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Circuit and bond polytopes on series–parallel graphs☆

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ABSTRACT

In this paper, we describe the circuit polytope on series–parallel graphs. We first show the existence of a compact extended formulation. Though not being explicit, its construction process helps us to inductively provide the description in the original space. As a consequence, using the link between bonds and circuits in planar graphs, we also describe the bond polytope on series–parallel graphs.

In an undirected graph, a *circuit* is a subset of edges inducing a connected subgraph in which every vertex has degree two. In the literature, a circuit is sometimes called *simple cycle*. Given a graph and costs on its edges, the *circuit problem* consists in finding a circuit of maximum cost. This problem is already NP-hard in planar graphs [1], yet some polynomial cases are known, for instance when the costs are non-positive.

Although characterizing a polytope corresponding to an NP-hard problem is unlikely, a partial description may be sufficient to develop an efficient polyhedral approach. Concerning the *circuit polytope*, which is the convex hull of the (edge-)incidence vectors of the circuits of the graph, facets have been exhibited by Bauer [2] and Coullard and Pulleyblank [3], and the cone has been characterized by Seymour [4]. Several variants of cardinality constrained versions have been studied, such as [5–8].

For a better understanding of the circuit polytope on planar graphs, a natural first step is to study it in smaller classes of graphs. For instance, in [3], the authors provide a complete description in Halin graphs.

Another interesting subclass of planar graphs are the series–parallel graphs. Due to their nice decomposition properties, many problems NP-hard in general are polynomial for these graphs, in which case it is quite

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standard to (try to) characterize the corresponding polytopes. Results of this flavor were obtained for various combinatorial optimization problems, such as the stable set problem [9], graph partitioning problem [10], 2-connected and 2-edge-connected subgraph problems [11,12], k-edge-connected problems [13], Steiner-TSP problem [14].

Since a linear time combinatorial algorithm solves the circuit problem in series–parallel graphs, an obvious question arising is the description of the corresponding polytope. Surprisingly, it does not appear in the literature, and we fill in this gap with Theorem 11.

The main ingredient for the proof of our main theorem is the existence of a compact extended formulation for the circuit polytope on series–parallel graphs. An *extended formulation* of a given polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : Bx + Cy \leq d\}$ whose projection onto the x variables $proj_x(Q) = \{x \in \mathbb{R}^n : \text{ there exists } y \in \mathbb{R}^m \text{ such that } (x, y) \in Q\}$ is P. The *size* of a polyhedron is the number of inequalities needed to describe it. An extended formulation is called *compact* when its size is polynomial. We refer to [15] for further insights on this topic.

The past few years, extended formulations proved to be a powerful tool for polyhedral optimization, and thus received a growing interest in the community. Indeed, describing a polytope directly in its original space is often pretty challenging, and by looking for an extended formulation one has more tools at disposal. As an example, for most combinatorial optimization polytopes in series–parallel graphs, Martin et al. [16] proposed a general technique to derive extended formulations from dynamic programming algorithms, but the corresponding descriptions in the original space remain unknown.

Recently, it has been shown that the perfect matching polytope admits no compact extended formulation [17]. It means, even if an optimization problem is polynomial, there may not exist such a formulation. Here, though we are not able to explicitly construct a compact extended formulation for the circuit polytope on series–parallel graphs, we show that there exists one, see Section 2.1.1. The construction process of this extended formulation relies on a straightforward inductive description of the circuits of series–parallel graphs, combined with a theorem of Balas [18,19]. It allows us to prove by induction that the circuit polytope on series–parallel graphs is completely described by three families of inequalities. We provide examples where exponentially many of these inequalities define facets, see Corollary 19. Thus, the circuit polytope on series–parallel graphs is another example of polytope having exponentially many facet-defining inequalities that admits a compact extended formulation.

A graph is series–parallel if and only if, given any planar drawing of the graph, its dual is series–parallel. The dual of a circuit is a *bond*, that is a cut containing no other nonempty cut. These bonds play an important role e.g. in multiflow problems [20]. By planar duality and the description of the circuit polytope on series–parallel graphs, we get the description of the bond polytope on series–parallel graphs, see Theorem 13.

The paper is organized as follows. In Section 1, we fix graph related notation and definitions, and review some known and new auxiliary results about circuits in series–parallel graphs. Section 2 deals with the circuit polytope on series–parallel graphs. First, we get a polyhedral description of the latter for non trivial 2-connected series–parallel graphs, by providing the existence of a compact extended formulation, and then inductively projecting it. By applying standard techniques, the polyhedral description for general series–parallel graphs follows, which has exponential size in general. In Section 3, using the planar duality, we describe the bond polytope on series–parallel graphs, and then we study facet-defining inequalities, which have counterparts for the circuit polytope as well.

1. Circuits in series–parallel graphs

Throughout, $G = (V, E)$ will denote a connected undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The graph *induced* by a subset W of V is the graph $G[W]$ obtained by removing the vertices of $V \setminus W$, and $\delta_G(W)$ is the set of edges having exactly one extremity in W. Given disjoint $U, W \subset V$, $\delta_G(U, W)$ is the set of edges having one extremity in each of U and W. When it is clear from the context, we will omit the subscript G. Given a set of edges $F \subseteq E$, $V(F)$ denotes the set of vertices incident to any edge of F. We denote by $A\Delta B = (A \cup B) \setminus (A \cap B)$ the symmetric difference of A and B.

A subset F of E is called a *cut* if $F = \delta_G(W)$ for some $W \subseteq V$. If $u \in W$ and $v \in V \setminus W$, the cut *separates* u and v. A cut defined by a singleton is a *star*. A *bond* is a cut containing no other nonempty cut. One can check that a nonempty cut $\delta_G(W)$ is a bond if and only if both $G[W]$ and $G[V \setminus W]$ are connected. In the literature, a bond is sometimes called a *central cut*. A *bridge* is an edge whose removal disconnects the graph, that is a bond of size one. Note that the symmetric difference of bonds is a cut.

A subset of edges is called a *cycle* if it induces a subgraph where every vertex has even degree. A connected cycle with every vertex of degree two is a *circuit*. If e is a circuit, it is called a *loop*. Let $\mathcal{C}(G)$ denote the set of circuits of G. Note that the symmetric difference of circuits is a cycle.

By definition, the emptyset is both a bond and a circuit.

When no removal of a single vertex disconnects a graph, the latter is said *2-connected*. Loops and bridges are called *trivial 2-connected* graphs. The *non trivial 2-connected components* of a graph are the maximal 2-connected subgraphs of the graph, *i.e.*, the components obtained after removing the loops and bridges.

A graph is *series–parallel* if all its non trivial 2-connected components can be built, starting from the circuit of length two C_2, by repeatedly applying the following operations: add a parallel edge to an existing edge; or subdivide an existing edge, that is replace the edge by a path of length two. This construction gives an inductive description of the circuits of such graphs.

