ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Robust location transportation problems under uncertain demands

V. Gabrel, M. Lacroix, C. Murat, N. Remli *

Lamsade, Université Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France

ARTICLE INFO

Article history:
Received 5 July 2010
Received in revised form 15 June 2011
Accepted 27 September 2011
Available online xxxx

Keywords:
Robust optimization
Interval uncertainty
Convex optimization
Mixed integer linear programming
Location transportation problem

ABSTRACT

In robust optimization, the *multi-stage* context (or dynamic decision-making) assumes that the information is revealed in stages. So, part of the decisions must be taken before knowing the real values of the uncertain parameters, and another part, called *recourse decisions*, is taken when the information is known. In this paper, we are interested in a robust version of the location transportation problem with an uncertain demand using a 2-stage formulation. The obtained robust formulation is a convex (not linear) program, and we apply a cutting plane algorithm to exactly solve the problem. At each iteration, we have to solve an NP-hard recourse problem in an exact way, which is time-consuming. Here, we go further in the analysis of the recourse problem of the location transportation problem. In particular, we propose a mixed integer program formulation to solve the quadratic recourse problem and we define a tight bound for this reformulation. We present an extensive computation analysis of the 2-stage robust location transportation problem. The proposed tight bound allows us to solve large size instances.

1. Introduction

Robust optimization is a recent methodology for handling problems affected by uncertain data, and where no probability distribution is available on the ambiguous parameters.

The goal of the robust optimization framework is to obtain a "robust" solution, which protects the decision maker against adverse realizations of the uncertainty. A specific definition of robustness depends on the modeling of the uncertainty, the location of the uncertainty within the problem (in the objective function and/or in the constraints), and the decision-making context (in particular, the way information is revealed over time). Within the problem under consideration in this paper, the uncertainty concerns the constraints, or in other words the feasibility of a solution.

According to the decision-making context, the robust approaches can be divided into two categories. The first category is suited to the *single-stage* context, where the decision-maker has to select a solution before knowing the real value of each uncertain parameter. When uncertainty affects the *feasibility* of a solution (because constraint coefficients are uncertain), robust optimization seeks to obtain a solution that will be feasible for any realization taken by the unknown coefficients (Soyster [15]); however, complete protection from adverse realizations often comes at the expense of a severe deterioration in the objective. This extreme approach can be justified in some engineering applications of robustness, such as robust control theory, but is less advisable in operations research, where adverse events such as low customer demand do not produce the high-profile repercussions that engineering failures – such as a doomed satellite launch or a destroyed unmanned robot – can have. To make the robust methodology appealing to business practitioners, robust optimization thus focuses on obtaining a solution that will be feasible for any realization taken by the unknown coefficients within a smaller, "realistic" set, called the *uncertainty set*. The specific choice of the set plays an important role in ensuring computational tractability of the robust problem and limiting deterioration of the objective at optimality, and must be thought through

* Corresponding author.
E-mail address: nabila.remli@dauphine.fr (N. Remli).

0166-218X/$ – see front matter © 2011 Published by Elsevier B.V.
doi:10.1016/j.dam.2011.09.015

carefully by the decision maker (see for example [2]). In the approach proposed by Bertsimas and Sim [6,5] the assumption is that only a subset of parameters should simultaneously reach their worst value. Thus, the uncertainty set is defined by the cardinality Γ of this subset, given by the decision maker, and called the budget of uncertainty.

The second category is suited to the *multi-stage* context (or dynamic decision-making) where the information is revealed in subsequent stages. The multi-stage approach in robust optimization was first introduced by Ben-Tal et al. [3], and initially focused on two-stage decision making on linear programs under an uncertain feasible set. The idea is to consider two sets of variables, such that the first set must be determined before the disclosure of the uncertainty, and that the second one can be computed after the uncertainty has been revealed. The second stage problem is known as the recourse problem. The 2-stage robust methodology was inspired from stochastic optimization. In the latter case, the uncertainty is described by probability laws, and one has to decide the "here-and-now" variables using an expected value of all possible recourse decisions. The "wait-and-see" variables are decided after the uncertainty has been disclosed. This argument is extended to robust optimization. Indeed, recalling that there are no probability estimations available on the uncertain parameters, one has to decide on the first stage variables making possible a recourse decision for any possible realization of the uncertain parameters in the uncertainty set.

In a recent paper, Bertsimas et al. [4] proposed a complete overview on the theory and the applications of robust optimization. In the specific application considered in this paper, which is the problem of choosing the location of warehouses and the transportation scheme from warehouses to customers, only the right hand side coefficients (representing the demands) are uncertain. This particular context has been studied by Thiele et al. [16]. The authors describe a two-stage robust approach to address general linear programs affected by an uncertain right hand side. The robust formulation they obtained is not a linear but a convex problem, and they propose a cutting plane algorithm to exactly solve the problem. Indeed, at each iteration, they have to solve an NP-hard recourse problem in an exact way, which is time-consuming.

In this paper, our objectives are to propose a robust version of the location transportation problem with uncertain demands, using a 2-stage formulation, and to apply the algorithm proposed by Thiele et al. [16]. Here, we go further in the analysis of the recourse problem of the location transportation problem. In particular, we define a tight bound for a mixed-integer reformulation.

The paper is organized as follows: in Section 2, the location transportation problem is introduced along with its corresponding 2-stage robust formulation. A mixed integer program is then proposed in Section 3 to solve the quadratic recourse problem with a tight bound. Finally, in Section 4, the results of numerical experiments are discussed.

2. Robust location transportation problem

We consider the following location transportation problem: a commodity has to be transported from each of m potential sources to each of n destinations. The sources capacities are $C_i,\ i = 1, \ldots, m$ and the demands at the destinations are $\beta_j,\ j = 1, \ldots, n$. To guarantee feasibility, we assume that the total sum of the capacities at the sources is greater than or equal to the sum of the demands at the destinations. The fixed and variable costs of supplying from source $i = 1, \ldots, m$ are f_i and d_i, respectively. The cost of transporting one unit of the commodity from source i to destination j is μ_{ij}. The goal is to determine which sources to open (r_i), the supply level y_i and the amounts t_{ij} to be transported such that the total cost is minimized. The mathematical formulation of the nominal location transportation problem is the following linear program (T):

$$(T)\begin{cases} \min & \displaystyle\sum_{i=1}^m d_i y_i + \sum_{i=1}^m f_i r_i + \sum_{i=1}^m \sum_{j=1}^n \mu_{ij} t_{ij} & \\ \text{s.t.} & \displaystyle\sum_{j=1}^n t_{ij} \le y_i & i = 1, \ldots, m \\ & \displaystyle\sum_{i=1}^m t_{ij} \ge \beta_j & j = 1, \ldots, n \\ & y_i \le C_i r_i & i = 1, \ldots, m \\ & r_i \in \{0, 1\}, \qquad y_i, \qquad t_{ij} \ge 0 & i = 1, \ldots, m,\ j = 1, \ldots, n. \end{cases}$$

Furthermore, it should be noted that the decision maker has to decide in two steps: first, the warehouses have to be located and filled, and after, once the demands are known, the routing of commodities is decided. According to this context, the transportation part of the problem has a significant importance when deciding the location part.

