Polyhedral Analysis and Branch-and-Cut for the Structural Analysis Problem

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Abstract. In this paper we consider the structural analysis problem for differential-algebraic systems with conditional equations. This consists, given a conditional differential-algebraic system, in verifying if the system is structurally solvable for every state, and if not in finding a state in which the system is structurally singular. We give an integer linear programming formulation for the problem. We also identify some classes of valid inequalities and characterize when these inequalities define facets for the associated polytope. Moreover, we devise separation routines for these inequalities. Based on this, we develop a Branch-and-Cut algorithm along with experimental results are presented.

Keywords: Differential-algebraic system, structural analysis, bipartite graph, matching, polytope, facet, Branch-and-Cut.

1 Introduction

Differential-algebraic systems (DAS) are used for modeling complex physical systems such as electrical networks or dynamic movements. Such a system can be given as $F(z, \dot{z}, t) = 0$, where z is the variable vector, t is time and $\dot{z}$ is the partial derivative of z with respect to time. A DAS is said to be *solvable* if it can be solved with numerical methods [9]. A necessary (but not sufficient) condition for a DAS to be structurally solvable is that there are as many equations as variables, and there exists a mapping between the equations and the variables in such a way that each equation is related to only one variable and each variable is related to only one equation. If this is satisfied, then we say that the system is *structurally solvable*. Otherwise, the system is said to be *structurally singular*. The *structural analysis problem* (SAP) of a DAS consists in checking whether or not the system is structurally singular.

In this paper we consider this problem from a polyhedral point of view. We give a linear integer programming formulation for the problem. We discuss the associated polytope and characterize some classes of facet defining inequalities.

Using this, we propose a Branch-and-Cut algorithm and present experiment results.

The structural analysis problem for DASs has been proved to be polynomial-time solvable by Murota [8]. Given a DAS, one can associate a bipartite graph $G = (U \cup V, E)$, called *incidence graph*, where U corresponds to the equations, V to the variables, and there is an edge $uv \in E$ between a node $u \in U$ and a node $v \in V$ if the variable corresponding to v appears in the equation corresponding to u. Murota [8] proved that a DAS is structurally singular if and only if its incidence graph does not contain a perfect matching.

In many practical situations, the form of an equation of a DAS, especially the variables that appear in it, may depend of a condition such as temperature changements in hydraulic systems. Such equation is called *conditional*. Therefore, from a conditional equation, we can obtain two different (non-conditional) equations with respect to the values of the conditions associated with it. We will suppose that all conditions are independent. A DAS containing conditional equations is called *conditional DAS* (CDAS). An assignment of the values *true* and *false* to the conditions of a DAS will be called a *state* of the system. Hence each state yields a non-conditional DAS and, therefore, verifying if a conditional DAS is structurally **singular reduces to verify whether there exists a state for which the incidence graph is free perfect matching**. A first and preliminary study of SAP with conditional equations is given in [3, 4, 7]. In [5] we have shown the NP-completeness of this problem.

The paper is organized as follows. In the following section we give a graph model for the SAP for CDAS and a linear integer programming formulation. Section 3 describe some classes of valid inequalities and discuss their facial structure. **In section 4 we present** separation routines for these inequalities. In section 5, we devise a Branch-and-Cut algorithm based on these results and present some experimental results. Some concluding remarks are given in section 6.

2 Formulations

First we give formulation in terms of graphs for the SAP for CDAS. Given a CDAS (with conditional and non conditional equations) with each equation we associate a node we will denote this set of nodes by U. Let U_c be the subset of nodes of U associated with the conditional equations. With each variable we associate a node and we will denote this set by V. We consider an edge uv between a vertex $u \in U_c$ and a vertex $v \in V$, called *true edge* (resp. *false edge*, *true/false edge*), if the variable associated with v appears in the equation associated with u, when the condition of this equation is supposed true (resp. false, both true and false). We consider an edge uv between a vertex $u \in U \setminus U_c$ and a vertex $v \in V$ if the variable associated with v appears in the non-conditional equation associated with u. These edges are also called true/false edges. We denote by $G = (U \cup V, E)$ the bipartite graph thus obtained, where E is the set of these edges. For a node $u \in U$, we denote by E_u^t (resp. E_u^f, E_u^{tf}) the set of true (resp. false, true/false) edges incident to u. Note that these sets are disjoints. Also note that the sets