Observation 1. *Let $G = (V, E)$ be a non trivial 2-connected series–parallel graph.*

(i) *If G is obtained from a graph H by subdividing an edge $e \in E(H)$ into e, f, then the circuits of G are obtained from those of H as follows:*
  - *C, for $C \in \mathcal{C}(H)$ not containing e,*
  - *$C \cup f$, for $C \in \mathcal{C}(H)$ containing e.*

(ii) *If G is obtained from a graph H by adding a parallel edge f to an edge $e \in E(H)$, then the circuits of G are obtained from those of H as follows:*
  - *C, for $C \in \mathcal{C}(H)$ not containing e,*
  - *C and $C \setminus e \cup f$, for $C \in \mathcal{C}(H)$ containing e,*
  - *$\{e, f\}$.*

A well-known characterization of cuts is that they are the sets of edges intersecting every circuit an even number of times. In series–parallel graphs, we have the following property [20].

Observation 2 ([20]). *In a series–parallel graph, a bond and a circuit intersect in zero or two edges.*

If the graph is also 2-connected, then this property becomes a characterization of circuits, see below. Note that the following does not hold if the series–parallel graph is not 2-connected.

Lemma 3. *In a non trivial 2-connected series–parallel graph, a set of edges is a circuit if and only if it intersects every bond in zero or two edges.*

Proof. We prove the non trivial direction. By contradiction, let G be a minimal counter-example, and let F be a set of edges intersecting every bond in zero or two edges that is not a circuit. First, suppose that G is build from H by adding a parallel edge f to an edge $e \in E(H)$. Necessarily, we have $f \in F$ as otherwise H would be a smaller counter-example. Similarly, $e \in F$. Suppose there exists $g \in F \setminus \{e, f\}$. Since G is planar and 2-connected, so is its dual. Any pair of edges in a 2-connected graph being contained in a circuit, the planar duality between circuits and bonds implies that there exists a bond B of G containing both g and e. Hence, B also contains f, which provides the contradiction $|F \cap B| \geq 3$. Now, assume that G is build from H by subdividing $e \in E(H)$ into $\{f, g\}$. Since G is 2-connected, $\{f, g\}$ is a bond, hence F contains either both f and g or none of them. In both cases, H is clearly a smaller counter-example, a contradiction. $\square$

For an ordering $v_1, \ldots, v_n$ of V such that $\delta(\{v_1, \ldots, v_i\})$ is a bond for all $i = 1, \ldots, n-1$, the partition $\mathcal{S} = \{S_1, \ldots, S_{n-1}\}$ of E defined by $S_\ell = \delta(v_\ell, \{v_{\ell+1}, \ldots, v_n\})$, for $\ell = 1, \ldots, n-1$, is a *star decomposition*. We will denote the initial star $\delta(v_1, \{v_2, \ldots, v_n\})$ by $I_{\mathcal{S}}$. Equivalently, a star decomposition is obtained by partitioning the edgeset by iteratively removing stars of the graph such that, at each step, the vertex to be removed is adjacent to some removed vertex, and the set of remaining vertices induces a connected graph. $I_{\mathcal{S}}$ is the unique element of the star decomposition which is a star of the original graph.

Using induction and the construction of non trivial 2-connected series–parallel graphs, one can see that in these graphs any vertex is the initial vertex of some star decomposition. In particular, such decompositions exist.

Lemma 4. *Given a star decomposition $\mathcal{S}$ of a series–parallel graph G, the following holds:*

(a) *a circuit intersects each member of $\mathcal{S}$ at most twice,*
(b) *a circuit does not intersect two members of $\mathcal{S}$ twice.*

Proof. Let C be a circuit of G and $v_1, \ldots, v_n$ an ordering of V such that $\mathcal{S} = \{S_1, \ldots, S_{n-1}\}$ with $S_\ell = \delta(v_\ell, \{v_{\ell+1}, \ldots, v_n\})$, for $\ell = 1, \ldots, n-1$.

Since every member of $\mathcal{S}$ is contained in a star of G and a circuit goes through each vertex at most once, Lemma 4(a) holds. Let us show Lemma 4(b) by contradiction, and let $i < j$ be such that $|S_i \cap C| = |S_j \cap C| = 2$ and $|S_k \cap C| \leq 1$ for all $k < j$, $k \neq i$. By construction of star decompositions, we have $C \cap S_\ell = \emptyset$, for all $\ell < i$, and $C \setminus (\bigcup_{\ell=1}^{j-1} S_\ell)$ is a path of which S_j contains two edges, hence $|\delta(\{v_1, \ldots, v_j\}) \cap C| = 4$. Since $\delta(\{v_1, \ldots, v_j\})$ is a bond, this contradicts Observation 2. $\square$

Two sequences of edge subsets $\mathcal{M} = (M_0, \ldots, M_k)$ and $\mathcal{N} = (N_1, \ldots, N_k)$ form a *star-cut collection* if $\{M_0, \ldots, M_k\} \subseteq \mathcal{S}$ and $M_0 = I_{\mathcal{S}}$, for some star decomposition $\mathcal{S}$ of G, and $M_i \Delta N_i$ is a cut of G, for $i = 1, \ldots, k$. Note that the elements of $\mathcal{N}$ are not required to be disjoint.

2. Circuit polytope on series–parallel graphs

Given a graph $G = (V, E)$ and $F \subseteq E$, $\chi^F \in \mathbb{R}^E$ denotes the incidence vector of F, that is χ_e^F equals 1 if $e \in F$ and 0 otherwise. Since there is a bijection between edge sets and their incidence vectors, we will often use the same terminology for both. Let $C(G)$ be the convex hull of the incidence vectors of the circuits of G, that is $C(G) = \text{conv}\{\chi^C : C \in \mathcal{C}(G)\}$. In this section, we give an external description of the circuit polytope on series–parallel graphs.

Note that the circuit polytope of the graph is the union of the circuit polytopes of its loops, bridges, and non trivial 2-connected components. Therefore, we start by studying the circuit polytope for this latter case, and then derive the description for general series–parallel graphs.

Throughout, we will use the following theorem of Balas [18,19]. His result holds for any finite union of polyhedra, yet we only state what we need in this paper, the union of two polytopes.

Theorem 5 (*Balas [18,19]*)**.** *Given two polytopes $P_1 = \{x \in \mathbb{R}^n : A^1 x \leq b^1\}$ and $P_2 = \{x \in \mathbb{R}^n : A^2 x \leq b^2\}$, we have $conv\{P_1 \cup P_2\} = proj_x(Q)$, where $Q = \{x = x^1 + x^2, A^1 x^1 \leq (1-\lambda) b^1, A^2 x^2 \leq \lambda b^2, 0 \leq \lambda \leq 1\}$.*

Note that Theorem 5 applied to integral polytopes yields an extended formulation which is also integral. Furthermore, it also implies the following.