In practice, the customers' demands are often estimated at the stage of construction of the warehouses. To be realistic, it is common to assume some uncertainty in these demands. We define the uncertainty set as being interval numbers for each one of them. Formally, the jth customer demand β_j belongs to $[\overline{\beta}_j - \hat{\beta}_j, \overline{\beta}_j + \hat{\beta}_j]$, where $\overline{\beta}_j \ge 0$ represents the nominal value of β_j and $\hat{\beta}_j \ge 0$ its maximum deviation. Clearly, each demand β_j can take on any value from the corresponding interval, regardless of the values taken by other coefficients. We denote as (T^β) the location transportation problem for a given $\beta \in [\overline{\beta} - \hat{\beta}, \overline{\beta} + \hat{\beta}]$, with a nonempty feasible set. Finally, we denote as $\text{opt}(T^\beta)$ the optimal value (bounded value) of (T^β) for a given β.

Please cite this article in press as: V. Gabrel, et al., Robust location transportation problems under uncertain demands, Discrete Applied Mathematics (2011), doi:10.1016/j.dam.2011.09.015

Because of the context of our problem, we apply a 2-stage robust approach to the uncertain problem (T^{β}). In fact, we recall that the decision maker has to size the capacities before knowing the demands and, once these demands are revealed, he has to satisfy them. Thus, according to this context, we define the r_i and y_i variables as first stage decisions, while the t_{ij} variables represent the recourse variables or the second stage decisions. The total costs of all the decisions should be minimized.

Furthermore, the decision maker wants to take decisions based on a realistic scenario and to avoid the worst case demands. The model chosen to represent the uncertainty set is the one suggested by [1,16,9]. This model is a natural adaptation of the original Bertsimas and Sim approach (see [6,5]), in which a parameter Γ, called *the budget of uncertainty*, is defined. The value of Γ represents the maximum range of the uncertain demands that can simultaneously deviate from their nominal values. As the uncertainty is on the right hand sides (demands), Γ belongs to $[0, n]$. For $\Gamma = 0$, every right hand side is equal to its nominal value, while $\Gamma = n$ leads to considering the problem with the greatest demands.

The aim of setting the parameter Γ in the robust formulation is to restrict the demands that are greater than the nominal ones. Hence, according to its predictions, the decision maker is free to choose any value of Γ in the interval $[0, n]$ and solve the 2-stage robust problem. Then he can decide to open the sources and fill the warehouses, although the actual demands are not known yet.

Nevertheless, when the demands are revealed, the decision maker must satisfy them, even if they are larger than those expected. Thus, we assume the total recourse hypothesis: given a solution y_i and r_i, a solution for the transportation problem exists whatever the demands are. This problem may concern, for example, the installation of power plants, where the worst demands must be handled.

Under all the assumptions above, the robust location transportation problem, denoted by $T_{rob}(\Gamma)$, is to choose the sources to open (with the r_i variables) and the amounts to store (with the y_i variables) such that the worst demand in the uncertainty set is satisfied with minimum cost. The robust problem is the following:

$$T_{rob}(\Gamma)\begin{cases} \min & \sum_{i=1}^{m} d_i y_i + \sum_{i=1}^{m} f_i r_i + \mathrm{opt}(R(y,\Gamma)) & \\ \text{s.t.} & y_i \le C_i r_i & i = 1,\dots,m \\ & \sum_{i=1}^{m} y_i \ge B & \\ & y_i \ge 0, \qquad r_i \in \{0,1\} & i = 1,\dots,m \end{cases}$$

where

$$B = \sum_{j=1}^{n} (\overline{\beta}_j + \hat{\beta}_j) \tag{1}$$

and $\mathrm{opt}(R(y,\Gamma))$ represents the optimum value of the recourse problem:

$$R(y,\Gamma)\left\{ \max_{\beta \in \mathcal{U}(\Gamma)} \quad \min_{\substack{\sum_{j=1}^{n} t_{ij} \le y_i,\ i=1,\dots,m \\ \sum_{i=1}^{m} t_{ij} \ge \beta_j,\ j=1,\dots,n \\ t_{ij} \ge 0,\ i=1,\dots,m,\ j=1,\dots,n}} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij} t_{ij} \right.$$

where the uncertainty set $\mathcal{U}(\Gamma)$ is defined by:

$$\mathcal{U}(\Gamma) = \left\{ \beta \in \mathbb{R}^n : \beta_j = \overline{\beta}_j + z_j \hat{\beta}_j,\ j = 1,\dots,n,\ z \in \mathcal{Z}(\Gamma) \right\} \tag{2}$$

and

$$\mathcal{Z}(\Gamma) = \left\{ z \in \mathbb{R}^n : \sum_{j=1}^{n} z_j \le \Gamma,\ 0 \le z_j \le 1,\ j = 1,\dots,n \right\}. \tag{3}$$

The constraint $\sum_{i=1}^{m} y_i \ge B$, where B is given by (1), is due to the total recourse considered in our formulation, and the assumption of satisfaction of the greatest demands.

At optimality $\mathrm{opt}(R(y,\Gamma))$ represents the transportation cost for a fixed capacity level y, and Γ worst deviations. Considering (2) and (3) we rewrite the recourse problem $R(y,\Gamma)$ as:

$$R(y,\Gamma)\left\{ \max_{\substack{\sum_{j=1}^{n} z_j \le \Gamma \\ 0 \le z_j \le 1,\ j=1,\dots,n}} \quad \min_{\substack{\sum_{j=1}^{n} t_{ij} \le y_i,\ i=1,\dots,m \\ \sum_{i=1}^{m} t_{ij} \ge \overline{\beta}_j + \hat{\beta}_j z_j,\ j=1,\dots,n \\ t_{ij} \ge 0,\ i=1,\dots,m,\ j=1,\dots,n}} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij} t_{ij} \right.$$

and then, by the strong duality theorem, one can replace the inner minimization problem by its dual in the recourse problem (since the problem is feasible for all the demands):

$$Q(y,\Gamma)\begin{cases}\max & -\sum_{i=1}^{m} y_i u_i + \sum_{j=1}^{n} \overline{\beta}_j v_j + \sum_{j=1}^{n} \hat{\beta}_j v_j z_j \\ \text{s.t.} & v_j - u_i \le \mu_{ij} \qquad i=1,\dots,m,\ j=1,\dots,n \\ & \sum_{j=1}^{n} z_j \le \Gamma \\ & 0 \le z_j \le 1 \qquad j=1,\dots,n \\ & u_i, \quad v_j \ge 0 \qquad i=1,\dots,m,\ j=1,\dots,n\end{cases}$$

where u_i, v_j are the dual variables. The recourse problem $Q(y,\Gamma)$ is now a maximization problem with quadratic objective function. We describe a way to handle it in Section 3.

