Flow-based mathematical formulation and strengthening cuts for the Cumulative CVRP

Sandra Ulrich NGUEVEU

CNRS ; LAAS ; 7 avenue du colonel Roche, F-31077 Toulouse Cedex 4, France,

Université de Toulouse ; UPS, INSA, INP, ISAE ; UT1, UTM, LAAS ; F-31077 Toulouse Cedex 4, France

Email: ngueveu@laas.fr

Mathieu LACROIX

LIPN (UMR 7030), Université Paris 13, 99, avenue J-B Clément, 93430 Villetaneuse, France

Email: lacroix@lipn.univ-paris13.fr

1 Introduction

The Cumulative Capacitated Vehicle Routing Problem (CCVRP) aims at minimizing the sum of arrival times at customers, instead of the classical route cost. Its applications arise mainly in humanitarian logistics and routing for relief, where the priority is the fast and equitable delivery of vital goods and supply, for example after a natural disaster [1]. It generalizes the NP-hard traveling repairman problem (TRP), which has already been proven harder to solve and to approximate than the TSP [2], by adding capacity constraints and a homogeneous vehicle fleet.

To the best of our knowledge, few papers in the literature address the CCVRP. Ngueveu *et al.* [4] proposed a time-based formulation (TF), identified specific properties, described two polynomial lower bounds and a memetic algorithm (MA) tested on instances from 50 to 200 nodes. The authors also showed that the objective function of the CCVRP can be expressed as the sum of the cost of each edge e, multiplied by a "coefficient", which is equal to 0 if e is unused, otherwise it is equal to one plus the number of customers visited after crossing edge e, before returning to the depot. Ribeiro and Laporte [5] presented an adaptive large neighborhood search heuristic (ALNS) that outperforms MA on most instances from 50 to 420 nodes. At each iteration of the ALNS, a subset of clients is removed from the current solution then reinserted in it using different heuristics. The authors used a request bank to store clients that could not be inserted due to capacity limitations, and penalized the objective function, to ensure exploration of a large neighborhood.

The remainder of the paper presents a new flow-based formulation that outperforms (TF) and is better suited for branch-and-cuts, as well as some valid strengthening cuts.

2 Flow-based formulation

Mathematical models for the CCVRP are defined on a complete non-oriented graph $G = (V, E, C)$ where $V = \{0, ..., n, n+1\}$ represents the sets of nodes, $V' = V\backslash\{0, n+1\}$ is the set of clients whereas nodes 0 and $n+1$ correspond to the depot. E is the set of edges and a cost $c_{ij} \in C$ is assigned to each edge $e = (i, j) \in E$ to represent the length or traveling time between nodes i and j. A fleet of R identical vehicles of capacity Q is located at the depot and available to satisfy the demand $q_i > 0$ of each client $i \in V'$. A route is performed by a single vehicle and each client is served exactly once.

Model (TF) from [4] uses decision variables t_i^k equal to the arrival time of vehicle k at client i and x_{ij}^k the binary variable equal to 1 if vehicle k crosses edge (i, j) from i to j.

$$\text{(TF)}\ \min F = \sum_{k=1}^{R} \sum_{i \in V'} t_i^k \tag{1}$$

subject to (s.t.)

$$\sum_{i \in V} x_{ij}^k = \sum_{i \in V} x_{ji}^k, \qquad \forall j \in V', \forall k \in \{1 \dots R\} \tag{2}$$

$$\sum_{k=1}^{R} \sum_{j \in V} x_{ij}^k = 1, \qquad \forall i \in V' \tag{3}$$

$$\sum_{i \in V'} \sum_{j \in V} x_{ij}^k q_i \leq Q, \qquad \forall k \in \{1 \dots R\} \tag{4}$$

$$\sum_{j \in V} x_{0j}^k = \sum_{i \in V} x_{i,n+1}^k = 1, \qquad \forall k \in \{1 \dots R\} \tag{5}$$

$$t_i^k + c_{ij} \leq t_j^k + (1 - x_{ij}^k)T, \qquad \forall i \in V \setminus \{n+1\}, \forall j \in V, \forall k \in \{1 \dots R\} \tag{6}$$

$$t_i^k \geq 0, \qquad \forall i \in V, \forall k \in \{1 \dots R\} \tag{7}$$

$$x_{ij}^k \in \{0, 1\}, \qquad \forall i \in V, \forall j \in V, i \neq j, \forall k \in \{1 \dots R\} \tag{8}$$

The sum of arrival times is to be minimized (1) whilst respecting the following constraints: a vehicle arriving at customer i must leave it (2); each customer is served by exactly one route (3) and vehicle capacity must be respected (4); the depot is at the beginning and at the end of each route (5); finally, constraints (6) compute the arrival times at each node and prevent subtours with the help of T defined as a large positive constant. This formulation has a polynomial number of variables and constraints, but suffers from a very weak linear relaxation which makes it less efficient for branch-and-bound algorithms, because optimal solutions of the linear relaxation of (TF) can have a cost of zero.

