The Uncapacitated Asymmetric Traveling Salesman Problem with Multiple Stacks

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Abstract. In the uncapacitated asymmetric traveling salesman with multiple stacks, we perform a hamiltonian circuit to pick up n items, storing them in a vehicle with k stacks satisfying last-in-first-out constraints, and then we deliver every item by performing a hamiltonian circuit. We are interested in the convex hull of the (arc-)incidence vectors of such couples of hamiltonian circuits.
For the general case, we determine the dimension of this polytope, and show that every facet of the asymmetric traveling salesman polytope defines one of its facets. For the special case with two stacks, we provide an integer linear programming formulation whose linear relaxation is polynomial-time solvable, and we propose new families of valid inequalities to reinforce this linear relaxation.

Keywords: uncapacitated asymmetric traveling salesman problem with multiple stacks, polytope, facets, formulation, valid inequalities

Introduction

The *Asymmetric Traveling Salesman Problem (ATSP)* consists of finding a hamiltonian circuit of minimum cost in a digraph. This problem is emblematic of the success of polyhedral approaches which consist in studying the convex hull of the (arc-)incidence vectors of the solutions. Although ATSP is NP-complete, it is possible to solve instances of quite large size [7] combining linear programming based methods and structural results about the polytope.

Many variants and extensions of ATSP have been considered [7]. Here, we are interested in the *uncapacitated asymmetric traveling salesman with multiple stacks*. We are given a vehicle with k stacks of infinite capacity. Starting from its depot, the vehicle has to pickup n items in a city, each one in a specific location, and then to deliver them to specific locations in another city. We consider that the two cities are far away from each other, hence the vehicle must do all the pick ups before performing all the deliveries. Moreover, no rearrangement of the content of the vehicle is allowed and the stacks satisfy a last-in-first-out policy.

More formally, the two cities are modeled by two cost vectors c^1 and c^2 on the arcs of a complete digraph $D = (V, A)$, where $V = \{0, \ldots, n-1\}$. The first one, c^1, represents the travel costs within the town where all the pickups are performed, and the latter one corresponds to the delivery town. The vertex 0 is the depot and the other vertices correspond to the locations of the items. Without loss of generality, we suppose that the i^{th} item is picked up at vertex i and must be delivered to vertex i.

A solution of our problem is a couple of two hamiltonian circuits, say (C^1, C^2), where C^1 is a *pickup hamiltonian circuit*, that is a trip of the vehicle in order to perform all the pickups, and C^2 is a *delivery hamiltonian circuit*. Moreover, there must exist a decomposition of the n items into k stacks in such a way that C^1 (C^2, respectively) iteratively stores (picks, respectively) each item at the top of a stack. Such a decomposition is called a *loading plan*, and two hamiltonian circuits for which a loading plan exists are called *k-consistent*. The cost of a solution is the sum of the travel costs associated with the arcs of both circuits, and the goal is to find two k-consistent hamiltonian circuits such that the cost is minimum.

Our problem is a relaxation of the *capacitated traveling salesman with multiple stacks* recently introduced by Petersen et al. in [9], where, in addition, stacks may not contain more than p items. They provided a mathematical formulation and then developped a local search algorithm to heuristically solve the problem on their set of instances. Since then, most published algorithms were tested on these instances. Later on, Petersen et al. proposed and compared different approaches to solve the problem in [10]. One of their ideas especially gives good results, and a similar approach if used by Alba et al. in [1] to derive a Branch-and-Cut algorithm, which is currently the best available for the general case. For the special case with two stacks, Carrabs et al. [5] designed an additive Branch-and-Bound algorithm. It strongly relies on the specific structure with two stacks and does not extend straightforwardly to the general case. A different approach was adopted by Lusby et al. in [8], where they check whether there are k-consistent hamiltonian circuits within the t best ones, for some t. Despite the variety of available approaches, the largest instances solved to optimality roughly have 25 items.

Incidentally, the work of Felipe et al. [6] is based on heuristic procedures using neighborhood searches, whereas Toulouse [12] derived an approximation scheme.

Furthermore, the problem yields a few captivating subproblems. For instance, deciding whether two hamiltonian circuits are k-consistent is NP-complete even if the capacity of each stack is a fixed number greater than 5, see [2]. It turns out that it becomes tractable if the capacity condition is relaxed, see Calvo et al. [3] and Casazza et al. [4]. In another direction, Bonomo et al. [2] proved that it is also polynomial if the number of stacks is fixed.