Now, the robust location transportation problem can be written as the following problem:

$$T'_{rob}(\Gamma)\begin{cases}\min & \sum_{i=1}^{m} d_i y_i + \sum_{i=1}^{m} f_i r_i + \mathrm{opt}(Q(y,\Gamma)) \\ \text{s.t.} & y_i \le C_i r_i \qquad i=1,\dots,m \\ & \sum_{i=1}^{m} y_i \ge B \\ & y_i \ge 0, \quad r_i \in \{0,1\} \qquad i=1,\dots,m\end{cases}$$

which is a minimization of a convex function under linear constraints (the convexity can easily be proved, see [16]). Using *Kelley's algorithm* [12] as in [16], we solve in Section 2.1 the problem $T'_{rob}(\Gamma)$.

2.1. Kelley's algorithm for the location transportation problem

We first remark that the optimum solution of the recourse problem $Q(y,\Gamma)$ is given by an extreme point of its feasible set. This allow us to rewrite the robust problem $T'_{rob}(\Gamma)$ as follows:

$$T''_{rob}(\Gamma)\begin{cases}\min & \sum_{i=1}^{m} d_i y_i + \sum_{i=1}^{m} f_i r_i + \alpha \\ \text{s.t.} & \alpha \ge -\sum_{i=1}^{m} y_i u_i^s + \sum_{j=1}^{n} \overline{\beta}_j v_j^s + \sum_{j=1}^{n} \hat{\beta}_j v_j^s z_j^s \qquad s=1,\dots,S \\ & \sum_{i=1}^{m} y_i \ge B \\ & y_i \le C_i r_i \qquad i=1,\dots,m \\ & \alpha, y_i \ge 0, \quad r_i \in \{0,1\} \qquad i=1,\dots,m\end{cases}$$

where (u_j^s, v_j^s, z_j^s) with $s \in S$ are the extreme points of the recourse problem $Q(y,\Gamma)$, for fixed y and Γ. In this formulation, we observe that the objective function of $T''_{rob}(\Gamma)$ is linear but the number of constraints is exponential.

The general idea of Kelley's algorithm is to iteratively generate a new extreme point of S (or possibly many points) by solving the recourse problem, and to add it (them) to the master problem until the optimal solution is found. Indeed, the algorithm stops when a lower bound L, defined by a relaxation of the problem $T''_{rob}(\Gamma)$, is equal to an upper bound U defined by the value of a feasible solution. Formally, the algorithm is presented in Algorithm 1.

Clearly, $\mathrm{opt}(T''_{rob}(\Gamma)^k)$ is a lower bound for the optimum of the robust problem, since it contains a subset of constraints of $T''_{rob}(\Gamma)$. Thus, as constraints are added to the master problem, the value of L will be non decreasing. On the other side, since the solution $(y^k, r^k, u^k, v^k, z^k)$ is feasible for $T''_{rob}(\Gamma)$, at each iteration k, we assign to U the minimum solution value over all generated solutions. Moreover, as the number of extreme points in S is bounded, the algorithm stops within a finite number of iterations. For completeness, a proof of convergence can be found in [12].

The most computationally difficult part of the algorithm is to solve the recourse problem in step 2. Indeed, it is a bilinear program subject to linear constraints, which is a class of convex maximization problems proven NP-hard (see [8,17]). Several authors have been interested in solving bilinear problems. Early works are those of Falk [7] and Konno [13], who proposed a cutting plane algorithm, improved by Sherali and Shetty in [14]. Instead of solving it in a direct way, we will reformulate $Q(y, \Gamma)$ as a mixed integer program. We present this formulation in the next section.

Algorithm 1 Kelley's algorithm for solving $T''_{rob}(\Gamma)$

Step 0: Initialization
Define $T''_{rob}(\Gamma)^1$ containing one extreme point (u^0, v^0, z^0) with $u^0 = v^0 = z^0 = 0$.
Set $L \leftarrow -\infty$, $U \leftarrow +\infty$ and $k \leftarrow 1$. Go to Step 1.
Step 1
Solve the master problem

$$T''_{rob}(\Gamma)^k \begin{cases} \min & \sum_{i=1}^m d_i y_i + \sum_{i=1}^m f_i r_i + \alpha \\ \text{s.t.} & \alpha \geq -\sum_{i=1}^m y_i u_i^l + \sum_{j=1}^n \overline{\beta}_j v_j^l + \sum_{j=1}^n \hat{\beta}_j v_j^l z_j^l \quad l = 0\ldots, k-1 \\ & \sum_{i=1}^m y_i \geq B \\ & y_i \leq C_i r_i \quad i = 1, \ldots, m \\ & y_i \geq 0, \quad r_i \in \{0, 1\} \quad i = 1, \ldots, m \end{cases}$$

and denote (y^k, r^k, α^k) its optimal solution.
Update $L \leftarrow \sum_{i=1}^m d_i y_i^k + \sum_{i=1}^m f_k r_i^k + \alpha^k$, and go to Step 2.
Step 2
For the fixed capacities y^k, solve the recourse problem $Q(y^k, \Gamma)$ and denote (u^k, v^k, z^k) its optimal solution.
Set $U \leftarrow \min\left\{U, \sum_{i=1}^m d_i y_i^k + \sum_{i=1}^m f_i r_i^k - \sum_{i=1}^m y_i^k u_i^k + \sum_{j=1}^n \overline{\beta}_j v_j^k + \sum_{j=1}^n \hat{\beta}_j v_j^k z_j^k\right\}$.
if $U = L$ **then**
 return (y^k, r^k, α^k) as an optimal solution to the problem $T''_{rob}(\Gamma)$;
else
 go to Step 3
end if
Step 3
Add the constraint

$$\alpha \geq -\sum_{i=1}^m y_i u_i^k + \sum_{j=1}^n \overline{\beta}_j v_j^k + \sum_{j=1}^n \hat{\beta}_j v_j^k z_j^k$$

to the master problem $T''_{rob}(\Gamma)^k$, $k \leftarrow k + 1$ and go to Step 1.

3. Mixed-integer program reformulation of the recourse problem

In the formulation of $T_{rob}(\Gamma)$, the budget of uncertainty Γ can be naturally interpreted as the number of constraints for which $\beta_j \neq \overline{\beta}_j$. Consequently, one can assume Γ to be integer. Moreover, with this assumption we can propose a MIP formulation of the problem $Q(y, \Gamma)$. This MIP formulation is based on the following proposition.