Corollary 6. *Given two polytopes P_1 and P_2, there exists an extended formulation of conv$\{P_1 \cup P_2\}$ whose size is two plus the sizes of P_1 and P_2.*

Later on, we shall use this corollary when P_2 is a vertex, in which case we get an extended formulation of conv$\{P_1 \cup P_2\}$ with two more inequalities than the one of P_1.

2.1. 2-connected series–parallel graphs

In this section, we describe the circuit polytope for non trivial 2-connected series–parallel graphs. The main ingredient of our proof is the existence of a compact extended formulation for this polytope, based on Observation 1. Though this extended formulation is not explicit, we use its construction process to prove inductively that the circuit polytope is described by the inequalities given in Theorem 10. Let us mention that there are examples where exponentially many of these inequalities are facet-defining, see Corollary 19.

In this section, $G = (V, E)$ is a non trivial 2-connected series–parallel graph.

2.1.1. Existence of a compact extended formulation

We show the existence of a compact extended formulation by induction on the construction of G. First, note that $C(C_2) = \text{conv}\{(0,0),(1,1)\} = \{x \in \mathbb{R}^2_+ : x_e = x_f, x_e + x_f \le 2\}$, where e and f denote the edges of C_2. Next, let us describe how to get an extended formulation for $C(G)$ when G is obtained from a 2-connected series–parallel graph H by either subdividing an edge or adding a parallel edge.

When G is obtained from H by subdividing an edge $e \in E(H)$ into e, f, the following immediately derives from Observation 1(i).

Observation 7. *Suppose G is obtained from H by subdividing an edge $e \in E(H)$ into e, f. Then, adding a variable x_f to any extended formulation of $C(H)$ and imposing $x_e = x_f$ provides an extended formulation for $C(G)$.*

When G is obtained from H by adding a parallel edge f to $e \in E(H)$, an extended formulation for $C(G)$ can be obtained as follows.

Lemma 8. *Suppose G is obtained from H by adding a parallel edge f to an edge $e \in E(H)$ and let $Q(H)$ be an integral polyhedron which is an extended formulation of $C(H)$. Then,*

(a) *The polytope $S(G)$ obtained by replacing x_e by $x_e + x_f$ in $Q(H)$ and setting $0 \le x_e$ and $0 \le x_f$ is an extended formulation of the convex hull of the incidence vectors of all the circuits of G different from $\chi^{e,f}$.*
(b) *The convex hull of $S(G)$ union $\chi^{e,f}$ is an extended formulation of $C(G)$.*

Proof. (a) Let $R(G)$ denote the convex hull of incidence vectors of all the circuits of G except $\{e, f\}$. By Observation 1(ii), since $proj_x Q(H) = C(H)$, we have $proj_x S(G) \cap \mathbb{Z}^m = R(G) \cap \mathbb{Z}^m$. Since $Q(H)$ is integral, so is $S(G)$, which implies the integrality of $proj_x S(G)$.

(b) By (a), $proj_x S(G)$ is integral, hence so is $\text{conv}\big\{proj_x S(G) \cup \chi^{e,f}\big\}$. Since the projection of the convex hull of a set of points is the convex hull of its projected points, $proj_x\big(\text{conv}\{S(G) \cup \chi^{e,f}\}\big)$ is integral, and we are done. □

Note that the operations involved in Observation 7 and Lemma 8 preserve integrality. By construction of non trivial 2-connected series–parallel graphs, and since $C(C_2)$ is integral, we get an extended formulation for $C(G)$ by repeatedly applying Observation 7 and Lemma 8. Moreover, the extended formulation given by Lemma 8(a) yields two new inequalities, and that applying Corollary 6 in Lemma 8(b) provides an extended formulation with two more inequalities. Thus, if G is obtained from H by adding a parallel edge, then an extended formulation for $C(G)$ has 4 more inequalities than an extended formulation for $C(H)$. Furthermore, if G is obtained from H by subdividing an edge, then an extended formulation for $C(G)$ has the size of an extended formulation for $C(H)$. The following corollary stems from these observations and the fact that $C(C_2)$ is described by 3 inequalities.

Corollary 9. *There exists an extended formulation for $C(G)$ of size $O(|E(G)|)$.*

We mention here that a polytope closely related to the circuit polytope is, given a vertex r, the *r-circuit polytope*, that is the convex hull of the circuits containing r. Indeed, the circuit polytope of a graph can be seen as the union of all its r-circuit polytopes. In series–parallel graphs, the latter have been thoroughly studied by Baïou and Mahjoub in [14] who provide, in particular, their description into the original space. Therefore, an explicit extended formulation for the circuit polytope on series–parallel graphs can be obtained by applying Balas' Theorem [18,19] for the union of polyhedra together with their description. However, since the description of the r-circuit polytope has exponentially many inequalities, this approach yields an exponential-size extended formulation. Moreover, projecting such a formulation to get a description into the original space usually requires tremendous efforts. In contrast, our approach allows to project step by step, which is done in the next section.

2.1.2. Description in the original space

In this section, we show that the inequalities (1)–(3) given below describe the circuit polytope on non trivial 2-connected series–parallel graphs, see Theorem 10. Throughout, for a sequence $\mathcal{M} = (M_0, \dots, M_k)$ of edge sets, $x(\mathcal{M})$ will stand for $\sum_{i=1}^{k} x(M_i)$.

$$x_e \geq 0 \qquad \text{for all } e \in E. \tag{1}$$

$$x_e \leq x(B \setminus e) \qquad \text{for all bonds } B \text{ of } G, \text{ for all } e \in B, \tag{2}$$

$$x(\mathcal{M}) - x(\mathcal{N}) \leq 2 \qquad \text{for all } \mathcal{M}, \mathcal{N} \text{ star-cut collections of } G, \tag{3}$$

Inequalities (1) are called *non-negativity constraints*, (2) are *bond constraints*, and (3) are *star-cut constraints*.

Theorem 10. $C(G) = \{x \in \mathbb{R}^m_+$ satisfying (2) and (3)$\}$.

Proof. Let us first show that (1)–(3) are valid for $C(G)$. Clearly, every incidence vector of a circuit satisfies the non-negativity constraints (1). The validity of bond constraints (2) comes from Observation 2. To show the validity of star-cut constraints (3), let $\mathcal{M}, \mathcal{N}$ be a star-cut collection and C a circuit of G. Since M_0 and $M_i \Delta N_i$ are cuts for $i \in \{1, \dots, k\}$, each of them intersects C an even number of times. Therefore, if C intersects $M_i \in \mathcal{M}$ at most once, then $\chi^C(M_i) - \chi^C(N_i) \leq 0$ if $i \geq 1$ and $\chi^C(M_i) = 0$ if $i = 0$. The validity of $x(\mathcal{M}) - x(\mathcal{N}) \leq 2$ follows since, by Lemma 4, at most one member of $\mathcal{M}$ intersects C twice, the other ones intersecting C at most once.