As one could suspect for a problem combining routing and loading aspects, the existing approaches tend to show that it is quite challenging to practically solve instances of decent sizes. Yet, we are far from a good understanding of the polyhedral structure of the problem, and it is reasonable to expect that results in this direction would lead to better algorithms, especially for the branching ones.

The main contribution of the present paper consists in a polyhedral study of the uncapacitated asymmetric traveling salesman with multiple stacks. In particular, we determine the dimension of the corresponding polytope and show that every facet of the asymmetric traveling salesman polytope defines one of its facets.

The paper is organized as follows. In Section 1, we give the definitions used throughout the paper. In Section 2, we reveal a close link between the asymmetric traveling salesman polytope and the convex hull of couples of k-consistent hamiltonian circuits. In Section 3, we focus on the special case with two stacks: we provide an integer linear programming formulation whose linear relaxation can be solved in polynomial time, as well as new families of valid inequalities.

1 Definitions

Throughout, $D = (V, A)$ will denote a complete directed graph with vertex set $V = \{0, \dots, n-1\}$. Let $a = (u, v)$ be an arc of A, u is the *tail* and v is the *head* of a. We will also denote a by uv. Given $X \subseteq V$ and $Y \subseteq V$, $A[X, Y]$ is the set of arcs having their tail in X and their head in Y. Let $A[X] = A[X, X]$, $\delta^+(X) = A[X, V \setminus X]$, $\delta^-(X) = A[V \setminus X, X]$ and $\delta(X) = \delta^+(X) \cup \delta^-(X)$. An arc of $\delta^+(X)$ (resp. $\delta^-(X)$) is *leaving* X (resp. *entering* X). A set $B \subseteq A$ of arcs is *covering* X if every vertex of X belongs to at least one arc of B. An *ij-path* is a path whose first vertex is i and last vertex is j. For pairwise distinct $i, j, k \in V \setminus \{0\}$, $\mathcal{P}^0_{ij}(D \setminus \{k\})$ denotes the set of ij-paths of $D \setminus \{k\}$ containing 0.

Given a hamiltonian circuit C and $i \neq j \in V \setminus \{0\}$, we will write $i \prec_C j$ if C visits 0, i and j in this order. Given $X, Y \subset V \setminus \{0\}$, $X \prec_C Y$ means that $x \prec_C y$ for all $x \in X$ and $y \in Y$. An *increasing sequence of size k for C* is a set of k vertices $v_1, \dots, v_k$ satisfying $v_j \prec_C v_{j+1}$ for $j = 1, 2, \dots, k-1$. Let Id_n denote the hamiltonian circuit $0, 1, \dots, n-1$ and $\overline{Id}_n$ its reverse $n-1, n-2, \dots, 0$.

We now give another definition of consistency, equivalent to the one we saw in the introduction. Given an integer k, two hamiltonian circuits C^1 and C^2 of D are *k-consistent* if and only if there exists a partition $\{V_1, \dots, V_k\}$ of $V \setminus \{0\}$ and a linear order S_h on the vertices of V_h for $h = 1, 2, \dots, k$, such that for all $i \neq j$ in V_h, $h = 1, 2, \dots, k$, with $i \prec_{S_h} j$, we have $i \prec_{C^1} j$ and $j \prec_{C^2} i$. We will write *consistent* instead of 2-consistent.

Given a subset $B \subseteq A$ of arcs, its *incidence vector* is a vector $\chi^B \in \{0, 1\}^{|A|}$ defined by $\chi^B_a = 1$ if $a \in B$, and $\chi^B_a = 0$ otherwise. Since there is a bijection between subsets of arcs and subsets of $\{0, 1\}^{|A|}$, we will often use the same terminology for both. For instance, a hamiltonian circuit C might denote either the subset of arcs or its incidence vector, depending on the context. Given $B \subseteq A$ and $x \in \mathbb{R}^{|A|}$, let $x(B) = \sum_{a \in B} x_a$.

