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Box-Total Dual Integrality and Edge-Connectivity

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Abstract Given a graph $G = (V, E)$ and an integer $k \geq 1$, the graph $H = (V, F)$, where F is a family of elements (with repetitions allowed) of E, is a k-edge-connected spanning subgraph of G if H cannot be disconnected by deleting any $k-1$ elements of F. The convex hull of incidence vectors of the k-edge-connected subgraphs of a graph G forms the k-edge-connected subgraph polyhedron of G. We prove that this polyhedron is box-totally dual integral if and only if G is series-parallel. In this case, we also provide an integer box-totally dual integral system describing this polyhedron.

Keywords Box-total dual integrality · k-edge connected subgraph · Polyhedron · Series-parallel graph

1 Introduction

Totally dual integral systems, introduced in the late 70's, are strongly connected to min-max relations in combinatorial optimization [34]. A rational system of linear inequalities $Ax \geq b$ is *totally dual integral (TDI)* if the maximization problem in the linear programming duality

$$\min\{c^\top x : Ax \geq b\} = \max\{b^\top y : A^\top y = c, y \geq \mathbf{0}\}$$

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admits an integer optimal solution for each integer vector c such that the optimum is finite. Every rational polyhedron can be described by a TDI system [28]. For instance, the polyhedron $\{x : Ax \geq b\}$ can be described by TDI systems of the form $\frac{1}{q}Ax \geq \frac{1}{q}b$ for certain positive q. However, a polyhedron is integer if and only if it can be described by a TDI system with only integer coefficients [23, 28]. Integer TDI systems yield min-max results that may have combinatorial interpretation.

A stronger property is box-total dual integrality: a system $Ax \geq b$ is *box-totally dual integral (box-TDI)* if $Ax \geq b, \ell \leq x \leq u$ is TDI for all rational vectors ℓ and u (possibly with infinite components). General properties of such systems can be found in Cook [12] and Chapter 22.4 of Schrijver [34]. Note that, although every rational polyhedron can be described by a TDI system, not every polyhedron can be described by a box-TDI system. A polyhedron which can be described by a box-TDI system is called a *box-TDI polyhedron*. As proved by Cook [12], every TDI system describing such a polyhedron is actually box-TDI.

Recently, several new box-TDI systems have been exhibited. Chen, Ding, and Zang [6] characterized box-Mengerian matroid ports. Ding, Tan, and Zang [18] characterized the graphs for which the Edmonds' system defining the matching polytope [21] is box-TDI. Ding, Zang, and Zhao [19] exhibited new subclasses of box-perfect graphs. Cornaz, Grappe, and Lacroix [14] provided several box-TDI systems in series-parallel graphs. Barbato, Grappe, Lacroix, Lancini, and Wolfler Calvo [3] gave the minimal box-TDI system with integer coefficients for the flow cone for series-parallel graphs. For these graphs, Chen, Ding, and Zang [7] provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron.

In this paper, we are interested in integrality properties of systems related to k-edge-connected spanning subgraphs. A *k-edge-connected spanning subgraph* of a graph $G = (V, E)$ is a graph $H = (V, F)$, with F being a collection of elements of E where each element can appear several times, that remains connected after the removal of any $k - 1$ edges.

These objects model a kind of failure resistance of telecommunication networks. More precisely, they represent networks which remain connected when $k - 1$ links fail. The underlying network design problem is the *k-edge-connected spanning subgraph problem* (*k-ECSSP*): given a graph G and positive edge costs, find a k-edge-connected spanning subgraph of G of minimum cost. Special cases of this problem are related to classical combinatorial optimization problems. The 2-ECSSP is a well-studied relaxation of the traveling salesman problem [24] and the 1-ECSSP is nothing but the well-known minimum spanning tree problem. While this latter is polynomial-time solvable, the k-ECSSP is **NP**-hard for every fixed $k \geq 2$ [27].

Different algorithms have been devised in order to deal with the k-ECSSP, such as branch-and-cut procedures [4][15], approximation algorithms [8][26], cutting plane algorithms [30], and heuristics [11]. In [36], Winter introduced a linear-time algorithm solving the 2-ECSSP on series-parallel graphs. Most of these algorithms rely on polyhedral considerations.