E_u^t and E_u^f are empty for all nodes $u \in U \setminus U_c$. Let $\pi = \{E_u^t, E_u^f, E_u^{tf} : u \in U\}$ be the partition of E induced by the true, false and true/false edges subsets. Let $E^{tf} = \cup_{u \in U} E_u^{tf}$ and $\mathcal{E} = \{E_u^t, E_u^f : u \in U_c\}$. If $\mathcal{F} \subseteq \mathcal{E}$, we let $H_{\mathcal{F}} = (U \cup V, E_{\mathcal{F}})$ be the subgraph of G induced by $E_{\mathcal{F}} = E^{tf} \cup (\cup_{F \in \mathcal{F}} F)$. Figure 1 shows the graph associated with the CDAS

$$\begin{array}{ll} eq_1: & \text{if } a > 0, \text{ then } \quad 0 = 4x_2^2 + 2\,\dot{x_3} + 4x_2 + 2, \quad \text{else} \quad 0 = \dot{x_2} + 4x_1, \\ eq_2: & \text{if } b > 0, \text{ then } \quad 0 = 2\,\dot{x_1} + 2x_2, \quad \text{else} \quad 0 = x_1 + \dot{x_3} + 1 \\ eq_3: & 0 = 6\,\dot{x_3} + 2x_1. \end{array} \qquad (1)$$

Here nodes u_1, u_2, u_3 are associated with equations eq_1, eq_2, eq_3 and nodes v_1, v_2, v_3 are associated with variables x_1, x_2, x_3 respectively.

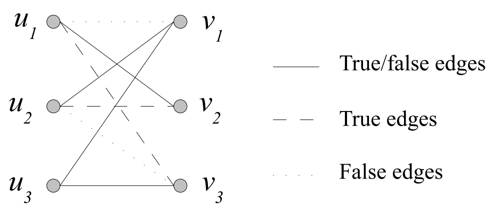

Fig. 1. Graph G associated with the CDAS (1).

Thus the SAP reduces to **find** $\mathcal{F} \subseteq \mathcal{E}$**, such** that
i) for all $u \in U_c$, either $E_u^t \notin \mathcal{F}$ or $E_u^f \notin \mathcal{F}$ or both,
ii) the number of edge sets in $\mathcal{F}$ is maximum,
iii) the graph $H_{\mathcal{F}}$ does not contain a perfect matching.
If the number of edge sets $|\mathcal{F}|$ is equal to $|U_c|$ then this means that we have found a state which yields a structurally singular system. Otherwise, there exists a perfect matching in the incidence graph associated with any state of the system and, thus, the CDAS is structurally solvable [4]. We will refer to this problem as the *free perfect matching subgraph problem (FPMSP).*
In what follows we shall give a formulation of the FPMSP as an integer linear program. With every edge set F of $\mathcal{E}$, we associate a binary variable x_F which takes 1 if F is contained in $\mathcal{F}$, and 0 otherwise. **Given a perfect matching** M of G, we denote by $F_M = \{F \in \mathcal{E} : F \cap M \neq \emptyset\}$ the family of edge sets of $\mathcal{E}$ that intersect M. The FPMSP is then equivalent to the following integer linear program (P).

$$\max \sum_{F \in \mathcal{E}} x_F$$

$$x_{E_u^t} + x_{E_u^f} \leq 1 \qquad \forall\, u \in U_c, \qquad (1)$$

$$\sum_{F \in F_M} x_F \leq |F_M| - 1 \qquad \forall\, M \in \mathcal{M}, \qquad (2)$$

$$x_F \geq 0 \qquad \forall\, F \in \mathcal{E}, \qquad (3)$$

$$x_F \leq 1 \qquad \forall\, F \in \mathcal{E}, \qquad (4)$$

$$x_F \in \{0, 1\} \qquad \forall\, F \in \mathcal{E}, \qquad (5)$$

where $\mathcal{M}$ is the set of perfect matchings of G. Inequalities (1) express the fact that at must one edge set among E_u^t and E_u^f may be take in $\mathcal{F}$. And inequalities (2) ensure that give a perfect matching M of G, all the edge sets intersecting M cannot be contained in $\mathcal{F}$.

3 Associated Polytope and valid inequalities

In this section, we give some valid inequalities for FPMSP and study their facial structure. Given a $\mathcal{F} \subseteq \mathcal{E}$, let $x^{\mathcal{F}} \in \{0,1\}^{\mathcal{E}}$ be the vector given by $x_F^{\mathcal{F}} = 1$ if $F \in \mathcal{F}$, and $x_F^{\mathcal{F}} = 0$ if not. $x^{\mathcal{F}}$ is called the *incidence vector* of $\mathcal{F}$. Let $P_{FPMSP}(G, U_c, \pi)$ be the convex hull of the solutions of program (P), that is, $P_{FPMSP}(G, U_c, \pi) = conv(\{x \in \{0,1\}^{\mathcal{E}} | x \text{ satisfies } (1), (2)\})$.