If $\mathcal{C}$ is a set of vectors, $conv(\mathcal{C})$ denotes its convex hull. $ATSP_n$ will be the convex hull of the hamiltonian circuits on n vertices, and, given an integer $k \geq 2$, let $\mathcal{P}_{k,n}$ be the convex hull of the vectors (χ^{C^1}, χ^{C^2}) where C^1 and C^2 are k-consistent hamiltonian circuits on n vertices. Note that if $k \geq n$ then $\mathcal{P}_{k,n} = ATSP_n \times ATSP_n$.

2 General results

In this section, we first recall well-known results on the traveling salesman polytope. Then, we characterize k-consistent hamiltonian circuits. To conclude, we reveal a polyhedral connection between $ATSP_n$ and $\mathcal{P}_{k,n}$, see Corollary 6 and Theorem 7.

2.1 Asymmetric traveling salesman polytope

Here, we recall two well-known results on the asymmetric traveling salesman polytope. We shall use them in the rest of the section.

Let x be a vector of $\mathbb{R}^{|A|}$. The inequalities (1)-(5) are clearly valid for $ATSP_n$.

$$\sum_{j \in V \setminus \{i\}} x_{ij} = 1 \quad \forall\, i \in V, \tag{1}$$

$$\sum_{i \in V \setminus \{j\}} x_{ij} = 1 \quad \forall\, j \in V, \tag{2}$$

$$\sum_{a \in \delta^+(W)} x_a \geq 1 \quad \forall\, \emptyset \subset W \subset V, \tag{3}$$

$$x_a \geq 0 \quad \forall\, a \in A, \tag{4}$$

$$x_a \leq 1 \quad \forall\, a \in A, \tag{5}$$

$$x_a \text{ integer} \quad \forall\, a \in A. \tag{6}$$

Inequalities (1) and (2) are the *outdegree constraints* and *indegree constraints*, they force an integral solution to enter and leave each vertex exactly once. The *subtour elimination constraints* (3) ensure that an integral solution does not contain any subtour. Inequalities (4) and (5) are the *trivial constraints* and inequalities (6) are the *integrality constraints*. These constraints are sufficient to formulate the ATSP, as indicated in the following theorem.

Theorem 1 ([7]) *A vector of* $\mathbb{R}^{|A|}$ *satisfying* (1)-(6) *is the incidence vector of a hamiltonian circuit.*

Let d_n denote the dimension of $ATSP_n$.

Theorem 2 ([7]) $d_n = n(n-3)+1$.

2.2 Consistency

The following result seems to be well-known, see [3] and [4]. Yet, our formulation of the characterization seems simpler so we provide our own proof.

Lemma 3 *Two hamiltonian circuits are* k*-consistent if and only if no* $k+1$ *vertices form an increasing sequence for both circuits.*

Proof. The necessity comes from the pigeon hole principle. To see the sufficiency, let C and C' be hamiltonian circuits and consider the permutation graph G associated to C and C' defined by $ij \in E$ if and only if i and j are visited in the same order by C and C'. Note that an increasing sequence of size $k+1$ for both circuits is precisely a clique of size $k+1$ in G. If no such sequence exists, then the size of a clique in G is at most k. Since a permutation graph is a perfect graph, we get $\chi(G) \leq k$, hence G is k-colorable. By definition of the permutation graph, since a color class is a stable set, assigning a stack to each color shows that C and C' are consistent. □

As observed in [3] and [4], since computing the chromatic number of a perfect graph is polynomial [11], the above proof implies that deciding whether two hamiltonian circuits are k-consistent is polynomial in the number of vertices.

2.3 Links with $ATSP_n$

In this section, we determine the dimension of $\mathcal{P}_{k,n}$ and show that every facet of $ATSP_n$ induces a facet of $\mathcal{P}_{k,n}$. Recall that $d_n = \dim(ATSP_n)$.

Claim 4 *Given $k \geq 2$, let C be a hamiltonian circuit and $\mathcal{C}$ the set of hamiltonian circuits k-consistent with C. Then,* $\dim(conv(\mathcal{C})) = d_n$.

Proof. Note that $\dim(conv(\mathcal{C})) \leq d_n$. Hence, since $\mathcal{P}_{2,n} \subseteq \mathcal{P}_{k,n}$ for all $k \geq 2$, it is enough to find $d_n + 1$ affinely independent circuits consistent with C. Without loss of generality, we may assume that $C = \overline{Id}_n$.