Proposition 1. *If Γ is an integer number then there exists an optimal solution (u^*, v^*, z^*) of $Q(y, \Gamma)$ such that $z_j^* \in \{0, 1\}$, $j = 1, \ldots, n$.*

Proof. Let us define the following polyhedra $\mathcal{Y} = \left\{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n : -u_i + v_j \leq \mu_{ij},\ u, v \geq 0\right\}$ and $\mathcal{Z}(\Gamma) = \left\{z \in \mathbb{R}^n : \sum_{j=1}^n z_j \leq \Gamma,\ 0 \leq z_j \leq 1,\ j = 1, \ldots, n\right\}$. As $Q(y, \Gamma)$ is a bilinear problem, we know that if the problem has a finite optimal value (which is guaranteed here because both polyhedra are bounded, by assumption) then an optimal solution (u^*, v^*, z^*) exists such that (u^*, v^*) is an extreme point of $\mathcal{Y}$ and z^* is an extreme point of $\mathcal{Z}(\Gamma)$ (see [11]). This implies that when Γ is an integer number, z^* are 0–1. □

From Proposition 1, and assuming that $\Gamma \in \mathbb{N}(\Gamma \leq n)$, we deduce that, at optimality either β_j is equal to its nominal value, $\overline{\beta}_j$, or its worst value, $\overline{\beta}_j + \hat{\beta}_j$. This consideration is not too restrictive for the decision maker, since Γ represents the number of worst demands among n.

Now, because of binary variables z_j we are able to linearize the problem $Q(y, \Gamma)$ by replacing each product $v_j z_j$ in the objective function with a new variable ω_j and adding constraints that enforce ω_j to be equal to v_j if $z_j = 1$, and zero otherwise (see [10]). The recourse problem becomes a mixed integer program:

Please cite this article in press as: V. Gabrel, et al., Robust location transportation problems under uncertain demands, Discrete Applied Mathematics (2011), doi:10.1016/j.dam.2011.09.015

$$Q'(y,\Gamma)\begin{cases} \max & -\sum_{i=1}^{m} y_i u_i + \sum_{j=1}^{n} \overline{\beta}_j v_j + \sum_{j=1}^{n} \hat{\beta}_j \omega_j \\ \text{s.t.} & v_j - u_i \le \mu_{ij} \qquad\qquad i=1,\dots,m,\ j=1,\dots,n \\ & \sum_{j=1}^{n} z_j \le \Gamma \\ & \omega_j \le v_j \qquad\qquad j=1,\dots,n \\ & \omega_j \le M z_j \qquad\qquad j=1,\dots,n \\ & z_j \in \{0,1\} \qquad\qquad j=1,\dots,n \\ & u_i, v_j, \omega_j \ge 0 \qquad\qquad j=1,\dots,n,\ i=1,\dots,m \end{cases}$$

where M is a sufficiently large constant, representing the bounds on the v_j variables.
For reducing the integrality gap, M needs to be as small as possible. The Corollary 1 sets a tight bound for M. First we prove Proposition 2.

Proposition 2. *The dual of the classical transportation problem can be written as follows:*

$$(D^*)\begin{cases} \max & -\sum_{i=1}^{m} y_i u_i + \sum_{j=1}^{n} \beta_j v_j \\ s.t. & -u_i + v_j \le \mu_{ij} \qquad i=1,\dots,m,\ j=1,\dots,n \\ & u_i, v_j \ge 0 \qquad i=1,\dots,m,\ j=1,\dots,n. \end{cases}$$

We set (u^*, v^*) as the optimal solution of the problem (D^*).

Let us consider an instance of the transportation problem, such that the demand of the first customer is equal to $\beta_1 - \hat{\beta}_1$ with $\hat{\beta}_1 > 0$. The dual (D') of such a problem is the following linear program:

$$(D')\begin{cases} \max & -\sum_{i=1}^{m} y_i u_i + (\beta_1 - \hat{\beta}_1) v_1 + \sum_{j=2}^{n} \beta_j v_j \\ s.t. & -u_i + v_j \le \mu_{ij} \qquad i=1,\dots,m,\ j=1,\dots,n \\ & u_i, v_j \ge 0 \qquad i=1,\dots,m,\ j=1,\dots,n. \end{cases}$$

There exists an optimal solution (u', v') of (D') such that $u' \le u^*$ and $v' \le v^*$.

Proof. In the simple case where (u^*, v^*) is also optimal for the problem (D'), then Proposition 2 is verified. We are interested in the opposite case. We set (u', v') as the optimal solution of (D') which does not satisfy Proposition 2. We define the solution (u'', v'') as follows:

$$u''_i = \min\{u^*_i, u'_i\} \quad \text{for all } i = 1,\dots,m$$
$$v''_j = \min\{v^*_j, v'_j\} \quad \text{for all } j = 1,\dots,n.$$

Let us prove that (u'', v'') is an optimal solution for the problem (D').
First, we prove that (u'', v'') is feasible. By contradiction, suppose that there exists i_1 and j_1 such that $-u''_{i_1} + v''_{j_1} > \mu_{i_1 j_1}$.

- If $u''_{i_1} = u^*_{i_1}$ then
$$u^*_{i_1} < -\mu_{i_1 j_1} + v''_{j_1}. \tag{4}$$
Moreover, by definition $v''_{j_1} \le v^*_{j_1}$, which implies that
$$-\mu_{i_1 j_1} + v''_{j_1} < -\mu_{i_1 j_1} + v^*_{j_1} \tag{5}$$
and from (4) and (5) we deduce that $u^*_{i_1} < -\mu_{i_1 j_1} + v^*_{j_1}$, which contradicts the feasibility of the solution (u^*, v^*).
- If $u''_{i_1} = u'_{i_1}$ then
$$u'_{i_1} < -\mu_{i_1 j_1} + v''_{j_1}. \tag{6}$$
By definition $v''_{j_1} \le v'_{j_1}$ and thus
$$-\mu_{i_1 j_1} + v''_{j_1} < -\mu_{i_1 j_1} + v'_{j_1} \tag{7}$$
and from (6) and (7) we deduce that $u'_{i_1} < -\mu_{i_1 j_1} + v'_{j_1}$, which contradicts the feasibility of the solution (u', v').

Thus, the solution (u'', v'') is feasible.

Let us denote now $f'(u, v)$ (respectively $f^*(u, v)$) as the value of a feasible solution (u, v) for the problem (D') (respectively (D^*)).