W.l.o.g., we may suppose that any graph $H_{\{F\}}$, $F \in \mathcal{E}$, does not contain a perfect matching. Indeed, suppose that there exists $u \in U_c$ such that, say, the graph $H_{\{E_u^t\}}$ contains a perfect matching. Then one can transform the instance of FPMSP into another one by removing E_u^t and adding E_u^f to E^{tf}. This corresponds to considering that the equation associated with u is no more conditional. A similar transformation can be done if F belongs to $\{E_u^f : u \in U_c\}$.

In [4], we have shown that $P_{FPMSP}(G, U_c, \pi)$ is full dimensional. We have also given necessary and sufficient conditions for the inequalities of (P) to define facets.

We will say that a perfect matching M of G dominates a perfect matching M' of G and we will write $M \succcurlyeq M'$ if $F_M \subseteq F_{M'}$. Observe that if M dominates M', then inequalities (2) associated with M' dominates that corresponding to M'. In consequence, perfect matchings of $G = (U \cup V, E)$ that may induce facet defining inequalities of type (2) must be minimal with respect to the $\succcurlyeq$ relation that is matchings which do no dominate any other perfect matching. Such a matching will be called *minimal matching*. The inequalities of type (2), induced by minimal matchings, will be called *minimal matching inequalities*.

In the following we are going to describe two further classes of valid inequalities, both are induced by matchings.

3.1 Close matchings inequalities

Given a true (resp. false) edge set $F \in \mathcal{E}$ incident to node $u \in U_c$, we denote by $\bar{F}$ the false (resp. true) edge set incident to u. That is, if $F = E_u^t$ (resp. $F = E_u^f$), then $\bar{F} = E_u^f$ (resp. $\bar{F} = E_u^t$). Let M be a minimal matching and $F' \in F_M$. A perfect matching $\hat{M}$ is called *close to* M w.r.t. F' if $F_{\hat{M}} \subseteq (F_M \setminus \{F'\}) \cup \{\bar{F}'\}$.

Proposition 1. *Let M be a minimal matching and $F' \in F_M$. If there exists a matching $\hat{M}$ close to M w.r.t. F', then the following inequality*

$$\sum_{F \in F_M} x_F + x_{\bar{F}'} \leq |F_M| - 1, \tag{6}$$

is valid.

Proof. By summing inequalities (2) associated with M and $\hat{M}$, inequality $x_{F'} + x_{\bar{F}'} \leq 1$ and inequalities $x_F \leq 1$ for all $F \in F_M \setminus (F_{\hat{M}} \cup \{F'\})$, we obtain the inequality $2\sum_{F \in F_M} x_F + 2x_{\bar{F}'} \leq 2|F_M| - 1$. By dividing by 2 and rounding down the right-hand side we obtain (6). □

These inequalities will be called *close matching inequalities*. In the following, we give necessary and sufficient conditions for these inequalities to be facet defining.

Theorem 1. *Inequality* (6) *defines a facet of* $P_{FPMSP}(G, U_c, \pi)$ *if and only if*
1) there exists $F^\star \in F_M \setminus \{F'\}$ *such that the graph* $H_{(F_M \cup \{\bar{F}'\}) \setminus \{F', F^\star\}}$ *is perfect matching free,*
2) for all $F \in F_M \setminus \{F'\}$*, at least one of the graphs* $H_{(F_M \cup \{\bar{F}\}) \setminus \{F\}}$ *and* $H_{(F_M \cup \{\bar{F}', \bar{F}\}) \setminus \{F', F\}}$ *is perfect matching free,*
3) for all $F \in \mathcal{E} \setminus F_M$ *such that* $\bar{F} \notin F_M$ *and the graph* $H_{(F_M \cup \{F\}) \setminus \{F'\}}$ *contains a perfect matching, there exists* $F^* \in F_M \setminus \{F'\}$ *such that at least one graph among* $H_{(F_M \cup \{F\}) \setminus \{F^*\}}$ *and* $H_{(F_M \cup \{\bar{F}', F\}) \setminus \{F^*, F'\}}$ *is perfect matching free.*

Proof. (⇒) 1) Suppose that for all $F^\star \in F_M \setminus \{F'\}$, the graph $H_{S_{F^\star}}$, where $S_{F^\star} = (F_M \cup \{\bar{F}'\}) \setminus \{F', F^\star\}$, contains a perfect matching, say, $M_{F^\star}$. Summing inequality (2) associated with $M_{F^\star}$ and inequalities $x_F \leq 1$ for all $F \in S_{F^*} \setminus F_{M_{F^\star}}$ yields the inequality

$$\sum_{F \in S_{F^\star}} x_F \leq |S_{F^\star}| - 1. \tag{7}$$

Now by summing inequalities (7) for all $F^\star \in F_M \setminus \{F'\}$, the close matching inequality (6) associated with M and F', $|F_M| - 2$ times the inequality $x_{F'} + x_{\bar{F}'} \leq 1$, dividing the resulting inequality by $|F_M| - 1$, and rounding down the right-hand side, we get the inequality

$$\sum_{F \in F_M} x_F + 2x_{\bar{F}'} \leq |F_M| - 1. \tag{8}$$

Clearly inequality (8) dominates (6) and thus the latter cannot define a facet.