Given a graph $G = (V, E)$, the convex hull of incidence vectors of all the families of E inducing a k-edge-connected spanning subgraph of G forms a polyhedron, hereafter called the *k-edge-connected spanning subgraph polyhedron* of G and denoted by $P_k(G)$. Cornuéjols, Fonlupt, and Naddef [16] gave a system describing $P_2(G)$ when G is series-parallel. Vandenbussche and Nemhauser [35] characterized in terms of forbidden minors the graphs for which this system describes $P_2(G)$. Chopra [10] described $P_k(G)$ for outerplanar graphs when k is odd. Didi Biha and Mahjoub [17] extended these results to series-parallel graphs for all $k \geq 2$. By a result of Baïou, Barahona, and Mahjoub [1], the inequalities in these descriptions can be separated in polynomial time, which implies that the k-ECSSP is solvable in polynomial time for series-parallel graphs.

When studying k-edge-connected spanning subgraphs of a graph G, we can add the constraint that each edge of G can be taken at most once. We denote the corresponding polyhedron by $Q_k(G)$. Barahona and Mahjoub [2] described $Q_2(G)$ for Halin graphs. Further polyhedral results for the case $k = 2$ have been obtained by Boyd and Hao [5] and Mahjoub [32][33]. Grötschel and Monma [29] described several classes of facets of $Q_k(G)$. Moreover, Fonlupt and Mahjoub [25] extensively studied the extremal points of $Q_k(G)$ and characterized the class of graphs for which this polytope is described by cut inequalities and $\mathbf{0} \leq x \leq \mathbf{1}$.

The polyhedron $P_1(G)$ is known to be box-TDI for all graphs [31]. For series-parallel graphs, the system given in [16] describing $P_2(G)$ is not TDI. Chen, Ding, and Zang [7] showed that dividing it by 2 yields a TDI system for such graphs. Actually, they proved that this system is box-TDI if and only if the graph is series-parallel.

Contributions. Our starting point is the result of Chen, Ding, and Zang [7]. First, their result implies that $P_2(G)$ is a box-TDI polyhedron for series-parallel graphs. However, this leaves open the question of the box-TDIness of $P_2(G)$ for non series-parallel graphs. More generally, for which integers k and graphs G is $P_k(G)$ a box-TDI polyhedron?

We answer this question by proving that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel. Note that this work is one of the first ones that proves the box-TDIness of a polyhedron without giving a box-TDI system describing it. Instead, our proof is based on the recent matricial characterization of box-TDI polyhedra given by Chervet, Grappe, and Robert [9].

By [34, Theorem 22.6], there exists a TDI system with integer coefficients describing $P_k(G)$. For series-parallel graphs, the system provided by Chen, Ding, and Zang [7] has noninteger coefficients. Moreover, the system given by Didi Biha and Mahjoub [17] describing $P_k(G)$ when k is even is not TDI. When $k \geq 2$ and G is series-parallel, which combinatorial objects yield an integer TDI system describing $P_k(G)$?

We answer this question by exhibiting integer TDI systems based on multicuts. When k is even, we use multicuts to provide an integer TDI system for $P_k(G)$ when G is series-parallel. Our proof relies on the standard constructive characterization of series-parallel graphs. When k is odd, we prove that the description of $P_k(G)$ given by Didi Biha and Mahjoub [17] based on multicuts is TDI if and only if the graph is series-parallel. For this case, our proof relies on new properties of the set of degree 2 vertices in simple series-parallel graphs stated in Lemma 2.3.

The box-totally dual integral characterization of $P_k(G)$ implies that these systems are actually box-TDI if and only if G is series-parallel. By definition of box-TDIness, adding $x \leq \mathbf{1}$ to these systems yields box-TDI systems for $Q_k(G)$ for series-parallel graphs.

Outline. In Section 2, we give the definitions and preliminary results used throughout the paper. In Section 3, we prove that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel. In Section 4, we provide a TDI system with integer coefficients describing $P_k(G)$ when G is series-parallel and $k \geq 2$ is even. In Section 5, we show the TDIness of the system given by Didi Biha and Mahjoub [17] that describes $P_k(G)$ for G series-parallel and $k \geq 3$ odd.

2 Definitions and Preliminary Results

This section is devoted to the definitions, notation, and preliminary results used throughout the paper.

2.1 Graphs and Combinatorial Objects

Given a set E, a *family* of E is a collection of elements of E where each element can appear multiple times. The incidence vector of a family F of E is the vector χ^F of $\mathbb{Z}_+^E$ such that e's coordinate is the multiplicity of e in F for all e in E. Since there is a bijection between families and their incidence vectors, we will often use the same terminology for both.