Before proving that (u'', v'') is optimal for (D'), let us prove that $v''_1 = v'_1$. By contradiction, suppose that $v''_1 \neq v'_1$ and thus $v''_1 = v^*_1$ and $v^*_1 < v'_1$. We have already supposed that (u^*, v^*) is not optimal for the problem (D'), then

$$f'(u^*, v^*) < f'(u', v'). \tag{8}$$

Furthermore, for each feasible solution (u, v) we have

$$f'(u, v) = -\sum_{i=1}^{m} y_i u_i + \sum_{j=1}^{n} \beta_j v_j - \hat{\beta}_1 v_1$$

$$f^*(u, v) = -\sum_{i=1}^{m} y_i u_i + \sum_{j=1}^{n} \beta_j v_j$$

which implies that

$$f'(u, v) = f^*(u, v) - \hat{\beta}_1 v_1 \tag{9}$$

and from (8) and (9) we obtain

$$f^*(u^*, v^*) - \hat{\beta}_1 v^*_1 < f^*(u', v') - \hat{\beta}_1 v'_1 \tag{10}$$

$$f^*(u^*, v^*) - f^*(u', v') < \hat{\beta}_1 v^*_1 - \hat{\beta}_1 v'_1. \tag{11}$$

If we suppose that $v^*_1 < v'_1$, then $\hat{\beta}_1 v^*_1 - \hat{\beta}_1 v'_1 < 0$. Thus, from (11)

$$f^*(u^*, v^*) - f^*(u', v') < 0 \tag{12}$$

which provides a contradiction to the fact that (u^*, v^*) is an optimal solution for (D^*). Thus, necessarily $v^*_1 \geq v'_1$ and

$$v''_1 = v'_1. \tag{13}$$

Let us prove now that (u'', v'') is an optimal solution for (D'). The cost of such a solution is equal to

$$f'(u'', v'') = -\sum_{i=1}^{m} y_i u''_i + (\beta_1 - \hat{\beta}_1) v''_1 + \sum_{j=2}^{n} \beta_j v''_j. \tag{14}$$

Let $\bar{I} \subseteq I$ be the subset of indices of $I = 1 \ldots m$ such that $i \in \bar{I}$ if $u''_i = u'_i$, and thus $u'_i \leq u^*_i$. And define $\bar{J} \subseteq J$ as being the subset of indices of $J = 1, \ldots, n$ such that $j \in \bar{J}$ if $v''_j = v'_j$, and thus $v'_j \leq v^*_j$. The cost of the solution (u'', v'') is

$$f'(u'', v'') = -\sum_{i \in \bar{I}} y_i u'_i + \sum_{j \in \bar{J} \setminus \{1\}} \beta_j v'_j + (\beta_1 - \hat{\beta}_1) v''_1 - \sum_{i \in I \setminus \bar{I}} y_i u^*_i + \sum_{j \in J \setminus \bar{J}} \beta_j v^*_j.$$

From (13) one can replace v''_1 by v'_1 and thus

$$\begin{aligned} f'(u'', v'') &= -\sum_{i \in I} y_i u'_i + (\beta_1 - \hat{\beta}_1) v'_1 + \sum_{j \in J \setminus \{1\}} \beta_j v'_j + \sum_{i \in I \setminus \bar{I}} y_i (u'_i - u^*_i) - \sum_{j \in J \setminus \bar{J}} \beta_j (v'_j - v^*_j) \\ &= f'(u', v') + \sum_{i \in I \setminus \bar{I}} y_i (u'_i - u^*) - \sum_{j \in J \setminus \bar{J}} \beta_j (v'_j - v^*_j). \end{aligned}$$

Suppose now that (u'', v'') is not an optimal solution of (D'), then the amount

$$A = \sum_{i \in I \setminus \bar{I}} y_i (u'_i - u^*_i) - \sum_{j \in J \setminus \bar{J}} \beta_j (v'_j - v^*_j) \tag{15}$$

should be strictly negative. We define the solution $(\tilde{u}, \tilde{v})$ as :

$$\begin{aligned} \tilde{u}_i &= \max\{u^*_i, u'_i\} \quad \text{for all } i = 1, \ldots, m \\ \tilde{v}_j &= \max\{v^*_j, v'_j\} \quad \text{for all } j = 1, \ldots, n. \end{aligned}$$

One can easily prove that the solution $(\tilde{u}, \tilde{v})$ is feasible (following the same reasoning as for (u'', v'')). The optimal value of $(\tilde{u}, \tilde{v})$ for the problem (D^*) is equal to

$$f^*(\tilde{u}, \tilde{v}) = -\sum_{i=1}^{m} y_i \tilde{u}_i + \sum_{j=1}^{n} \beta_j \tilde{v}_j$$

$$= -\sum_{i\in\bar{I}} y_i u_i^* - \sum_{i\in I\setminus\bar{I}} y_i u_i' + \sum_{j\in\bar{J}} \beta_j v_j^* + \sum_{j\in J\setminus\bar{J}} \beta_j v_j'$$
$$= -\sum_{i\in I} y_i u_i^* + \sum_{j\in J} \beta_j v_j^* - \sum_{i\in I\setminus\bar{I}} y_i(u_i' - u_i^*) + \sum_{j\in J\setminus\bar{J}} \beta_j(v_j' - v_j^*)$$
$$= f^*(u^*, v^*) - \sum_{i\in I\setminus\bar{I}} y_i(u_i' - u_i^*) + \sum_{j\in J\setminus\bar{J}} \beta_j(v_j' - v_j^*)$$
$$= f^*(u^*, v^*) - A.$$

Assuming $A < 0$ contradicts the optimality of the solution (u^*, v^*) for (D^*). Thus, $A \geq 0$ and $f'(u'', v'') \geq f'(u', v')$. In fact, $A = 0$ and the solution $(\tilde{u}, \tilde{v})$ is optimal for (D^*). We conclude that the solution (u'', v'') is feasible and optimal for (D') and verifies Proposition 2. □

Now, we can state the following corollary to set the values of M in the problem $Q'(y, \Gamma)$.

Corollary 1. *In the problem $Q'(y, \Gamma)$, we can replace each constraint*

$$w_j \leq M z_j$$

*by the **valid** constraint*

$$w_j \leq v_j^*(n) z_j$$

where $v_j^*(n)$, $j = 1, \ldots, n$, is the optimal solution value of v variables in $Q'(y, n)$.