Let us prove the theorem by induction. The first step of the induction comes from $C(C_2) = \{x \in \mathbb{R}^2_+ :$ satisfying (2) and $x_e + x_f \leq 2\}$ and the fact that $\{\{e, f\}\}, \emptyset$ forms a star-cut collection, where $C_2 = \{e, f\}$. Suppose now that $C(H)$ is given by inequalities (1)–(3) for a non trivial 2-connected series–parallel graph H, and let us show that $C(G)$ is also described by (1)–(3) when G is obtained from H by subdividing an edge or by adding a parallel edge in H.

First, remark that if G is obtained from H by subdividing e into e, f, then $C(G)$ is given by the inequalities of $C(H)$ and $x_e = x_f$. The inequalities of $C(H)$ of type (1)–(3) remain of the same type in G, and $x_e = x_f$ is implied by the two inequalities of type (2) associated with the bond $\{e, f\}$.

Now, let G be obtained from H by adding a parallel edge f to $e \in E(H)$. By the induction hypothesis, we have $C(H) = \{x^H \in \mathbb{R}^{m-1} : A^H x^H \leq b^H\}$ where A^H is given by the non-negativity (1), bond (2), and star-cut (3) constraints for H. Denote by $\bar{A}^H$ and $\bar{x}^H$ the matrix and vector obtained from A^H and x^H by, respectively, removing the column A_e^H corresponding to e and the component x_e^H. The application of Lemma 8(a) introduces a new variable y and provides the following description of $S(G)$:

$$\{(x^H, y) \in \mathbb{R}^{m-1} \times \mathbb{R} : \bar{A}^H \bar{x}^H + A_e^H(x_e^H + y) \leq b^H, 0 \leq x_e^H, 0 \leq y\}.$$

Lemma 8(b) implies that $C(G)$ is the convex hull of the union of $S(G)$ and $\chi^{\{e,f\}}$. Let us apply Theorem 5 to $P_1 = S(G)$ and $P_2 = \{\chi^{\{e,f\}}\}$. The latter being a vertex, we can get rid of x^1 and x^2 to get the following extended formulation of $C(G)$, where $\bar{x}$ denotes the vector x after the removal of x_e and x_f.

$$\{(\bar{x}, x_e, x_f, \lambda) \in \mathbb{R}^{m-2} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \bar{A}^H \bar{x} + A_e^H(x_e + x_f - 2\lambda) \leq (1-\lambda)b^H, \lambda \leq x_e, \lambda \leq x_f, 0 \leq \lambda \leq 1\}.$$

To project it by Fourier–Motzkin's method [21], we only need to consider the inequalities where λ appears, and since A^H is given by (1)–(3) for H, we may write them down explicitly, implicitly using the fact that if e belongs to a cut of G, then so does f, and conversely:

$$\begin{aligned}
&0 &&\leq \lambda && && (4)\\
&-1 &&\leq -\lambda && && (5)\\
&-x_h &&\leq -\lambda && \text{for } h = e, f && (6)\\
&x_\ell - x(B \setminus \ell) &&\leq -2\lambda && \text{for all bonds } B \text{ of } G \text{ containing } e, f \text{ and } \ell \in B \setminus \{e, f\} && (7)\\
&x_e + x_f - x(D \setminus \{e, f\}) &&\leq 2\lambda && \text{for all bonds } D \text{ of } G \text{ containing } e, f && (8)\\
&x(\mathcal{M}) - x(\mathcal{N}) &&\leq 2 - 2(\alpha_e(\mathcal{M}, \mathcal{N}) + 1)\lambda && \text{for all star-cut collections } \mathcal{M}, \mathcal{N} \text{ of } G, \text{ with } \alpha_e(\mathcal{M}, \mathcal{N}) \geq 0, && (9)
\end{aligned}$$

where $\alpha_e(\mathcal{M}, \mathcal{N}) = |\{N \in \mathcal{N} : e \in N\}| - |\{M \in \mathcal{M} : e \in M\}|$.

We now prove that the inequalities obtained by projecting out λ are either contained or implied by the non-negativity constraints (1) and bond constraints (2) and star-cut constraints (3) for G, which implies our theorem. Recall that, to get rid of λ, one has to combine every inequality where λ's coefficient is negative with every inequality where it is positive [21]. Combinations with $0 \leq \lambda$ immediately give rise to inequalities of type (1), (2) or (3) for G. Thus, it remains to combine (8) with every other inequality.

First, remark that adding twice inequality (5) to any inequality (8) leads to an inequality obtained by adding non-negativity constraints and the star-cut constraint $x(M_0) \leq 2$ where M_0 is a star of G containing e, f. Moreover, adding (8) to twice (6) gives $x_h - x(D \setminus h) \leq 0$ for all bonds D containing e, f, and $h \in \{e, f\}$, which are inequalities of type (2).

Adding (8) to (7) gives $x_\ell \leq x(B \setminus \{e, f, \ell\}) + x(D \setminus \{e, f\})$. If D contains ℓ, the latter is a sum of non-negativity constraints (1). Otherwise, $B \Delta D$ is a cut contained in $B \cup D \setminus \{e, f\}$ and thus contains a bond J containing ℓ but not e, f, since a cut is a disjoint union of bonds. Hence, the inequality is the sum of $x_\ell \leq x(J \setminus \ell)$ and non-negativity constraints (1).

For a bond D containing e, f and a star-cut collection $\mathcal{M} = (M_0, \ldots, M_k)$, $\mathcal{N} = (N_1, \ldots, N_k)$ with $\alpha_e(\mathcal{M}, \mathcal{N}) \geq 0$, combining (8) and (9) gives $x(\mathcal{M}) - x(\mathcal{N}) + (\alpha_e(\mathcal{M}, \mathcal{N}) + 1)(x_e + x_f - x(D \setminus \{e, f\})) \leq 2$. If e and f belong to a member of $\mathcal{M}$, then $\alpha_e(\mathcal{M}, \mathcal{N}) + 1 = |\{N \in \mathcal{N} : e \in N\}|$. Moreover, considering separately the elements of $\mathcal{N}$ containing e and f from the other ones, the inequality can be rewritten as:

$$x(\mathcal{M}) - \sum_{N \in \mathcal{N} : e, f \in N} \Big(x(N \setminus \{e, f\}) + x(D \setminus \{e, f\})\Big) - \sum_{N \in \mathcal{N} : e, f \notin N} x(N) \leq 2.$$

Since $x(N\Delta D) \le x(N \setminus \{e,f\}) + x(D \setminus \{e,f\})$ for all $N \in \mathcal{N}$ containing e and f, the above inequality is implied by $x(\mathcal{M}) - x(\mathcal{N}') \le 2$, where $\mathcal{N}' = (N_1', \dots, N_k')$ with N_i' equals $N_i \Delta D$ if $e \in N_i$ and N_i otherwise, for $i = 1, \dots, k$. Moreover, since D and $M_i \Delta N_i$ are cuts, so is $M_i \Delta N_i'$, for $i = 1, \dots, k$, as the symmetric difference of two cuts is a cut. Therefore, $\mathcal{M}, \mathcal{N}'$ is a star-cut collection.