2) Suppose there exists $F_1 \in F_M \setminus \{F'\}$ such that both the graphs H_{S_1} and H_{S_2} where $S_1 = (F_M \cup \{\bar{F}_1\}) \setminus \{F_1\}$ and $S_2 = (F_M \cup \{\bar{F}', \bar{F}_1\}) \setminus \{F', F_1\}$ contain perfect matchings M_1 and M_2 respectively. By summing inequalities (2) associated with matchings M_1 and M_2, inequalities $x_F \leq 1$ for all $F \in (S_1 \setminus F_{M_1}) \cup (S_2 \setminus F_{M_2})$, inequality $x_{F'} + x_{\bar{F}'} \leq 1$, dividing by 2 the resulting inequality and rounding down the right-hand side, we get the inequality

$$\sum_{F \in F_M \setminus \{F_1\}} x_F + x_{\bar{F}_1} + x_{\bar{F}'} \leq |F_M| - 1. \tag{9}$$

Now summing inequality (9), the close matching inequality (6) associated with M and F', inequality $x_{F_1} + x_{\bar{F}_1} \leq 1$, dividing by 2 the resulting inequality and rounding down the right-hand side, we get the inequality

$$\sum_{F \in F_M} x_F + x_{\bar{F}_1} + x_{\bar{F}'} \leq |F_M| - 1. \tag{10}$$

Clearly inequality (10) dominates (6), and thus the latter cannot define a facet.

3) Suppose there exists $F_2 \in \mathcal{E} \setminus F_M$ such that $\bar{F}_2 \notin F_M$, the graph H_S where $S = F_M \cup \{F_2\} \setminus \{F'\}$ contains a perfect matching, say M_S, and for all $F^* \in F_M \setminus \{F'\}$, the graphs $H_{S_1^*}$, $H_{S_2^*}$ contain perfect matchings M_1^* and M_2^*, where $S_1^* = F_M \cup \{F_2\} \setminus \{F^*\}$ and $S_2^* = F_M \cup \{\bar{F}', F_2\} \setminus \{F^*, F'\}$. By summing inequality (2) associated with the matching M_1^* and M_2^*, inequalities $x_F \leq 1$ for all $F \in (S_1^* \setminus F_{M_1^*}) \cup (S_2^* \setminus F_{M_2^*})$, inequality $x_{F'} + x_{\bar{F}'} \leq 1$, dividing by 2 the resulting inequality and rounding down the right-hand side we get the inequality

$$\sum_{F \in F_M \setminus \{F^*\}} x_F + x_{F_2} + x_{\bar{F}'} \leq |F_M| - 1. \tag{11}$$

Now by summing inequality (2) associated with matching M_S, inequalities $x_F \leq 1$ for all $F \in S \setminus F_{M_S}$, inequalities (11) for all $F^\star \in F_M \setminus \{F'\}$, the close matching inequality (6) associated with M and F', dividing by $|F_M|$ the resulting inequality and rounding down the right-hand side we get the inequality

$$\sum_{F \in F_M} x_F + x_{F_2} + x_{\bar{F}'} \leq |F_M| - 1. \tag{12}$$

Clearly inequality (12) dominates (6), and hence the latter cannot define a facet.

($\Leftarrow$) Now we suppose that conditions 1), 2), 3) are satisfied. Let us denote by $ax \leq \alpha$ inequality (6) associated with M and F'. Let $bx \leq \beta$ be a facet defining inequality of $P_{FPMSP}(G, U_c, \pi)$ such that $\{x \in P_{FPMSP}(G, U_c, \pi) : ax = \alpha\} \subseteq \{x \in P_{FPMSP}(G, U_c, \pi) : bx = \beta\}$. We will show that $b = \rho a$ for some $\rho \in \mathbb{R}$.

By condition 1), there exists $F^\star \in F_M \setminus \{F'\}$ such that the graph $H_{S^\star}$ is perfect matching free where $S^\star = (F_M \cup \{\bar{F}'\}) \setminus \{F', F^\star\}$. Hence $S^\star$ is a solution of FPMSP. Since the matching is minimal, the set $S'^\star = F_M \setminus \{F'\}$ is also a solution of FPMSP. Moreover we have that $ax^{S^\star} = ax^{S'^\star} = \alpha$. Hence $bx^{S^\star} = bx^{S'^\star}$. This implies that $b(F') = b(\bar{F}')$.