Proof. From Proposition 2, we deduce that the values of u_i^* and v_j^* for $i = 1, \ldots, m$, $j = 1, \ldots, n$ are upper bounds for respectively u_i and v_j variables in all instances of the transportation problem where one or many demands decrease. Indeed, one can build a sequence of one by one decreasing demands and apply successively Proposition 2. In the problem $Q'(y, \Gamma)$, we recall that when $\Gamma = n$ then all demands $j = 1, \ldots, n$ are equal to their highest values $\overline{\beta}_j + \hat{\beta}_j$. In this case, the problem is equivalent to the dual of a classical transportation problem. When Γ decreases, some of the demands will also be decreasing. Thus, we deduce the bound $v_j^*(n)$ for the problem $Q'(y, \Gamma)$. □

Remark 1. In Algorithm 2.1, the bounds values $v^*(n)$ are computed at each iteration. Indeed, a new vector y^k is generated at each iteration k. Thus, the recourse problem is solved twice at Step 2:

- the first time we solve the problem with $\Gamma = n$, in order to compute $v^*(n)$ values. This is done quickly, since the problem $Q'(y^k, n)$ is a linear program.
- the second time we solve $Q'(y^k, \Gamma)$ using the $v^*(n)$ bounds.

In the next section, we are interested in numerical experiments, performed on the robust location transportation problem.

4. Numerical experiments

Several series of tests were performed on the robust location transportation problem. The experimental results are divided into two groups: first, we focus on the recourse problem and we explore the time limits of the tight bound defined in Section 3. The second group of tests concerns the overall problem solution.

4.1. The recourse problem

The first series of tests were realized for various values of the parameters of the recourse problem $Q'(y, \Gamma)$, namely the number of sources, the number of demands, the amounts available at each source, the nominal and the greatest demands at each destination and the transportation costs. To be closer to reality, we choose to set the number of demands greater than the number of sources. All other numbers are randomly generated as follows: for all $j = 1, \ldots, n$, the nominal demand $\overline{\beta}_j$ belongs to $[10, 500]$, and the deviation $\hat{\beta}_j = p_j \overline{\beta}_j$, such that p_j represents the percentage of maximum increase of each demand j. We take p_j in $[0.1, 0.5]$, which ensures $\hat{\beta}_j$ to be strictly positive. For these series of tests we assume that the amounts y_i at each source $i = 1, \ldots, m$ are obtained by an equal distribution of the sum of the maximum demands. Finally, the transportation costs are in the interval $[1, 1000]$.

The problem $Q'(y, \Gamma)$ was solved with CPLEX 11.2. For each (n, m), ten instances have been generated. Table 1 shows the results: the average of the running time and the percentage of solved instances. The running time results of the bound previously mentioned (see Section 3) and a large arbitrary constant M (we set M equal to $\mathrm{opt}(Q'(y, n))$) are compared such that the computation was stopped after 35 min.

The results described in Table 1 show that the computing time obtained by setting M to the bound $v^*(n)$ is significantly lower than the arbitrary bound. Moreover, we remark that the running time increases for the value of Γ around $n/2$ (see Fig. 1(a)). Fig. 1(b) illustrates the evolution of the objective value versus Γ for a sample $m = 100$ and $n = 250$. The curve

Table 1
Running time results of the recourse problem.

$n \times m$	Γ (%)	Running time (s)		Solved instances	
		M	$v^*(n)$	M	$v^*(n)$
	25	2.63	0.76	10/10	10/10
250 × 10	50	1178.14	14.05	5/10	10/10
	75	215.24	0.97	9/10	10/10
	25	14.86	2.09	10/10	10/10
250 × 50	50	434.85	4.94	9/10	10/10
	75	645.52	6.80	7/10	10/10
	25	8.57	1.82	10/10	10/10
250 × 100	50	18.93	2.69	10/10	10/10
	75	46.05	12.66	10/10	10/10
	25	4.99	1.00	10/10	10/10
500 × 10	50	1051.82	1050.25	5/10	5/10
	75	843.92	2.29	6/10	10/10
	25	10.49	1.60	10/10	10/10
500 × 50	50	223.63	5.52	9/10	10/10
	75	612.10	11.94	8/10	10/10
	25	12.03	1.90	10/10	10/10
500 × 100	50	10.98	2.26	10/10	10/10
	75	170.57	21.34	10/10	10/10
	25	35.59	3.26	10/10	10/10
1000 × 10	50	1472.13	1260.21	3/10	4/10
	75	1530.47	10.42	3/10	10/10

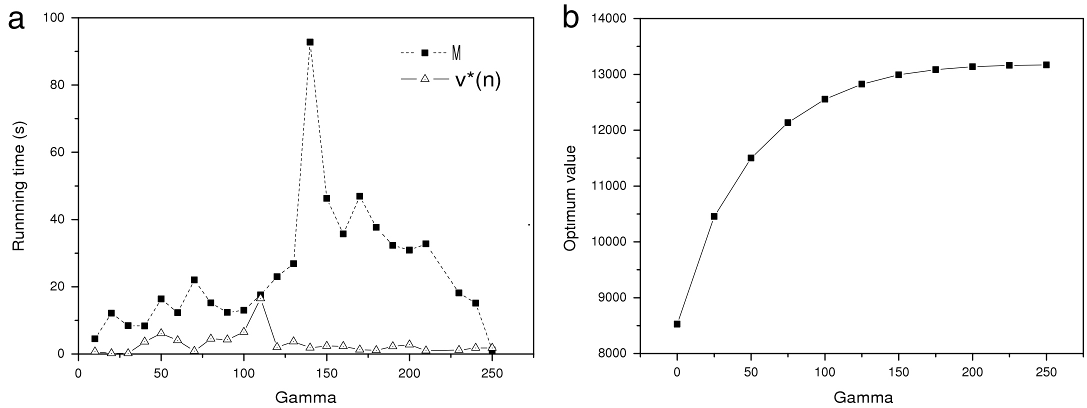

Fig. 1. A sample $m = 100$ and $n = 250$: (a) Running time vs. Γ. (b) Optimal value vs. Γ.

obtained is an increasing concave function, where $\mathrm{opt}(Q'(y, \Gamma))$ increases quickly with small values of Γ and slowly with high values. This is due to the model itself, since whenever Γ increases, the most influential uncertain parameters will be chosen.

Additional experiments have been performed on the uncertain transportation problem using the bound $v^*(n)$, in order to determine the limit size of the problem that can be solved within one hour of CPU time. Fig. 2(a) shows the numerical results for the number of the demands set to $n = 500$. We observe that the running time increases with the number of sources m. Indeed, for $m = 10$ the problem takes a few seconds to be solved. An average of 20 min is needed for instances with $m = 80$ and $\Gamma = 60\%$ of total deviation, and one hour for those with $m \in [300, 500]$ and $\Gamma = 50\%$.

Fig. 3(a) shows the limit running time, such that for $n = 1000$ uncertain demands, all instances containing $m = 10$ sources are solved within one hour, whatever the value of Γ between 10% and 100%. For $100 \leq m \leq 500$ the CPLEX solver is not able to reach the optimum within this time for $\Gamma = 50\%$, and for $m \geq 600$ there are memory limits with the solver.