Clearly, if $n \leq 3$, then two hamiltonian circuits are consistent. So the claim holds for $n = 3$. Consider the case $n = 4$. The five hamiltonian circuits 0123, 0132, 0312, 0213 and 0231 are consistent with $C = 0321$ and are affinely independant.

Suppose now that the claim holds for $n \geq 4$ and let us show that it holds for $n+1$. By the induction hypothesis, there exist $d_n + 1$ affinely independant hamiltonian circuits consistent with $\overline{Id}_n$. Inserting the vertex n at the end of all these circuits provides $d_n + 1$ affinely independant hamiltonian circuits of $n+1$ vertices consistent with $\overline{Id}_{n+1}$, each of them containing the arc $(n, 0)$. We now complete the set $\mathcal{C}$ by inserting in sequence $2n-2$ additional hamiltonian circuits consistent with $\overline{Id}_{n+1}$. In order to ensure that $\mathcal{C}$ only contains independant circuits, we add to $\mathcal{C}$ at each iteration a circuit S_{ij} associated with an arc ij which belong to S_{ij} but not to any other circuit of $\mathcal{C}$. The hamiltonian circuits S_{ij} are given below.

- $S_{(n-1,0)} = 0, 2, 3, \ldots, n-2, n, 1, n-1$. Since $n \geq 4$, $S_{(n-1,0)}$ does not contain the arc $(0, n)$.
- $S_{(i,0)} = 0, i+1, i+2, \ldots, n, 1, 2, \ldots, i$, for $i = 1, 2, \ldots, n-2$.
- $S_{(0,n)} = 0, n, 1, 2, \ldots, n-1$.
- $S_{(n,i)} = 0, 1, \ldots, i-1, n, i+1, i+2, \ldots, n-1$, for $i = 2, 3, \ldots, n-1$.

Since $|\mathcal{C}| = d_n + 1 + 2n - 2 = n(n-3) + 2 + 2(n-1) = d_{n+1} + 1$, the claim is proved. □

Lemma 5 *Given $k \geq 2$, if $\mathcal{D} = \{D_1, \dots, D_t\}$ is a set of affinely independent hamiltonian circuits, then there exists an affinely independent set $\{(C_i, C'_i) : i = 1, \dots, d_n + t\}$ where C_i and C'_i are k-consistent hamiltonian circuits and $C_i \in \mathcal{D}$ for $i = 1, \dots, d_n + t$.*

Proof. By Claim 4, there exist affinely independant hamiltonian circuits $C'_1, \dots, C'_{d_n+1}$ that are k-consistent with D_t. Let C''_i be a hamiltonian circuit that is k-consistent with D_i, for $i = 1, \dots, t-1$. For $j = 1, \dots, d_n + 1$, let $V_j = (D_t, C'_j)$ and for $j = 1, \dots, t-1$, let $V_{j+d_n+1} = (D_j, C''_j)$. By construction, $V_1, \dots, V_{d_n+t}$ are affinely independent. □

When $k = 1$, fixing the pickup hamiltonian circuit fixes the delivery one, hence $\dim(\mathcal{P}_{1,n}) = d_n$. For $k \geq 2$, the dimension of $\mathcal{P}_{k,n}$ immediately follows from Lemma 5.

Corollary 6 *Given $k \geq 2$, $\dim(\mathcal{P}_{k,n}) = 2d_n$.*

In fact, $\mathcal{P}_{k,n}$ and $ATSP_n$ also share some polyhedral structure, as shown in the following.

Theorem 7 *Every facet of $ATSP_n$ defines a facet of $\mathcal{P}_{k,n}$.*

Proof. If $k = 1$, then the result is clear by the remark above Corollary 6. Suppose that $k \geq 2$. Let $F = \{x \in \mathbb{R}^{|A|} : cx = d\}$ be a facet of $ATSP_n$, there exists d_n affinely independent hamiltonian circuits that belong to F. Let $\mathcal{C}$ be a family of $2d_n$ affinely independant vectors given by Lemma 5, and let $F' = \{(x_1, x_2) \in \mathbb{R}^{|A|} \times \mathbb{R}^{|A|} : cx_1 = d\}$. Note that every $(C^1, C^2) \in \mathcal{C}$ belongs to F', therefore, by Corollary 6, F' defines a facet of $\mathcal{P}_{k,n}$. □

3 Focus on two stacks

In this section, we focus on the special case of the uncapacitated asymmetric traveling salesman problem with two stacks. First, we derive an integer linear programming formulation for the problem. Then, we show that its linear relaxation is polynomial-time solvable. Finally, we propose three families of valid inequalities for $\mathcal{P}_{2,n}$ in order to reinforce the linear relaxation.