Let $F_1 \in F_M \setminus \{F'\}$. Clearly, $S_1 = F_M \setminus \{F_1\}$ is a solution. Furthermore, we have $ax^{S_1} = \alpha$. Thus $bx^{S_1} = bx^{S'^\star}$, and hence $b(F_1) = b(F')$. Consequently, we have $b(F) = \rho$ for all $F \in F_M \cup \{\bar{F}'\}$ for some $\rho \in \mathbb{R}$.

Let $F_2 \in \mathcal{E} \setminus F_M$ such that $\bar{F}_2 \notin F_M$. Clearly, F_2 is different from $\bar{F}'$. Let $S_2 = (F_M \cup \{F_2\}) \setminus \{F'\}$. If H_{S_2} is perfect matching free, then S_2 is a solution of FPMSP. As $ax^{S_2} = \alpha$, and thus $bx^{S_2} = bx^{S'^\star}$, this implies that $b(F_2) = 0$. If H_{S_2} contains a perfect matching, by condition 3), there exists $F_3 \in F_M \setminus \{F'\}$ such that one of the graphs H_{S_3} and $H_{S_3'}$ is perfect matching free, where $S_3 = (F_M \cup \{F_2\}) \setminus \{F_3\}$ and $S_3' = (F_M \cup \{F_2, \bar{F}'\}) \setminus \{F_3, F'\}$. Hence at least one of the sets S_3 and S_3' is a solution of FPMSP. If S_3 (resp. S_3') is a solution of FPMSP, as $ax^{S_3} = \alpha$ (resp. $ax^{S_3'} = \alpha$) and hence $bx^{S_3} = bx^{S'^\star}$ (resp. $bx^{S_3'} = bx^{S'^\star}$), we obtain that $b(F_2) = 0$. Thus we have $b(F) = 0$ for all $F \in \mathcal{E} \setminus F_M$ such that $\bar{F} \notin F_M$.

Finally, consider $F_4 \in \mathcal{E} \setminus F_M$ such that $\bar{F}' \neq F_4$ and $\bar{F}_4 \in F_M$. By condition 2), at least one of the graphs H_{S_4} and $H_{S_4'}$ is perfect matching free, where $S_4 = (F_M \cup \{F_4\}) \setminus \{\bar{F}_4\}$ and $S_4' = (F_M \cup \{F_4, \bar{F}'\}) \setminus \{\bar{F}_4, F'\}$. Thus either S_4

or S'_4 or both are solutions for FPMSP. If S_4 (resp. S'_4) is a solution of FPMSP, as $ax^{S_4} = \alpha$ (resp. $ax^{S'_4} = \alpha$) and hence $bx^{S_4} = bx^{S'^{\star}}$ (resp. $bx^{S'_4} = bx^{S'^{\star}}$), we obtain that $b(F_4) = 0$. Therefore $b(F) = 0$ for all $F \in \mathcal{E} \setminus F_M$ such that $\bar{F} \in F_M$ and $F \neq \bar{F}'$.

Overall, we have that $b = \rho a$, which ends the proof. □

3.2 k-Multiple matching inequalities

We denote by Θ **the set of non-empty subsets** of $\mathcal{E}$ verifying condition i). That is, a set $\theta \subseteq \mathcal{E}$ is in Θ if there does not exist a node $u \in U_c$ such that $\{E^t_u, E^f_u\} \subseteq \theta$. Let $\theta \in \Theta$, $\theta' \subseteq \theta$ and k be an integer in $\{1, ..., |\theta'| + 1\}$. We say that graph H_θ is $k-$multiple matching w.r.t. θ' if after removal of any $k-1$ edge sets of θ' in H_θ, the new graph still contains a perfect matching.

Proposition 2. *Let $\theta \in \Theta$, $\theta' \subseteq \theta$ and $k \in \{1, ..., |\theta'| + 1\}$. If the graph H_θ is $k-$multiple matching w.r.t. θ', then the following inequality*

$$\sum_{F \in \theta'} x_F + k \sum_{F \in \theta \setminus \theta'} x_F \leq |\theta'| + k(|\theta \setminus \theta'| - 1), \tag{13}$$

is valid for $P_{FPMSP}(G, U_c, \pi)$.