Given a graph $G = (V, E)$ and the incidence vector $z \in \mathbb{Z}_+^E$ of a family F of E, $G(z)$ denotes the graph (V, F).

Let $G = (V, E)$ be a loopless undirected graph. Two edges of G are *parallel* if they share the same endpoints, and G is *simple* if it does not have parallel edges. A graph is 2-*connected* if it cannot be disconnected by removing a vertex. The graph obtained from two disjoint graphs by identifying two vertices, one of each graph, is called a *1-sum*. A 2-connected graph is *trivial* if it is composed of a single edge. We denote by K_n the complete graph on n vertices, that is the simple graph with n vertices and one edge between each pair of vertices. Given an edge e of G, we denote by $G \setminus e$ (respectively G/e) the graph obtained by removing (respectively contracting) the edge e, where *contracting* an edge uv consists in removing it and identifying u and v. Similarly, we denote by $G \setminus v$ the graph obtained form G by removing the vertex v, and by $G[W]$ the *graph induced by* W, that is, the graph obtained by removing all vertices not in the vertex subset W. Given a vector $x \in \mathbb{R}^E$ and a subgraph H of G, we denote by $x_{|H}$ the vector obtained by restricting x to the components associated with the edges of H.

A subset of edges of G is called a *circuit* if it induces a connected graph in which every vertex has degree 2. Given a subset U of V, the *cut* $\delta(U)$ is the set of edges having exactly one endpoint in U. A *bond* is a minimal nonempty cut. Given a partition $\{V_1, \ldots, V_n\}$ of V, the set of edges having endpoints in two distinct V_i's is called a *multicut* and is denoted by $\delta(V_1, \ldots, V_n)$. We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G. For every multicut M, there exists a unique partition $\{V_1, \ldots, V_{d_M}\}$ of vertices of V such that $M = \delta(V_1, \ldots, V_{d_M})$, and $G[V_i]$ is connected for all $i = 1, \ldots, d_M$. We say that d_M is the *order* of M and $V_1, \ldots, V_{d_M}$ are the *classes* of M. Multicuts are characterized in terms of circuits, as stated in the following.

Lemma 2.1 ([13]) *A set of edges M is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G.*

We denote the symmetric difference of two sets S and T by $S \triangle T$. It is well-known that the symmetric difference of two cuts is a cut. Moreover, the following result holds.

Observation 2.2 *Let G be a graph, v be a degree 2 vertex of G, and M be a multicut such that $|M \cap \delta(v)| = 1$. Then, $M \cup \delta(v)$ and $M \triangle \delta(v)$ are multicuts. Moreover, $d_{M \cup \delta(v)} = d_M + 1$, and $d_{M \triangle \delta(v)} = d_M$.*

A graph is *series-parallel* if its nontrivial 2-connected components can be constructed from a circuit of length 2 by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. Equivalently, series-parallel graphs are those having no K_4-minor [20].

By construction, simple nontrivial 2-connected series-parallel graphs have at least one degree 2 vertex. Moreover, these vertices satisfy the following.

Lemma 2.3 *For a simple nontrivial 2-connected series-parallel graph, at least one of the following holds:*

(i) two degree 2 vertices are adjacent,
(ii) a degree 2 vertex belongs to a circuit of length 3,
(iii) two degree 2 vertices belong to the same circuit of length 4.

Proof We proceed by induction, the base case is K_3 for which (i) holds.

Let G be a simple 2-connected series-parallel graph. Since G is simple, it can be built from a series-parallel graph H by subdividing an edge e into a path f, g. Let v be the degree 2 vertex added with this operation. By the induction hypothesis, either H is not simple, or one among (i), (ii), and (iii) holds for H. Hence, there are four cases.

Case 1: H is not simple. Since G is simple, e is parallel to exactly one edge h. Hence, f, g, h is a circuit of G length 3 containing v, thus (ii) holds for G.
Case 2: (i) holds for H. Then, it holds for G.
Case 3: (ii) holds for H. Let C be a circuit of H of length 3 containing a degree 2 vertex, say w. If $e \notin C$, then (ii) holds for G. Otherwise, by subdividing e, we obtain a circuit of length 4 containing v and w, and hence (iii) holds for G.
Case 4: (iii) holds for H. Let C be a circuit of H of length 4 containing two degree 2 vertices. If $e \notin C$, then (iii) holds for G. Otherwise, by subdividing e, we obtain a circuit of length 5 containing three degree 2 vertices. Then, at least two of them are adjacent, and so (i) holds for G. □

2.2 Box-Total Dual Integrality

Let $A \in \mathbb{R}^{m \times n}$ be a full-row rank matrix. This matrix is *equimodular* if all its $m \times m$ non-zero determinants have the same absolute value. The matrix A is *face-defining for* a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if $\text{aff}(F) = \{x \in \mathbb{R}^n : Ax = b\}$ for some $b \in \mathbb{R}^m$, where $\text{aff}(F)$ denotes the affine hull of F. Such matrices are the *face-defining matrices of P*.