3.1 Formulation

Our formulation consists in gathering inequalities of two traveling salesman polytope and the following *consistency constraint*, see Claim 9.

$$\sum_{h=1,2} \sum_{a \in P^h} x_a^h \leq |P^1| + |P^2| - 1 \qquad \begin{array}{l} \forall\ i \neq j \neq k \neq i \in V \setminus \{0\}, \\ \forall\ P^1, P^2 \in \mathcal{P}^0_{ij}(D \setminus \{k\}). \end{array} \tag{7}$$

Let $\mathcal{P}$ be the set of vectors $(x^1, x^2) \in \mathbb{R}^{|A|} \times \mathbb{R}^{|A|}$ such that x^h satisfies (1)-(6) for $h = 1, 2$ and (x^1, x^2) satisfies (7). Note that $\mathcal{P}$ is a set of integral vectors.

Lemma 8 $\mathcal{P}_{2,n} = conv(\mathcal{P})$.

Proof. Let us show that a vector (x^1, x^2) corresponds to the incidence vector of a couple of consistent hamiltonian circuits if and only if (x^1, x^2) satisfies (7) and x^h satisfies (1)-(6) for $h = 1, 2$. The necessity follows from Theorem 1 and the following claim.

Claim 9 *Two hamiltonian circuits C^1 and C^2 are not consistent if and only if there exist pairwise distinct vertices $i, j, k \in V \setminus \{0\}$ such that the path P^h of C^h from i to j contains 0 but not k, for $h = 1, 2$.*

Proof. Note that P^h contains 0 if and only if $j \prec_{C^h} k \prec_{C^h} i$, hence the result follows from Lemma 3. □

For the sufficiency, let $(x^1, x^2) \in \mathcal{P}$. By Theorem 1, the arc set $C^h = \{a \in A : x^h_a = 1\}$ is a hamiltonian circuit for $h = 1, 2$. Note that (7) implies that C^1 and C^2 do not contain two ij-paths $P^1 \subset C^1$ and $P^2 \subset C^2$ covering 0 but not k for all pairwise distinct vertices $i, j, k \in V \setminus \{0\}$. Claim 9 implies that C^1 and C^2 are consistent, finishing the proof. □

By Lemma 8, the uncapacitated asymmetric traveling salesman problem with two stacks can be formulated by:

$$\mathcal{F} = \min_{(x^1,x^2)\in\mathcal{P}} c^1x^1 + c^2x^2.$$

Lemma 10 *The linear relaxation of $\mathcal{F}$ can be solved in polynomial time.*

Proof. We just need to show that the separation problem associated with constraints (3) and (7) is polynomial for any vector $(\bar{x}^1, \bar{x}^2) \in [0,1]^{|A|} \times [0,1]^{|A|}$ such that $\bar{x}^h$ satisfies constraints (1) and (2) for $h = 1, 2$. The separation of the subtour elimination constraints consists in the computation of a polynomial number of minimum cuts. Therefore, it is polynomial-time solvable. Consider the separation problem associated with the consistency constraints (7). Let $\tilde{x}^h = 1 - \bar{x}^h$ for $h = 1, 2$. Inequalities (7) can be rewritten as

$$\sum_{h=1,2} \sum_{a\in P^h} \tilde{x}^h_a \geq 1 \quad \begin{array}{l} \forall\, i \neq j \neq k \neq i \in V \setminus \{0\}, \\ \forall\, P^1, P^2 \in \mathcal{P}^0_{ij}(D \setminus \{k\}). \end{array}$$

Given three pairwise distinct vertices i, j, k of $V \setminus \{0\}$, the separation problem associated with i, j and k then reduces to find P^1 and P^2 belonging to $\mathcal{P}^0_{ij}(D \setminus \{k\})$ such that the cost $w = \tilde{x}^1(P^1) + \tilde{x}^2(P^2)$ is minimum. If $w < 1$, then the inequality (7) associated with i, j, k, P^1 and P^2 is violated by $(\bar{x}^1, \bar{x}^2)$. Otherwise, this latter satisfies all the consistency inequalities associated with i, j and k.