Suppose now that no member of $\mathcal{M}$ contains e and f. Applying the previous argument leads to the inequality $x(\mathcal{M}) - x(\mathcal{N}') + x_e + x_f - x(D \setminus \{e,f\}) \le 2$, where $\mathcal{M}, \mathcal{N}'$ is the star-cut collection defined above. Moreover, there exists M_{k+1} containing e, f such that $\{M_0, \dots, M_k, M_{k+1}\}$ is contained in a star decomposition. Let $\tilde{\mathcal{M}} = (M_0, \dots, M_{k+1})$ and $\tilde{\mathcal{N}} = (N_1', \dots, N_{k+1}')$, where $N_{k+1}' = D \Delta M_{k+1}$. The symmetric difference being associative, $M_{k+1} \Delta N_{k+1}$ equals D, and hence $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ is a star-cut collection of G. Moreover, the associated star-cut constraint (3) implies the inequality obtained by combination of (8) and (9). □

Let us mention a few simple constraints implied by the ones of Theorem 10. First, whenever $\{k, \ell\}$ is a bond, we have $x_k = x_\ell$, which is implied by the inequalities (2) for $\{k, \ell\}$. These will turn out to be the only hyperplanes containing $C(G)$. We postpone the proof of this fact to Section 3.3, see Corollary 18 . For every edge $uv \in E$, $\delta_G(u)$ is a bond since G is 2-connected. Then, we obtain the inequality $x_{uv} \le 1$ by adding x_{uv} to each side of $x_{uv} \le x(\delta_G(u) \setminus uv)$ and by applying $x(\delta_G(u)) \le 2$, which is a special case of (3). We also mention that, given a bond B, the inequality $x(B) \le 2$ is implied by a suitable star-cut constraint.

We will see at the end of Section 3 a family of examples where exponentially many of the inequalities of Theorem 10 define facets.

2.2. General series–parallel graphs

In this section, we provide a polyhedral description of the circuit polytope on general series–parallel graphs, see Theorem 11. The result is obtained by applying a standard union technique and the fact that the circuit polytope of a graph is the convex hull of the union of the circuit polytopes of its 2-connected components.

Theorem 11. *Let G be a series–parallel graph, $G_1, \dots, G_k$ its non trivial 2-connected components, $\mathfrak{L}$ its set of loops, and $\mathfrak{B}$ its set of bridges. Then*

$$C(G) = \left\{ \begin{array}{ll} x \in \mathbb{R}^m_+ \text{ satisfying (2)}, \ x(\mathfrak{B}) = 0 \text{ and} & \\ \displaystyle\sum_{i=1}^{k} (x(\mathcal{M}_i) - x(\mathcal{N}_i)) + 2x(\mathfrak{L}) \le 2, & \begin{array}{l}\text{for all } i = 1, \dots, k, \\ \text{for all star-cut collections } \mathcal{M}_i, \mathcal{N}_i \text{ of } G_i\end{array} \end{array} \right\}.$$

Proof. We prove the result by induction on the number of 2-connected components.

Let us see the first step. Since no bridge b belongs to a circuit, its circuit polytope is described by $\{x_b = 0\}$. Moreover, the circuit polytope of a loop ℓ is described by $\{0 \le 2x_\ell \le 2\}$. Finally, the circuit polytope of a non trivial 2-connected series–parallel component is given by Theorem 10.

Suppose that the result holds for two series–parallel graphs I and $H = \bigcup_{i=1}^{k-1} H_i$, where I is 2-connected and $H_i, i = 1, \dots, k-1$ are the 2-connected components of H, and let $G = I \bigcup (\cup_{i=1}^{k-1} H_i)$ be the graph obtained by identifying a vertex of I and a vertex of H. Then, $C(G) = \mathrm{conv}\{C(H) \cup C(I)\}$. Remark that $C(H) = \{x \in \mathbb{R}_+^{E(H)} : A_H x_H \le b_H\}$ and $C(I) = \{x \in \mathbb{R}_+^{E(I)} : A_I x_I \le b_I\}$ live in different spaces. Extend them to polytopes of $\mathbb{R}^{E(H)} \times \mathbb{R}^{E(I)}$ by setting the new coordinates to zero, and apply Theorem 5 to get $C(G) = proj_x\{x = (x_H, x_I), A_H x_H \le \lambda b_H, A_I x_I \le (1-\lambda) b_I, 0 \le \lambda \le 1\}$.

Let us get rid of λ in the above extended formulation. Combinations with $0 \le \lambda$ or $\lambda \le 1$ immediately give desired inequalities. It remains to combine $a_H x_H \le \lambda b_H$ and $a_I x_I \le (1-\lambda) b_I$ when $b_H \ne 0$ and $b_I \ne 0$.

Since in this case the induction hypothesis says that both inequality are of the new type, we get $b_H = b_I = 2$, thus the resulting inequality is $a_H x_H + a_I x_I \le b_H$, and the theorem follows. □

Since every 2-connected component of a series–parallel graph has a compact extended formulation, using the Theorem of Balas [18,19] for the union of several polytopes, one can extend Corollary 9 as follows.

Corollary 12. *If G is series–parallel, there exists a compact extended formulation of $C(G)$ in size $O(|E|)$.*

3. The bond polytope on series–parallel graphs

In this section, as a consequence of Theorem 11 and the planar duality, we describe the bond polytope on series–parallel graphs. We also provide examples where the latter contains exponentially many facets. Before stating these results, we introduce a few definitions.

3.1. Definitions

Given a series–parallel graph G, we denote its set of bonds by $\mathcal{B}(G)$, and the convex hull of their incidence vectors by $B(G)$.

If G is a non trivial 2-connected series–parallel graph, an *open nested ear decomposition* [22] $\mathcal{E}$ of G is a partition of $E(G)$ into a sequence $E_0, \dots, E_k$ such that E_0 is a circuit of G and the *ears* E_i, $i \in \{1, \dots, k\}$, are paths with the following properties:

- the two endpoints of each ear are distinct and appear in an E_j with $j < i$,
- no interior point of an ear E_i belongs to E_j for all $j < i$,
- if two ears E_i and $E_{i'}$ have both their endpoints in the same E_j, then any two paths contained in E_j, one between the endpoints of E_i and the other between the endpoints of $E_{i'}$, are either disjoint or contained one in another.