Our new mathematical model for the CCVRP relies on flow variables to compute optimal "coefficients" of edges in the objective-function (similar for example to [3] for the TRP). It uses decision variables f_{ij} equal to the amount of flow circulating from i to j on edge $e = (i, j)$ and the binary variable x_e equal to 1 if edge e is used, to obtain:

$$\text{(FF)} \ \min F = \sum_{i \in V \setminus \{n+1\}} \sum_{j \in V' \setminus \{i\}} c_{ij} f_{ij} \tag{9}$$

s.t.

$$\sum_{e \in \delta(i)} x_e = \begin{cases} R & \text{if } \ i \in \{0, n+1\} \\ 2 & \text{otherwise.} \end{cases}, \qquad \forall i \in V \tag{10}$$

$$\sum_{e \in S} x_e \geq 2 \left\lceil \frac{\sum_{i \in S} q_i}{Q} \right\rceil, \qquad \forall S \subseteq V' \tag{11}$$

$$\sum_{k \in V \setminus \{i\}} f_{ik} - \sum_{k \in V \setminus \{i\}} f_{ki} = \begin{cases} n & \text{if } \ i = 0 \\ -1 & \text{otherwise.} \end{cases}, \qquad \forall i \in V \setminus \{n+1\} \tag{12}$$

$$f_{i0} = f_{i,n+1} = f_{n+1,i} = 0, \qquad \forall i \in V' \tag{13}$$

$$f_{i_e,j_e} + f_{j_e,i_e} \leq \gamma x_e, \qquad \forall e = (i_e, j_e) \in E \tag{14}$$

$$f_{ij} \geq 0, \qquad \forall i \in V, \ \ \forall j \in V' \setminus \{i\} \tag{15}$$

$$x_e \in \{0, 1\}, \qquad \forall e \in E \tag{16}$$

The objective function (9) minimizes the solution cost equal to the sum of the flow costs, since the flow f_{ij} passing from i to j corresponds to one plus the number of clients that follow edge $e = (i, j)$ before the return to the depot. Constraints (10) are *degree constraints* ensuring that two edges are incident to each client, R edges are incident to the starting depot 0 and R edges are incident to the ending depot $n + 1$. Constraints (11) are the *capacity constraints*. Constraints (12) correspond to the *flow conservation constraints*: the depot generates n units of flow and each client consumes one unit of flow. Constraints (13) forbids flow returning to the depot. Finally, constraints (14) link the binary and real decision variables by forbidding flow on unused edges, with the help of a high constant γ which corresponds to the maximum number of customers delivered by a vehicle.

Formulation (FF) has a polynomial number of variables but an exponential number of constraints. Its efficiency lies in the fact that the total amount of flow generated by the depot and circulating through the graph is predefined. In addition, (FF) could be partitioned into 2 parts: a CVRP (10)-(11) and a flow computation problem (12)-(13), both connected through linking constraints (14). Therefore it is possible to apply on (FF) all known valid cuts from the CVRP literature. However, preliminary computational results showed that such straightforward branch-and-cut is not efficient, because the linear relaxation tends to produce very fractional solutions to satisfy the linking constraints (14).

3 Strengthening cuts and resulting branch-and-cut

Additional inequalities are added in order to strengthen the linear relaxation of (FF). The circulation of flow is forced on edges used with equations (17). Moreover, the flow from any customer entering i is limited by (18).

$$f_{ij} + f_{ji} \geq x_e, \qquad \forall e = (i,j) \in E, j \neq n+1 \tag{17}$$

$$\sum_{k \in V' \setminus \{i\}} f_{ki} \leq (\gamma - 1)(1 - x_{(0,i)}), \qquad \forall i \in V' \tag{18}$$

The branch-and-cut algorithm designed for (FF) starts from an initial linear relaxation $(\text{FF})^{(0)}$ with strengthening constraints (17) but without capacity constraints (11). Violated capacity cuts are identified with CVRPSEP[6]. Branching is applied only on binary variables x_e. A priority level of 2 is assigned to each variable $x_e \in \delta(0)$ and 1 to each variable $x_e \in E \setminus \delta(0)$, because edges connected to the starting depot carry the highest amount of flow and thus indirectly have a higher impact on the objective-function. The branching is applied on the most fractional variable among the ones with the highest priority level. If the variable chosen x'_e belongs to $\delta(0)$, then the additional constraint (18) is added to strengthen the generated subproblem that has assigned value 1 to x'_e.

4 Conclusion

The paper introduces a flow-based formulation for the CCVRP, that outperforms time-based models from the literature. Valid cuts are identified to strengthen its linear relaxation. Computational analysis of the resulting branch-and-cut algorithm will be presented.

References

[1] A.M. Campbell, D. Vandenbussche and W. Hermann, "Routing for relief efforts", *Transportation Science* 42(2), 127-145, 2008.

[2] A. Blum, P. Chalasani, D. Coppersmith, B. Pulleyblank, P. Raghavan and M. Sudan, "The minimum latency problem", *Proceedings STOC '94*, 163-171, 1994.

[3] M. Fischetti, G. Laporte and S. Martello, "The delivery man problem with cumulative matroids." *Operations Research*, 6, 1055-1064, 1993.

[4] S.U. Ngueveu, C. Prins and R. Wolfler Calvo, "An effective memetic algorithm for the cumulative capacitated vehicle routing problem", *Computers and Operations Research* 37, 1877-1885, 2010.

[5] G. M. Ribeiro and G. Laporte, "An adaptive large neighborhood search heuristic for the cumulative capacitated vehicle routing problem", *Computers and Operations Research* 39, 728-735, 2012.

[6] J. Lysgaard. *Separation routines for the CVRP*, `http://www.hha.dk/~lys/`, 2003.