For $h = 1, 2$, let P^h_{i0} and P^h_{0j} be respectively an $i0$-path and a $0j$-path of $D \setminus \{k\}$ and set $P^h = (P^h_{i0}, P^h_{0j})$. If $\tilde{x}^h(P^h) < 1$, then P^h belongs to $\mathcal{P}^0_{ij}(D \setminus \{k\})$. Indeed, otherwise, there would exist a vertex $v \in V \setminus \{i, j, 0\}$ such that P^h

contains two arcs a_1 and a_2 both leaving v or entering v. Since we have supposed that $\tilde{x}^h(P^h) < 1$, we have $\tilde{x}^h_{a_1} + \tilde{x}^h_{a_2} < 1$, which implies that $\bar{x}^h_{a_1} + \bar{x}^h_{a_2} > 1$. Therefore, $\bar{x}^h$ violates (1) or (2), a contradiction.

The separation problem of consistency inequalities (7) associated with i, j and k then reduces to compute four minimum paths where the arc costs are given by $(\bar{x}^1, \bar{x}^2)$. As the costs are non-negative, the separation problem is polynomial-time solvable. □

3.2 Valid inequalities

We propose three families of valid inequalities for $\mathcal{P}_{2,n}$. They are obtained by deriving structures where, if one of the hamiltonian circuits is a path, then either the other one cannot be a path, see Lemmas 11 and 14, or the other one cannot be the disjoint union of two paths, see Lemma 13. For small instances, these inequalities define facets of $\mathcal{P}_{2,n}$, we leave the question open whether it holds in general.

In all the figures, a vertex set depicted in gray (white, respectively) represents a complete subgraph (stable set, respectively).

***P_3*-subgraph inequalities** A subgraph $H = (U, B)$ of D is a *P_3-subgraph* if $U \neq V$ and there exists a partition $\mathcal{U} = \{U_1, U_2, U_3\}$ of U such that B is composed of $A[U_i], i = 1, 2, 3$ and every arc from U_1 to $U_2 \cup U_3$ and from U_2 to U_3. The partition $\mathcal{U}$ is said *associated with* H. Figure 1 shows a P_3-subgraph.

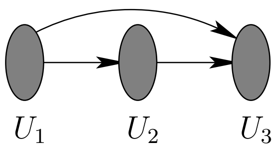

Fig. 1. A P_3-subgraph

Lemma 11 *Given a P_3-subgraph (U, B), the inequality*

$$x^1(B) + x^2(B) \leq 2(|U| - 1) - 1 \tag{8}$$

is valid for $\mathcal{P}_{2,n}$.

Proof. Let $\mathcal{U} = \{U_1, U_2, U_3\}$ be the partition associated with H and $U_4 = V \setminus U$. Since $U_4 \neq \emptyset$, every hamiltonian circuit C satisfies $\chi^C(B) \leq |U| - 1$. If there is equality, then $C \cap B$ is a path covering U. Due to the structure of H, C contains no arcs from U_1 to U_3. Since C is a hamiltonian circuit, it implies that $C \cap A[U_i]$ is a path covering U_i for $i = 1, \ldots 4$, and C contains exactly one arc from U_i to U_{i+1} for $i = 1, \ldots, 4$ (where $U_5 = U_1$). Let $i \in \{1, \ldots, 4\}$ be such that $0 \in U_i$, and

$\mathcal{U}' = \{U_k, k \neq i\}$. Denote $\mathcal{U}' = \{U'_1, U'_2, U'_3\}$. Let (U', B') be the P_3-subgraph defined on $\mathcal{U}'$. Since $C \cap B'$ is a path covering U', we may suppose, without loss of generality, that C contains no arc from U'_1 to U'_3. Now, since $0 \notin U'$, we have $U'_1 \prec_C U'_2 \prec_C U'_3$.