We will denote by $C_{\mathcal{E}}$ the unique circuit of an open nested ear decomposition $\mathcal{E}$. Two sequences of edge subsets $\mathcal{F} = (F_0, F_1, \dots, F_k)$ and $\mathcal{P} = (P_1, \dots, P_k)$ form an *ear-cycle collection* if $\{F_0, F_1, \dots, F_k\}$ is contained in an open nested ear decomposition $\mathcal{E}$ of G, $F_0 = C_{\mathcal{E}}$, and $F_i \Delta P_i$ is a cycle for $i = 1, \dots, k$. Note that the elements of $\mathcal{P}$ are not required to be disjoint.

A graph H is a *minor* of G if H arises from G by contractions and deletions of edges and deletions of vertices, where *contracting* an edge uv of E corresponds to deleting e and identifying u and v. A graph is series–parallel if and only if it does not contain a K_4-minor [23], where K_4 denotes the complete graph on four vertices.

3.2. The bond polytope on series–parallel graphs

K_4 being its own dual, a graph is series–parallel if and only if, given any planar drawing of the graph, its dual is series–parallel. It is immediate that the circuits of such a graph are precisely the bonds of its dual, thus the bond polytope of the graph is the circuit polytope of its dual. Then, applying Theorem 11 provides a description of the bond polytope on series–parallel graphs.

Given a circuit C and $e \in C$, $x_e \le x(C \setminus e)$ is a *circuit constraint*, and given an ear-cycle collection, $x(\mathcal{F}) - x(\mathcal{P}) \le 2$ is an *ear-cycle constraint*.

Theorem 13. *Let G be a series–parallel graph, $G_1, \dots, G_k$ its non trivial 2-connected components, $\mathfrak{L}$ its set of loops and $\mathfrak{B}$ its set of bridges.*

$$B(G) = \left\{ \begin{array}{ll} x \in \mathbb{R}^m_+ \textit{ satisfying } x_e \le x(C \setminus e) \textit{ for all circuits } C \textit{ and } e \in C,\ x(\mathfrak{L}) = 0 \textit{ and} & \\ \displaystyle\sum_{i=1}^{k} (x(\mathcal{F}_i) - x(\mathcal{P}_i)) + 2x(\mathfrak{B}) \le 2, \begin{array}{l} \textit{for all } i = 1, \dots, k, \\ \textit{for all ear-cycle collections } \mathcal{F}_i, \mathcal{P}_i \textit{ of } G_i \end{array} & \end{array} \right\}.$$

Proof. Fix a planar drawing of G, and let $\tilde{G}$ be the dual graph of G. The edgesets of G and $\tilde{G}$ are in bijection, and $\tilde{e}$ will denote the edge of $E(\tilde{G})$ corresponding to $e \in E(G)$. As noted above, the bond polytope of G is precisely the circuit polytope of $\tilde{G}$. First, recall that bridges of G are in bijection with loops of $\tilde{G}$, and conversely. Then, by Theorem 11, to get the desired result, we just need to show that the bond polytope on non trivial 2-connected series–parallel graphs is given by non-negativity, circuit and ear-cycle constraints.

Let G be a non trivial 2-connected series–parallel graph. Then, $\tilde{G}$ is also a non trivial 2-connected series–parallel graph. Since, by Theorem 10, $C(\tilde{G})$ is described by (1)–(3), and by the bijection between circuits in G and bonds in $\tilde{G}$, we only have to show that the ear-cycle constraints are valid for $B(G)$ and that a star-cut collection of $\tilde{G}$ is an ear-cycle collection of G.

To see the validity of the constraints, let us show that, given an open nested ear decomposition $\mathcal{E} = \{E_0, \dots, E_k\}$ and a bond B of G,

$$(*) \text{ if } |B \cap E| = 2 \text{ for some } E \in \mathcal{E}, \text{ then } |B \cap F| \le 1 \text{ for all } F \in \mathcal{E} \setminus E.$$

First, note that, by Observation 2 and the fact that an ear is always contained in a circuit, we have $|B \cap E| \le 2$, for all $E \in \mathcal{E}$. Now, suppose that $E_i, E_j \in \mathcal{E}$ both intersect B twice, with $i < j$. Denote by u and v the extremities of E_j and let e be an edge of $E_i \cap B$. The graph induced by the edges of $E_0 \cup \ldots \cup E_i \cup \{uv\}$ is 2-connected so it contains a circuit containing e and uv. Replacing uv by the ear E_j, we get that G contains a circuit C containing e and E_j. Therefore, $|C \cap B| \ge 3$, yet B is a bond, a contradiction to Observation 2. Therefore $(*)$ holds.

Then, with arguments similar to those proving the validity of star-cut constraints for the circuit polytope (see Theorem 10), we get the validity of the ear-cycle constraints by $(*)$ and the fact that a circuit and a cut intersect each other an even number of times.

We now prove by induction on the number of edges of G that a star decomposition of $\tilde{G}$ corresponds to an open nested ear decomposition of G. We will use edge subdivision and parallel addition operations, thus note that these two operations are dual one of each other. As the dual of C_2 is C_2, one can easily check that a star decomposition of C_2 corresponds to an open nested ear decomposition in its dual.

If G is obtained from H by subdividing an edge $e \in E(H)$ into e, f, then $\tilde{G}$ is obtained from $\tilde{H}$ by adding a parallel edge $\tilde{f}$ to $\tilde{e}$. By induction, any star decomposition $\mathcal{S}$ of $\tilde{H}$ corresponds to an ear decomposition $\mathcal{E}_{\mathcal{S}}$ of H. Adding e to the suitable set of $\mathcal{S}$ (which is the first extremity of e appearing in the star decomposition) gives a star decomposition of $\tilde{G}$, which straightforwardly corresponds to the ear decomposition of G obtained from $\mathcal{E}_{\mathcal{S}}$ by replacing e by $\{e, f\}$ in the member of $\mathcal{E}_{\mathcal{S}}$ containing e.

