gamma, \Delta \vdash t[u/x] : B}$$

Dual light affine logic

$$\frac{\Theta; \Gamma \vdash t : A}{\Theta; \Gamma \vdash t : \forall X.A} \qquad \frac{\Theta; \Gamma \vdash t : \forall X.A}{\Theta; \Gamma \vdash t : A[B/X]}$$

Theorem. *The functions definable by λ-terms of type $\mathbf{S}' \multimap \S^k \mathbf{S}'$ in $\mathbf{DLAL}$ are exactly the polytime functions.*

- However, there's an issue of *intensional expressiveness*: although every polytime function $f : \{0,1\}^* \to \{0,1\}^*$ admits a $\mathbf{DLAL}$-typable λ-term t computing it, t is most likely to be very contrived, $i.e.$, it may look nothing like the λ-term you would write to compute f.

- A system with similar properties, STA (Soft Type Assignment), based on affine $\mathbf{SLL}$ instead of $\mathbf{LLL}$, was introduced by Gaboardi and Ronchi Della Rocca (2007). It suffers from a similar problem.

The sub-elementary hierarchy within $\mathbf{ELL}$

- Baillot (2011) has shown how, using fixpoints in (affine) $\mathbf{ELL}$ types, one may obtain the following characterization (we stipulate 0-$\mathbf{EXP}$=$\mathbf{P}$):

  Theorem. n-$\mathbf{EXP}$ *(with* $n \geq 0$*) is the class of languages decidable by* $\mathbf{ELL}$ *proof nets of type* $!\mathbf{S} \vdash !^{2+n}\mathbf{B}$*, where* $\mathbf{B} = \forall X.X \multimap X \multimap X$*.*

- Later, Laurent has shown how to obtain the same characterization in the untyped framework (which was our choice in this lecture).

- The idea is that, to know the value of a Boolean (the “answer”) one may stop normalizing at the depth where the Boolean is.

The categorical perspective

- Quick recap on categorical models of $\mathbf{MELL}$:
  - a $*$-autonomous category $(\mathcal{L}, \otimes, 1, \perp)$;
  - a monoidal comonad $(!, \mathsf{dig}, \mathsf{der})$ on $\mathcal{L}$...
  - ... such that every $!A$ is a commutative comonoid;
  - (and the free !-coalgebra and the comonoid structure interact nicely).

- A model of $\mathbf{ELL}$ drops the condition that ! is a comonad. The ! functor of $\mathbf{LLL}$ further drops the monoidality requirement.

- Paradox: although being a model of $\mathbf{ELL}$ is "easier", in practice it is hard to find a *strict* one! In fact, the simplest way to model exponentials is to construct $!A$ as the free commutative comonoid, which automatically yields a model of $\mathbf{MELL}$ (a *Lafont category*, as most practical models).

Objects with involutions

- Let $\mathcal{C}$ be your favorite category. An *object with involutions* is a pair (A, s) such that A is an object of $\mathcal{C}$ and $s = (s_k)_{k \in \mathbb{Z}}$ is a family of involutions of A ($i.e.$, $s_k \circ s_k = id_A$ for all $k \in \mathbb{Z}$).

- A morphism between objects with involutions (A, s), (B, t) is a morphism $f : A \to B$ of $\mathcal{C}$ such that $t_k \circ f \circ s_k = f$ for all $k \in \mathbb{Z}$.

- Objects with involutions of $\mathcal{C}$ and their morphisms form a category $\mathrm{Inv}\mathcal{C}$. Moreover, if $\mathcal{C}$ is a model of $\mathbf{MELL}$, then so is $\mathrm{Inv}\mathcal{C}$.

- Define an endofunctor of $\mathrm{Inv}\mathcal{C}$ by $\S(A, s) = (A, (s_{k-1})_{k \in \mathbb{Z}})$, and acting as the identity on morphisms. If we define $!' = ! \circ \S$, we obtain a strict model of $\mathbf{ELL}$ (plus paragraph): $!'$ is a monoidal functor which is not a comonad but such that $!'A$ is a commutative comonoid.

Further reading

- Characterization of space classes: $\mathbf{PSPACE}$ (Gaboardi, Marion, Ronchi 2008), $\mathbf{L}$ (Schöpp 2007). The classes $\mathbf{L}$ and $\mathbf{coNL}$ have also being characterized using GoI5 (von Neumann algebras) by Girard (2010) and Aubert and Seiller (2013).

- Systems related to bounded linear logic (Dal Lago and Hofmann 2010, Dal Lago and Gabardi 2011).

- Sematic proofs of soundness, via intuitionistic realizability (Dal Lago and Hofmann 2008) or classical (Krivine's) realizability/forcing (Brunel 2013).

- Tons of other stuff, just ask me.