Proof. Suppose there exists a solution S of FPMSP such that the incidence vector x^S verifies

$$\sum_{F \in \theta'} x^S_F + k \sum_{F \in \theta \setminus \theta'} x^S_F > |\theta'| + k(|\theta \setminus \theta'| - 1). \tag{14}$$

As $\sum_{F \in \theta'} x^S_F \leq |\theta'|$ we then have $k \sum_{F \in \theta \setminus \theta'} x^S_F \geq k(|\theta \setminus \theta'| - 1) + 1$. Thus, $\sum_{F \in \theta \setminus \theta'} x^S_F = |\theta \setminus \theta'|$ and, in consequence, $\sum_{F \in \theta'} x^S_F \geq |\theta'| - k + 1$. This implies $|\theta^*| \leq k - 1$, where $\theta^* = \theta' \setminus S$. By an assumption, it follows that graph $H_{\theta \setminus \theta^*}$ contains a perfect matching. As $\theta \setminus \theta^* \subseteq S$, the graph H_S contains a perfect matching contradicting the fact that S is a solution of FPMSP. □

Proposition 3. *Inequality* (13) *does not define a facet of $P_{FPMSP}(G, U_c, \pi)$ if one of the following conditions is satisfied*
1) H_θ is $k'-$multiple matching w.r.t. θ', for $k' > k$,
2) there exists $F \in \theta \setminus \theta'$ such that H_θ is $k-$multiple matching w.r.t. $\theta' \cup \{F\}$,
3) there exists $\tilde{\theta} \subset \theta$ and $\tilde{\theta}' \subseteq \tilde{\theta}$ such that $\tilde{\theta} \setminus \tilde{\theta}' \subseteq \theta \setminus \theta'$, $\tilde{\theta}' \subseteq \theta'$ and $H_{\tilde{\theta}}$ is $k-$multiple matching w.r.t. $\tilde{\theta}'$.

Proof. Suppose that condition 1) is satisfied. In this case, the $k-$multiple matching inequality (13) associated with θ', θ and k' is valid for the $P_{FPMSP}(G, U_c, \pi)$. This implies that the incidence vectors of the solutions of the face F_k, induced by inequality (13) associated with θ', θ and k, verify $\sum_{F \in \theta'} x_F = |\theta'|$ and $\sum_{F \in \theta \setminus \theta'} x_F = |\theta \setminus \theta'| - 1$. As consequence, there solutions also verify with equality inequality (13) associated with θ', θ and k'. This implies that these solutions belong to the face $F_{k'}$ defined by this inequality. As $F_k \subseteq F_{k'}$ and the two inequalities (13) associated with θ', θ and k' and θ', θ and k are not equivalent, this implies that F_k is not a facet.

Now suppose that condition 2) is satisfied, that is there exists $F \in \theta \setminus \theta'$ such that H_θ is $k-$multiple matching w.r.t. $\theta' \cup \{F\}$. Observe that inequality (13) associated with θ', θ and k is the sum of inequality (13), associated with $\theta' \cup \{F\}$, θ and k, and $k-1$ times inequality $x_F \leq 1$. Thus the former one cannot define a facet.

Finally, suppose that condition 3) is satisfied. Then there exists $\tilde{\theta} \subset \theta$ and $\tilde{\theta}' \subseteq \tilde{\theta}$ such that $\tilde{\theta} \setminus \tilde{\theta}' \subseteq \theta \setminus \theta'$, $\tilde{\theta}' \subseteq \theta'$ and $H_{\tilde{\theta}}$ is $k-$multiple matching w.r.t. $\tilde{\theta}'$. It is easy to see that inequality (13) associated with θ', θ and k is the sum of inequality (13) associated with $\tilde{\theta}'$, $\tilde{\theta}$ and k, inequalities $x_F \leq 1$ for $F \in \theta' \setminus \tilde{\theta}'$ and k times inequalities $x_F \leq 1$ for $F \in (\theta \setminus \theta') \setminus (\tilde{\theta} \setminus \tilde{\theta}')$. Thus former one cannot define a facet. □

4 Separation

The separation problem for a classe of inequalities $ax \leq b$ consists, given a solution $\bar{x} \in R^{\mathcal{E}}$, in determining whether $\bar{x}$ satisfies $ax \leq b$, and if not in finding an inequality violated by $\bar{x}$. In this section we discuss the separation problem for the classes of inequalities presented in section 3.

First we consider the minimal matching inequalities. We show that the separation problem for these inequalities can be solved in polynomial time. The separation is performed in two steps. In the first one, we look for a matching inequality (2), if there exists any, violated by $\bar{x}$. This can be done in polynomial time [3]. So suppose that there exists a perfect matching, say M, such that inequality (2) associated with M, is violated by $\bar{x}$. If there does not exist such an inequality then, clearly, there is no a minimal matching inequality violated by $\bar{x}$. In a second step, we will try to strengthen M by determining a minimal matching M' with $F_{M'} \subseteq F_M$. Remark that $\bar{x}$ also violates the inequality (2) associated with M'. This second step reduces to computing a perfect matching in H_{F_M}, containing a maximum number of true/false edges. This also can be done in polynomial time [2].