Now, if C^1 and C^2 were consistent hamiltonian circuits violating (8), we would have $\chi^{C^h}(B) = |U| - 1$ for $h = 1, 2$. By the above observations, we would have $U'_1 \prec_{C^h} U'_2 \prec_{C^h} U'_3$ for $h = 1, 2$, a contradiction to Lemma 3. □

***P*$_4$-subgraph inequalities** A subgraph $H = (U, B)$ of D is a *P_4-subgraph* if there exists a partition $\mathcal{U} = \{U_0, U_1, U_2, U_3\}$ of U such that $0 \in U_0$, B is composed of $A[U_0], A[U_1 \cup U_2]$ and every arc from U_0 to U_1 and from U_2 to U_3, and $\mathcal{U}$ satisfies $|U_1| = |U_3| = 1$ or $|U_2| = 1$. We denote $|U_0| + |U_1| + |U_2|$ by ℓ_H. The partition $\mathcal{U}$ is said *associated with* H. Figure 2 shows a P_4-subgraph.

Claim 12 *Let C be a hamiltonian circuit and $H = (U, B)$ a P_4-subgraph and $\mathcal{U} = \{U_0, U_1, U_2, U_3\}$ its associated partition. If $|C \cap B| = \ell_H - 1$, then there exists $v_1 \prec_C v_2 \prec_C v_3$ with $v_i \in U_i$ for $i = 1, 2, 3$.*

Proof. Note that, since $|U_2| = 1$ or $|U_3| = 1$, C contains at most one arc from U_2 to U_3 because C is hamiltonian.

If C contains no such arc, then $C \cap B$ is a path covering $U_0 \cup U_1 \cup U_2$. Since $0 \in U_0$ and there are no arcs from $U_1 \cup U_2$ to U_0 in H, we have $U_1 \cup U_2 \prec_C V \backslash (U_0 \cup U_1 \cup U_2)$. Moreover, there are no arcs from U_0 to U_2, hence there exists $v_i \in U_i, i = 1, 2, 3$ such that $v_1 \prec_C v_2 \prec_C v_3$.

If C contains an arc $v_2 v_3$ for $v_i \in U_i, i = 2, 3$, then $C \cap B = P \cup P'$ where P and P' are two disjoint paths satisfying $P \cup P' = U_0 \cup U_1 \cup U_2 \cup \{v_3\}$. We may assume $v_2 v_3 \in P$. Due to the structure of H, $v_2 v_3$ is the last arc of P. If P intersects U_1, then there exists $v_1 \in U_1$ such that $v_1 \prec_C v_2 \prec_C v_3$. Otherwise, we have $0 \in P'$ and since there is no arc from U_1 to U_0 in H, there exists $v_1 \in U_1 \cap P'$, which implies that $v_1 \prec_C P$. In particular, we have $v_1 \prec_C v_2 \prec_C v_3$. □

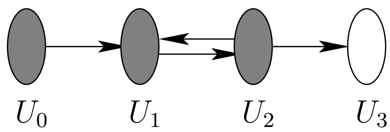

Fig. 2. A P_4-subgraph

Lemma 13 *Given a P_4-subgraph $H = (U, B)$, then the inequality*

$$x^1(B) + x^2(B) \leq 2(\ell_H - 1) \tag{9}$$

is valid for $\mathcal{P}_{2,n}$.

Proof. Let $\mathcal{U} = \{U_0, U_1, U_2, U_3\}$ be the partition associated with H. Since in H there are no arcs in U_3 or leaving U_3, if C is a hamiltonian circuit, then $\chi^C(B) \le \ell_H$. Suppose that C^1 and C^2 are consistent hamiltonian circuits and violate (9). We may assume that $\chi^{C^1}(B) = \ell_H$, and there exists $v_3 \in U_3$ such that $C^1 \cap B$ is a path covering $U_0 \cup U_1 \cup U_2 \cup \{v_3\}$. Note that at least one of U_1 and U_2 is a singleton. Then, since $0 \in U_0$ and there are no arcs from U_0 to U_2, we have $U_1 \prec_{C^1} U_2 \prec_{C^1} U_3$.

If $\chi^{C^2}(B) = \ell_H$, then every triplet $v_i \in U_i, i = 1,2,3$ contradicts Lemma 3. Therefore $\chi^{C^2}(B) = \ell_H - 1$ and Claim 12 applies to C^2 and H. Then, there exist $v_i \in U_i, i = 1,3$ such that $v_1 \prec_{C^2} v_2 \prec_{C^2} v_3$, and Lemma 3 is contradicted. □

W_5-subgraph inequalities A subgraph $H = (U, B)$ of D is a W_5-*subgraph* if $U \neq V$ and there exists a partition $\mathcal{U} = \{0, i, j, U_1, U_2\}$ of U such that B is composed of $A[U_1 \cup \{0\}]$, $A[U_2]$, ij, $j0$, every arc from $U_1 \cup \{j\}$ to $U_2 \cup \{i\}$ and from U_2 to $\{0, i\}$. The partition $\mathcal{U}$ is said *associated with* H. Figure 3 shows a W_5-subgraph.