If G is obtained from H by adding a parallel edge f to $e \in E(H)$, then $\tilde{G}$ is obtained from $\tilde{H}$ by subdividing $\tilde{e}$ into $\tilde{e}, \tilde{f}$. Let u be the vertex that is common to $\tilde{e}$ and $\tilde{f}$, and v, w the other ends of $\tilde{e}$ and $\tilde{f}$. Let $\mathcal{S} = \{\delta_{\tilde{G}_1^+}(v_1), \dots, \delta_{\tilde{G}_{n-1}^+}(v_{n-1})\}$ be a star decomposition of $\tilde{G}$. We may suppose, without loss of generality, that u and v or u and w are consecutive in the star decomposition. Indeed, otherwise, $u = v_i$ for some $i \in \{2, \dots, n-1\}$, and since G_i^+ and G_i^- are connected, exactly one of v, w is in G_i^- that is, equals some v_j for $j < i$. In this case, the star decomposition $\mathcal{S}'$ obtained from $\mathcal{S}$ by removing u, and then inserting

u right after v_j, without changing the rest, gives the same partition of E as $\mathcal{S}$. Thus $\mathcal{S}$ and $\mathcal{S}'$ are in bijection with the same partition of the edgesets of G.

Since u and one of v, w appear consecutively in the star decomposition, contracting them gives a star decomposition of $\tilde{H}$. By induction, the latter corresponds to an ear decomposition $\mathcal{E}$ of H. Now, possibly having exchanged the role of e and f because of the contraction, $\mathcal{E} \cup \{f\}$ is an ear decomposition of G, and we are done.

To finish the proof, if suffices to apply Theorem 11 and the fact that the dual of a cut is a cycle. □

It turns out that there are more ear-cycle collections of G than star-cut collections of $\tilde{G}$, and it is unclear which restrictions are to be made in order to get a bijection. As a consequence, if $B(G)$ can be deduced from $C(\tilde{G})$ in a rather simple manner, the converse seems more challenging.

3.3. Facet-defining inequalities

Determining directly which inequalities are facet-defining for the circuit polytope is not that easy. Surprisingly, the bond polytope is much simpler to study polyhedrally. The main reason is that we can safely remove parallel edges, see Observation 14. Thus Theorem 13 is not only a standard planar duality result, but also a tool to prove polyhedral results for the original polytope.

First, we provide the dimension of the bond polytope, see Lemma 15. Then, we characterize which of the non-negativity and circuit constraints are facet-defining, see Lemma 16. Unfortunately, it seems challenging to exhibit the structures for which ear-cycle inequalities define facets. We provide an example where exponentially many of them are facet-defining, see Claim 17.

Seen the structure of the inequalities given by Theorem 13, it is enough to study the facet-defining ones for non trivial 2-connected series–parallel graphs. In this section, let $G = (V, E)$ be such graph.

Observation 14. *The set of bonds of G is unchanged if we remove parallel edges.*

Proof. Whenever two edges of the graph are parallel, every bond contains either both or none. □

By the above observation and the construction of non trivial 2-connected series–parallel graphs, we may assume there are no parallel edges. This is emphasized by the following lemma.

Lemma 15. *The dimension of $B(G)$ is the number of edges of the graph obtained from G by removing every parallel edge.*

Proof. By Observation 14, we may assume that G has no parallel edges. Then, since the emptyset is a bond, that is $0 \in B(G)$, the result is equivalent to the existence of $|E(G)|$ linearly independent bonds of G. Clearly, $\dim B(G) \leq |E(G)|$. We prove the result by induction on $|E(G)|$, noting that $\dim B(C_3) = 3$.

Since G has no parallel edges, G is obtained from a non trivial 2-connected series–parallel graph H by subdividing an edge e into f, g.

If H contains no parallel edges, then the induction hypothesis gives a family of $\dim B(H) = |E(H)| = |E(G)| - 1$ linearly independent bonds of H. Replacing e by f, for each member of $\mathcal{F}$ containing e, and then adding $\{f, g\}$, gives a linearly independent family of $|E(G)|$ bonds of G, and we are done.

Since G did not contain parallel edges, if H does, then these parallel edges are $\{e, h\}$ for some $h \in E(G) \setminus \{f, g\}$. In this case, we have $B(H) \subseteq \{x_e = x_h\}$. By the induction hypothesis, there exists a family $\mathcal{L}$ of $\dim B(H) = |E(H)| - 1 = |E(G)| - 2$ linearly independent bonds of H. We may assume that $e \in B$ for some $B \in \mathcal{L}$. Define $D = B \setminus e \cup f$, we get the family $\mathcal{L} \cup D \cup \{f, g\}$ of bonds of G. Let us prove that they are linearly independent, by showing that the corresponding matrix A has full column rank. Since $D = B \setminus e \cup f$, by basic column operations we get that A has the same rank as the matrix whose columns are composed of χ^e, χ^f and the elements of $\mathcal{L}$. Thus A has full column rank if and only if the matrix obtained from $\mathcal{L}$ by deleting the coordinate corresponding to e has. It is indeed the case because $\mathcal{L}$ is a family of linearly independent circuits of H, and they all satisfy $x_e = x_h$. $\square$

The following lemma characterizes which of the non-negativity and circuit inequalities are facet-defining.

Lemma 16. *The inequality*

1. $x_\ell \geq 0$ *defines a facet of* $B(G)$ *if and only if* ℓ *is not contained in a triangle.*
2. $x_\ell \leq x(C \setminus \ell)$ *defines a facet of* $B(G)$ *if and only if* C *has no chord and* $|C| \geq 3$.

Proof. By Observation 14, we may assume that G has no parallel edges. We prove both results by a maximality argument.

(1) First, suppose that ℓ is contained in a triangle, say $\{\ell, e, f\}$. The two circuit inequalities $x_e - x_f - x_\ell \leq 0$ and $-x_e + x_f - x_\ell \leq 0$ give $x_\ell \geq 0$ so the latter is not facet-defining.

Suppose now that ℓ is not contained in a triangle. Consider the face F defined by $x_\ell \geq 0$ and suppose that it is not a facet, that is, there exists a face F' defined by an inequality $ax \leq b$ of $B(G)$ such that $F \subseteq F'$. Since $\emptyset \in F$, we have $b = 0$. For every edge uv non incident to ℓ, the bonds $\delta(u)$, $\delta(v)$ and $\delta(\{u, v\})$ belong to F, implying that $a(\delta(u)) = a(\delta(v)) = a(\delta(\{u, v\})) = 0$, leading to $a_{uv} = 0$. Finally, for any edge $f = uv$ incident to ℓ at node u, $\delta(v) \in F$. By hypothesis, f is the only edge of $\delta(v)$ incident to ℓ, implying that $a_f = 0$. Thus, $a = \rho\chi_\ell$, for some $\rho < 0$ and F defines a facet.

(2) If $|C| = 2$, then C is two parallel edges e and f, and $B(G) \subseteq \{x_e = x_f\}$. If C has a chord c, let C' and C'' be the two circuits defined by $C' \cup C'' \setminus c = C$, and assume $c \in C'$. Then, the inequality $x_\ell \leq x(C \setminus \ell)$ is obtained applying the circuit inequalities for ℓ and C' and then for c and C'', $x_\ell \leq x(C' \setminus \ell) = x(C' \setminus \{\ell, c\}) + x_c \leq x(C' \setminus \{\ell, c\}) + x(C'' \setminus c) = x(C \setminus \ell)$.