4.1 Separation of close matching inequalities

The heuristic separation problem for close matching inequalities (6) begins with a minimal matching M where the associated inequality is violated. The separation algorithm consists in finding for all $F \in F_M$ if there exists a close perfect matching to F_M and F, i.e. we verify if the graph $H_{(F_M \setminus \{F\}) \cup \{\bar{F}\}}$. For this we use M-augmenting path [2], this algorithm compute in $O(m)$. If for a $F \in F_M$ there exists a close perfect matching to F_M and F then the associated inequality is violated. This separation problem reduces to compute $|F_M|$ M-augmenting paths in bipartite graphs and can be solved in polynomial time.

4.2 Separation of 2-multiple matching inequalities

The heuristic separation problem for 2-multiple matching inequalities (9) is based on flows in a particular graph. Let ω be the subset of $\mathcal{E}$ given by, $E_u^t \in \omega$ if $\bar{x}_{E_u^t} \geq 0.5$ and $E_u^f \in \omega$ otherwise, for all $u \in U_c$. This heuristique searches, if there exists, $\theta \subseteq \omega$ and $\theta' \subseteq \theta$ such that the 2-multiple matching inequality associated with θ and θ' is violated. Let $E_u = (E_u^t \cup E_u^t) \cap \omega$ and $n = |U|$. We search n particular matchings, these matchings corresponding to the possibility to delete any one edge set of θ'. These matching also corresponding to the minimum cost flow of value n^2 in the directed graph $D = (V_D, A_D)$ defined as follows. For each $u \in U_c$, we add the nodes $w_u^1, w_u^2, w_u^3, w_u^4$ in V_D, for each $v \in V$ we add the node $w_v \in V_D$, for each $u \in U \setminus U_c$ we add the node $w_u \in V_D$ and we add the nodes s and t in V_D. For each $u \in U_c$ we add the arcs :

1. (w_u^1, t), (w_u^3, w_u^1), with cost 0 and a capacity n,
2. (w_u^2, w_u^1), with cost x'_{E_u} and a capacity n,
3. (w_u^4, w_u^2), with cost x'_{E_u} and a capacity 1,
4. (w_u^4, w_u^2), with cost 0 and a capacity $n-1$.

For each $E_u \in \omega$ and for each $uv \in E_u$ we add the arc (w_v, w_u^4) with cost 0 and a capacity n. For all $u \in U_c$ and for all $uv \in E^{tf}$ we add the arc (w_v, w_u^3) with cost 0 and a capacity n. For all $u \in U \setminus U_c$ and for all $uv \in E^{tf}$ we add the arc (w_v, w_u) with cost 0 and a capacity n. Finally, for all $u \in U \setminus U_c$ and for all $v \in V$ we add the arcs (s, w_v) and (w_u, t) with cost 0 and a capacity n. For each $u \in U_c$ if the flow is non-null on (w_u^2, w_u^1) add E_u in θ and if the flow is non-null on (w_u^2, w_u^1) and (w_u^3, w_u^1) add E_u in θ'. If the cost of the minimum weight flow f between s and t of value n^2 is less than 1 then we have found a violated 2-multiple-inequality (9) associated with θ and θ'.

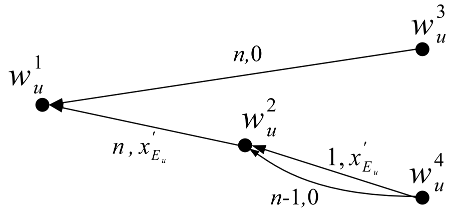

Fig. 2. Subset of nodes corresponding to node u.

5 Branch-and-Cut algorithm

Branch-and-Cut methods consist in a combination of a cutting plane technique and Branch-and-Bound algorithm. In this section, we present a Branch-and-Cut algorithm for the SAP for CDAS. Our aim is to address the algorithmic applications of the model and the theoretical results presented in the previous sections. To start the optimization, we consider the following linear program given by the inequalities, that is

$$\max\{\sum_{F\in\mathcal{E}} x_F | x \in [0,1]^{\mathcal{E}} satisfies (1)\}$$

An important task in the Branch-and-Cut algorithm is to determine whether or not an optimal solution of the linear relaxation of the FPMSP is feasible. An optimal solution $\bar{x}$ of the linear relaxation is feasible for the FPMSP if $\bar{x}$ is integer and $\bar{x}$ satisfies the matching inequalities. Thus verifying if $\bar{x}$ is feasible for FPMSP can be done in polynomial time. If an optimal solution $\bar{x}$ of the linear relaxation of the FPMSP is not feasible, the Branch-and-Cut algorithm generate a minimal matching inequality, if there exists we search all close matching inequalities based on the minimal matching inequality and add these inequalities, and we generate a 2-multiple matching inequality, valid for our problem and violated by $\bar{x}$. We remark inequalities are global (i.e. valid in all the Branch-and-Cut tree).