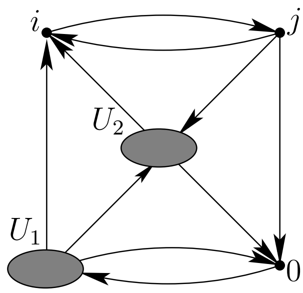

Fig. 3. A W_5-subgraph

Lemma 14 *Given a W_5-subgraph (U, B), the inequality*

$$x^1(B) + x^2(B) \le 2(|U| - 1) - 1 \qquad (10)$$

is valid for $\mathcal{P}_{2,n}$.

Proof. Let $\mathcal{U} = \{0, i, j, U_1, U_2\}$ be the partition associated with H.

Claim 15 *Let C be a hamiltonian circuit. If $|C \cap B| = |U| - 1$, then at least one of the following holds.*

(i) $U_1 \prec_C i \prec_C j$,
(ii) *there exist* $v_1 \in U_1$ *such that* $v_1 \prec_C i \prec_C j \prec_C V \setminus U$.

Proof. By contradiction, assume that C satisfies neither (i) nor (ii). Since $|C \cap B| = |U| - 1$, $C \cap B$ is a path P_1 covering U. Note that $P_2 = C \setminus P_1$ is a path covering $V \setminus U$.

Suppose that (i,j) does not belong to C. Since no arc of B except (i,j) leaves i and enters j, and there is no path from j to any vertex of U_1 using only arcs of $B \setminus \delta(0)$, P_1 is a path starting from j, passing by 0 and then covering U_1 before reaching i. We then deduce that $v \prec_C i \prec_C j$ for every vertex of U_1, hence C satisfies (i), a contradiction. Therefore, (i,j) belongs to C.

Suppose $0v_1 \in C$ for some $v_1 \in U_1$. In this case, we have $v_1 \prec_C i \prec_C j$. Since C does not satisfy (i), there exists $v_2 \in U_1 \setminus v_1$ such that $j \prec_C v_2$. Then, C contains a jv_2-path Q which does not cover 0. Thus Q contains P_2 because in $B \setminus \delta(0)$ there is no path from j to any vertex of U_1. It implies that $j \prec_C V \setminus U$, hence C satisfies (ii), a contradiction. Therefore, C contains no arc from 0 to U_1.

Therefore P_1 is a path ending at 0. Moreover, since there is no arc of $B \setminus A[0, U_1]$ entering U_1, we have $v \prec_C v'$ for all $v \in U_1$ and all $v' \in U_2 \cup \{i,j\}$. Since $(i,j) \in C$, C satisfies (i), a contradiction. □

Suppose that (C^1, C^2) are consistent hamiltonian circuits violating (10). Due to the degree constraints, we have $\chi^{C^h}(B) = |U| - 1$ for $h = 1, 2$, and Claim 15 applies.

Since $V \setminus U \neq \emptyset$, if C^1 and C^2 both satisfy Claim 15 (ii), then $i \prec_{C^1} j \prec_{C^1} V \setminus U$ contradicts Lemma 3. Hence we may assume that C^1 satisfies Claim 15 (i). Since C^2 satisfies either Claim 15 (i) or (ii), there exists $v_1 \in U_1$ such that $v_1 \prec_{C^2} i \prec_{C^2} j$. Moreover, we also have $v_1 \prec_{C^1} i \prec_{C^1} j$ and Lemma 3 contradicts the compatibility of C^1 and C^2. □

4 Future work

In this paper, we gave preliminary results towards a better understanding of the polyhedral structure of the uncapacitated asymmetric traveling salesman with multiple stacks. One of our goals is to derive an efficient Branch and Bound algorithm for the problem, and, at the moment, a key intermediary result would be a polynomial separation algorithm for the inequalities we proposed.

Keeping in mind that our problem is a relaxation of the capacitated version, we consider the above directions to be necessary steps before tackling the general case.

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