Suppose that C has no chord, $|C| \geq 3$, and $F' = \{x_\ell \leq x(C \setminus \ell)\} \subseteq \{ax \leq b\} = F$, where F is facet-defining. Since $0 \in F'$, we have $b = 0$. Let $uv \in E$ with $u, v \notin V(C)$. Since $\{uv\} = (\delta_G(u) \cup \delta_G(v)) \setminus \delta_G(\{u, v\})$, and $\delta_G(u), \delta_G(v), \delta_G(\{u, v\})$ are bonds, and are contained in F', we have

$$(\#)\ a_{uv} = 0, \text{ for all } uv \in E \text{ such that } u, v \notin V(C).$$

Denote the vertices of C by $\{v_1, \ldots, v_k\}$ where $\ell = v_k v_1$ and $v_i v_{i+1} \in C$ for $i = 1, \ldots, k-1$. Let $u \in V \setminus V(C)$. Note that there are at most two edges between u and $\{v_1, \ldots, v_k\}$. If there is exactly one, say uv_i, then, by $(\#)$ and $\delta_G(u) \in F'$, we have $a_{uv_i} = 0$. If there are two, say uv_i and uv_j, since G is series–parallel, every uv_i-path not containing v_j does not intersect $V(C)$. Therefore, since G is 2-connected, there exists a bond $B = \delta_G(W)$ containing uv_i and ℓ such that $B' = \delta_G(W \cup \{u\})$ is also a bond. Since the edges of $B \Delta B'$ are uv_i, uv_j, and edges not in $\delta_G(C)$, and then by $B, B' \in F'$ and $(\#)$, we get $a_{uv_i} = a_{uv_j}$. By $\delta_G(u) \in F'$, we have $a_{uv_i} + a_{uv_j} = 0$. Therefore, $a_{uv_i} = a_{uv_j} = 0$. Since C had no chord, we proved $a_{uv} = 0$ whenever $uv \notin C$.

To finish the proof, since G is 2-connected, there exists a bond B_i containing ℓ and $v_i v_{i+1}$ for all $i = 1, \ldots, k-1$. By Observation 2, $B_i \cap C = \{\ell, v_i v_{i+1}\}$, thus $B_i \in F'$. Therefore, we have $a_\ell = -a_f$ for all $f \in C \setminus \ell$. Since $ax \leq 0$ is valid for the bond $\delta_G(v_2)$, we have $0 \geq a_{v_1 v_2} + a_{v_2 v_3} = 2a_{v_1 v_2}$, thus we may assume $a_\ell = 1$, and then we get $F' = F$. $\square$

We now provide a family of series–parallel graphs where an exponential number of ear-cycle constraints are facet-defining.

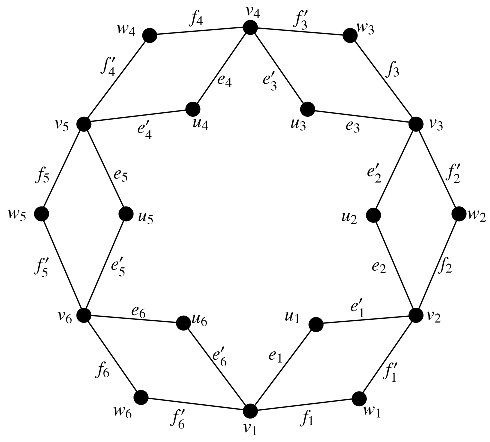

Fig. 1. An example of graph obtained from C_6 by parallel addition and subdivision of all the edges.

Example

The graph G_k we consider is built from the circuit on k edges C_k where a parallel edges is added to every edge and then all edges are subdivided. Fig. 1 shows the construction of such a graph from C_6. Denote by e_i and e_i' (resp. f_i and f_i') the edges obtained by subdividing one parallel edge (resp. the other one). Let u_i (resp. w_i) be the vertex incident to e_i and e_i' (resp. f_i and f_i') for all $i = 1, \dots, k$.

Claim 17. *$x(C) \leq 2$ is facet-defining for $B(G_k)$ if C is a circuit of $2k$ edges.*

Proof. Without loss of generality, suppose that $C = \{e_i, e_i' : i = 1, \dots, k\}$. Let F' be the face induced by the inequality and suppose that $F' \subseteq F$ where F is a facet induced by the constraint $ax \leq b$. Since $\{e_i, e_i'\} \in F'$, we have $a_{e_i} + a_{e_i'} = b$ for $i = 1, \dots, n$. Moreover, $\{e_i, f_i, f_j, e_j\}$ and $\{e_i', f_i, f_j, e_j\}$ belong to F', for all $j \neq i$, from which we get $a_{e_i} = a_{e_i'}$. Combining these two remarks give $a_{e_i} = a_{e_i'} = b/2$, for $i = 1, \dots, n$. Now, since both $\{e_i, f_i, f_j, e_j\}$ and $\{e_i, f_i', f_j, e_j\}$ belong to F', we have $a_{f_i} = a_{f_i'} = 0$, for $i = 1, \dots, n$. The emptyset being a bond, we have $b \geq 0$. In fact, $b > 0$ because otherwise $(a, b) = 0$. Therefore, without loss of generality, we may assume that $b = 2$. Then, we get $F = F'$, and we are done. $\square$

If we set $E_i = \{e_i, e_i'\}$ and $F_i = \{f_i, f_i'\}$ for all $i = 1, \dots, k$, one can also prove that the inequalities $x(E_j \cup F_j) + \sum_{i \neq j}(x(M_i) - x(E_i \cup F_i \setminus M_i)) \leq 2$ for all $j \in \{1, \dots, k\}$, where $M_i \in \{E_i, F_i\}$ for all $i = 1, \dots, k$ are facet-defining. In fact, together with the inequalities of Claim 17 and Lemma 16, this gives all the facet-defining inequalities for the example. However, other examples show that ear-cycle constraints are not always that nicely structured.

Let us mention some dual consequences of the results of Section 3.3 for the circuit polytope on series–parallel graphs.

First, we get its dimension by planar duality and Lemma 15.

Corollary 18. *Let F be a minimal set of edges intersecting every size two bond of $G = (V, E)$. Then, the dimension of $C(G)$ is $|E \setminus F|$.*

Moreover, Claim 17 implies the following.

Corollary 19. *There are examples for which exponentially many of the star-cut constraints (3) define facets of the circuit polytope on series–parallel graphs.*

Thus, the circuit polytope on series–parallel graphs is another example of polytope having exponentially many facet-defining inequalities that admits a compact extended formulation.

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