Finally, as $v^*(n)$ is a very tight bound, one wants to know the behavior of the relaxation of the mixed-integer program. Figs. 2(b) and 3(b) show the integrality gap for the instances corresponding to $n = 500$ and $n = 1000$ demands respectively. We remark that this ratio is increasing with the number of sources m, reaching its maximum when Γ is around 20% of total deviation. Furthermore, the extra cost generated by the linear relaxation varies between 0% and 13% (compared with the exact solution). For instance, if the decision maker is interested in knowing the worst optimal value for 70% of the total deviation from the nominal problem (which represents the most difficult instances), one can solve the linear relaxation in a few seconds and have only 3% of extra cost at most. This represents a considerable saving of time.

Table 2
Running time results of the robust location transportation problem.

$n \times m$	Γ(%)	# iter.	Time (s)	Master(%)	Recourse(%)	Solved inst.
10 × 10	10	8.6	0.05	10	90	10/10
	20	9.2	0.06	8.8	91.2	10/10
	30	10	0.06	8.8	91.2	10/10
	40	10.1	0.06	7.0	93.0	10/10
	50	10	0.05	5.0	95.0	10/10
	60	10	0.06	4.2	95.8	10/10
	70	10.3	0.04	6.0	94.0	10/10
	80	9.4	0.04	7.1	92.9	10/10
	90	9.9	0.03	10.1	89.9	10/10
	100	8.8	0.02	50.0	50.0	10/10
70 × 70	10	117.9	124.14	28.9	71.1	10/10
	20	130.3	263.14	19.6	80.4	10/10
	30	124.4	570.62	6.3	93.7	10/10
	40	141.7	804.67	7.1	92.9	10/10
	50	153.9	964.01	6.0	94.0	10/10
	60	159.3	1085.50	7.1	92.9	10/10
	70	170.4	702.98	10.3	89.7	10/10
	80	184	351.58	26.3	73.7	10/10
	90	163.5	141.07	49.6	50.4	10/10
	100	146.4	57.67	84.1	15.9	10/10
100 × 100	10	194.4	760.07	40.7	59.3	10/10
	20	180.2	1359.50	37.1	62.9	10/10
	30	208.1	1842.72	30.3	69.7	9/10
	40	231.1	2837.36	23.0	77.0	9/10
	50	235.9	4087.44	13.1	86.9	8/10
	60	237.7	4285.29	6.6	93.4	7/10
	70	228.1	3261.86	13.5	86.5	8/10
	80	245.2	2890.07	27.0	73.0	7/10
	90	265.6	1292.35	70.8	29.2	10/10
	100	242.2	633.17	93.6	6.4	10/10
150 × 150	10	261.7	2647	19.4	80.6	10/10
	20	302.4	5499.08	12.8	87.2	8/10
	30	298.4	6316.47	6.4	93.6	6/10
	40	231	6375.56	3.6	96.4	6/10
	50	181	6677.38	1.5	98.5	4/10
	60	242	7042.79	2.5	97.5	3/10
	70	–	7200	10.1	89.9	0/10
	80	222.6	6216.67	22.6	77.4	5/10
	90	286.1	3723.10	56.9	43.1	7/10
	100	306.5	2374.98	93.4	6.6	7/10

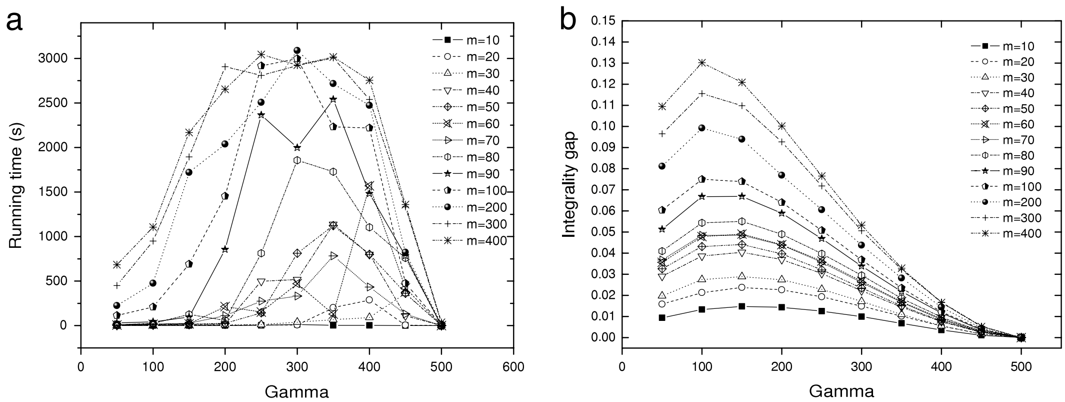

Fig. 2. Tests $n = 500$: (a) Running time vs. Γ. (b) Integrality gap vs. Γ.

4.2. *The location transportation tests results*

We now present the final tests performed on the 2-stage robust location transportation problem $T''_{rob}(\Gamma)$. In addition to the previous data interval generation (the nominal and greatest demands and the transportation costs), we include the location part of the problem. Indeed, for each source i, $i = 1, \ldots, m$, the capacities C_i are generated in [200, 700]. The fixed

Please cite this article in press as: V. Gabrel, et al., Robust location transportation problems under uncertain demands, Discrete Applied Mathematics (2011), doi:10.1016/j.dam.2011.09.015

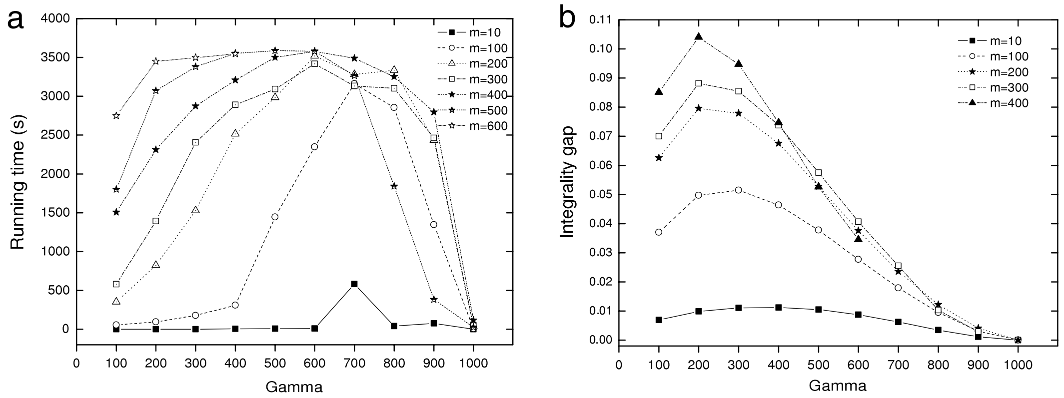

Fig. 3. Tests $n = 1000$: (a) Running time vs. Γ. (b) Integrality gap vs. Γ.