We remark that if the relaxation solution is less than $|U_c|$ then the CDAS is structurally solvable. In this case we stop the algorithm. We deduce, of this remark, the gap is not interesting for our problem.

The Branch-and-Cut algorithm has been implemented in C++ using ABACUS library [1] to manage the Branch-and-Cut tree and CPLEX 11.0 as LP-solver. To solve the minimum weight perfect matching and the minimum cost flow we use the LEMON Graph Library [6].

The algorithm was tested on a Pentium core 2 duo 2.66 GHz with 2 Gb RAM. We fixed the CPU time limit to 1h. Results are presented here for randomly generated instances. The tests are performed for systems with up to $n = 65$ conditional equations and 5 non-conditional equations. Recall that the corresponding bipartite graphs have 2(n+5) nodes. The systems are considered in such a way that each equation has between $k-1$ and $k+1$ variables where k is a given parameter. Our tests were performed for $k \in \{5, 7\}$. Five instances were tested for each problem and we provide the average results. We compare two variants, the first use only the matching inequalities and the second use minimal matching inequalities, close matching inequalities and $2-$multiple matching inequalities. The results are given in Table 1. The entries in the tables are :

n : the number of conditional equations,
k : the integer indicating that the number of variables in each equation is between $k-1$ and $k+1$,
o/p : the number of problems solved to optimality over the number of instances tested,
CPU : the total CPU time in seconds,
No : the number of generated nodes in the Branch-and-Cut tree,
Ct : the number of generated inequalities,

where the columns o/p1, CPU1, No1 and Ct1 are associated with the first variant and the columns o/p2, CPU2, No2 and Ct2 are associated with the second variant.

n	k	o/p1	CPU1	No1	Ct1	o/p2	CPU2	No2	Ct2
35	5	2/5	2452	37045	3943	5/5	0.2	18	2
35	7	0/5	>1h	59238	6182	5/5	0.8	68	13
45	5	3/5	1631	22627	2499	5/5	0.2	6	0
45	7	0/5	>1h	53468	5407	5/5	1.2	72	7
55	5	2/5	2565	29885	3202	5/5	2.6	204	8
55	7	0/5	>1h	47986	4944	5/5	1.8	68	2
60	5	2/5	2576	28345	3136	5/5	1.8	108	6
60	7	0/5	>1h	44913	4666	5/5	4.4	217	33
65	5	3/5	2529	27463	3204	5/5	0.6	39	0
65	7	0/5	>1h	42143	4384	5/5	46.6	1978	221

Table 1. Randomly generated instances

n	k	o/p2	CPU2	No2	Ct2
150	5	10/10	45	4	421
200	5	10/10	251	18	1158
250	5	10/10	239	8	805
300	5	10/10	264	7	657
350	5	10/10	636	11	971
400	5	10/10	175	2	203
450	5	10/10	423	4	428
900	5	10/10	2934	6	653
1000	5	10/10	3012	5	558
1100	5	9/10	9738	13	1319
1200	5	8/10	9829	8	999
1300	5	9/10	12818	4	680

Table 2. The big instances

From Table 1, we show the efficiency of inequalities (6) and (13) for solving the SAP. Remark for instances $n = 45$, $k = 5$ and $n = 65$, $k = 5$, all the solutions have been found at the root node. We can solve all instances in less than 50 seconds using the inequalities (6) and (13) and only 12 instances over 50 have been solved to optimality using the first variant of the algorithm. We have tested our algorithm on bigger instances. For this, we have extended the time limit to 5 hours. We consider 10 instances for each line. The results are given in Table 2.

We have solved instances with up to 1300 conditional equations, which correspond to graph having 2610 nodes. The biggest instances cannot be solved because due to memory lack. We can see the number of nodes in the tree is really small. We have on average less than 20 nodes in our Branch-and-Cut tree. This shows the efficiency of the Branch-and-Cut algorithm we propose.

6 Concluding remarks

In this paper, we consider the SAP for conditional DAS. We introduce new valid inequalities for the integer linear formulation given in [4]. We give necessary or necessary and sufficient conditions for these inequalities to be facet defining. We also provide efficient heuristics in order to separate these inequalities. All the results have been used in order to devise an efficient Branch-and-Cut algorithm for the SAP for conditional DAS. Instances of quite large size have been solved to optimality using this algorithm.

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