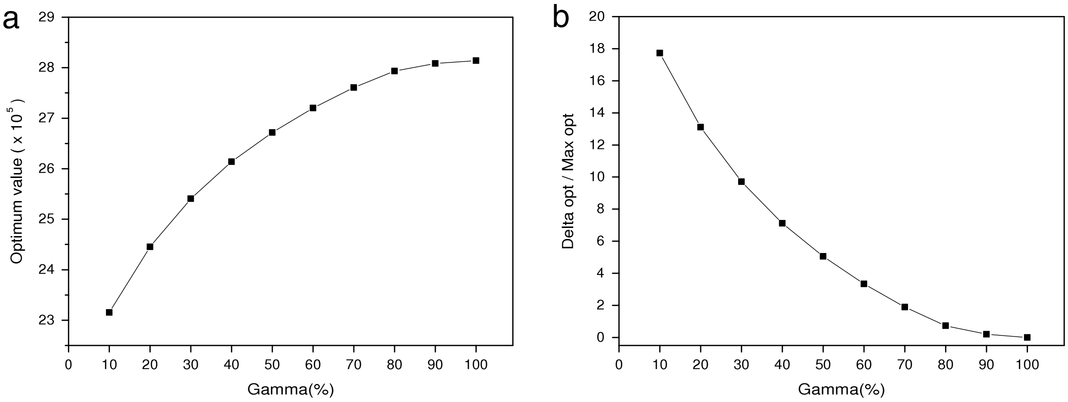

Fig. 4. A sample $n = m = 50$: (a) Optimum value vs. Γ. (b) $\frac{\Delta\text{Opt}}{\text{MaxOpt}}$ vs. Γ.

and variable costs are respectively in the intervals [100, 1000], [10, 100]. Kelley's algorithm (Section 2.1) is applied to solve the problem, and the computation is stopped after 2 h.

Fig. 4(a) shows the total costs versus Γ for one sample $m = 50$ and $n = 50$. The curve has the same shape as the total cost of the recourse problem (see Fig. 1(b)). Indeed, as the value of Γ increases the total costs also increase. Fig. 4(b) illustrates, for the same sample, the relative gap, in percentage, between the costs of the optimal solution of $T''_{rob}(\Gamma)$ and the worst case value (single stage problem with $\Gamma = n$) versus Γ. We remark that the decision maker can realize a significant profit up to 18% on the transportation costs, while ensuring the feasibility of the problem for any demand, including the largest one. This benefit is larger as Γ is smaller.

Running time results are summarized in Table 2 for different sizes of the problem. We set the number of sources equal to the number of demands, fixed to $n = m = 10$, 70, 100 and 150. As expected, the running time, as well as the number of iterations, increases when the problem becomes larger. For $n = m = 10$, 50 and 70 the problem requires less than 20 min to be solved. From $n = m = 100$ the running time is much greater. It is over two hours for a few values of Γ. When $n = m = 150$, this time limit is reached for almost all samples for $\Gamma \in [30\%, 80\%]$.

Moreover, we observe in column 5 and 6 the percentage of time shared by the master problem and by the recourse problem respectively. Such results confirm that the most difficult part in Kelley's algorithm concerns the recourse problem, especially when Γ is around $n/2$. Finally, when $\Gamma = 100\%$ the recourse problem becomes easy (in fact, equivalent to a linear program), which explains the result on these rows.

5. Conclusion

The aim of this paper is to solve a robust version of the location transportation problem. The 2-stage robust approach, presented here, offers more flexibility to the decision maker than the single-stage approach. Indeed, there are some recourse variables to be decided once the uncertainty has been revealed. Furthermore, the uncertain set is less conservative than the classical worst case approach since it considers a limited number of demands deviating from their nominal values. Previously, the 2-stage formulation has been considered by [1,16]. Nevertheless, the limit size of solved instances was performed for at most 50 uncertain parameters. Here, we present the first (to our knowledge) extensive computation

Please cite this article in press as: V. Gabrel, et al., Robust location transportation problems under uncertain demands, Discrete Applied Mathematics (2011), doi:10.1016/j.dam.2011.09.015

analysis on a particular recourse problem of the location transportation problem. As shown, the recourse problem is the most difficult part of the 2-stage robust optimization, and the tight bound we propose allows us to solve large size instances.

References

[1] A. Atamtürk, M. Zhang, Two-stage robust network flow and design under demand uncertainty, Operations Research 55 (4) (2007) 662–673.
[2] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robust Optimization, Princeton University Press, 2009.
[3] A. Ben-Tal, A. Goryashko, E. Guslitzer, A. Nemirovski, Adjustable robust solutions of uncertain linear programs, Mathematical Programming. Series A 99 (2004) 351–376.
[4] D. Bertsimas, D. Brown, C. Caramanis, Theory and applications of robust optimization, CoRR abs/1010.5445.
[5] D. Bertsimas, M. Sim, Robust discrete optimization and network flows, Mathematical Programming, Series B 98 (2003) 49–71.
[6] D. Bertsimas, M. Sim, The price of robustness, Operations Research 52 (1) (2004) 35–53.
[7] J.E. Falk, A linear max–min problem, Mathematical Programming 5 (1973) 169–188.
[8] C. Floudas, P. Parlados, State of the Art in Global Optimization, Kluwer, Dordrecht, Boston, London, 1995. Ch. Global Optimization of Separable Concave Functions Under Linear Constraints with Totally Unimodular Matices.
[9] V. Gabrel, C. Murat, Robustness and duality in linear programming, Journal of the Operational Research Society 61 (8) (2010) 1288–1296.
[10] F. Glover, E. Woolsey, Converting the 0–1 polynomial programming problem to a 0–1 linear program, Operations Research 22 (1974) 180–182.
[11] R. Horst, H. Tuy, Global Optimization: Deterministic Approaches, Springer-Verlag, Berlin, Heidelberg, New York, 1996. Ch. Special Problems of Concave Minimization.
[12] J.E. Kelley, The cutting-plane method for solving convex programs, Society for Industrial and Applied Mathematics 8 (4) (1960) 703–712.
[13] H. Konno, A cutting plane algorithm solving bilinear programs, Mathematical Programming 11 (1976) 14–27.
[14] H.D. Sherali, C. Shetty, A finitely convergent algorithm for bilinear programming problems using polar cuts and disjunctive face cuts, Mathematical Programming 19 (1980) 14–31.
[15] A.L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming, Operations Research 21 (5) (1973) 1154–1157.
[16] A. Thiele, T. Terry, M. Epelman, Robust linear optimization with recourse, Tech. Rep., 2009.
[17] S.A. Vavasis, Nonlinear Optimization: Complexity Issues, Oxford University Press, Inc., New York, NY, USA, 1991.