Theorem 2.4 ([9, Theorem 1.4]) *Let P be a polyhedron. Then, the following statements are equivalent:*

(i) P is box-TDI.

(ii) *Every face-defining matrix of P is equimodular.*
(iii) *Each face of P has an equimodular face-defining matrix.*

In Theorem 2.4, the equivalence of conditions (*ii*) and (*iii*) follows from the following observation.

Observation 2.5 ([9, Observation 4.10]) *Let F be a face of a polyhedron. If a face-defining matrix for F is equimodular, then so are all the face-defining matrices for F.*

We will also use the following.

Observation 2.6 *Let $A \in \mathbb{R}^{I \times J}$ be a full row rank matrix and $j \in J$. If A is equimodular, then so are following two matrices:*

(i) $\begin{bmatrix} A \\ \pm\chi^j \end{bmatrix}$ *if it is full row-rank,*
(ii) $\begin{bmatrix} A & \mathbf{0} \\ \pm\chi^j & \pm 1 \end{bmatrix}$.

Observation 2.7 ([9, Observation 4.11]) *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $F = \{x \in P : Bx = b\}$ be a face of P. If B has full-row rank and $n - \dim(F)$ rows, then B is face-defining for F.*

2.3 The k-Edge-Connected Spanning Subgraph Polyhedron

Note that $P_k(G)$ is the dominant of the convex hull of incidence vectors of all the families of E containing at most k copies of each edge and inducing a k-edge-connected spanning subgraph of G. Since the dominant of a polyhedron is a polyhedron, $P_k(G)$ is a full-dimensional polyhedron even though it is the convex hull of an infinite number of points.

From now on, we assume that $k \geq 2$. Didi Biha and Mahjoub [17] gave a complete description of $P_k(G)$ for all k, when G is series-parallel.

Theorem 2.8 ([17]) *Let G be a series-parallel graph and h be a positive integer. Then, $P_{2h}(G)$ is described by:*

$$(1) \begin{cases} x(D) \geq 2h \quad \text{for all cuts } D \text{ of } G, & \text{(1a)} \\ x \geq \mathbf{0}, & \text{(1b)} \end{cases}$$

and $P_{2h+1}(G)$ is described by:

$$(2) \begin{cases} x(M) \geq (h+1)d_M - 1 \quad \text{for all multicuts } M \text{ of } G, & \text{(2a)} \\ x \geq \mathbf{0}. & \text{(2b)} \end{cases}$$

Since the incidence vector of a multicut $\delta(V_1, \ldots, V_\ell)$ of order ℓ is the half-sum of the incidence vectors of the bonds $\delta(V_1), \ldots, \delta(V_\ell)$, we can deduce another description of $P_{2h}(G)$.

Corollary 2.9 *Let G be a series-parallel graph and h be a positive integer. Then, $P_{2h}(G)$ is described by:*

$$(3)\begin{cases} x(M) \geq hd_M \quad \text{for all multicuts } M \text{ of } G, & \text{(3a)} \\ x \geq \mathbf{0}. & \text{(3b)} \end{cases}$$

We call constraints (2a) and (3a) *partition constraints.* A multicut M is *tight for a point* of $P_k(G)$ if this point satisfies with equality the partition constraint (2a) (respectively (3a)) associated with M when k is odd (respectively even). Moreover, M is *tight for a face F* of $P_k(G)$ if it is tight for all the points of F.

The following results give some insights on the structure of tight multicuts.

Theorem 2.10 ([17, Theorem 2.3 and Lemma 3.1]) *Let x be a point of $P_{2h+1}(G)$, and let $M = \delta(V_1, \ldots, V_{d_M})$ be a multicut tight for x. Then, the following hold:*

(i) if $d_M \geq 3$, then $x\left(\delta(V_i) \cap \delta(V_j)\right) \leq h+1$ for all $i \neq j \in \{1, \ldots, d_M\}$.
(ii) $G \setminus V_i$ is connected for all $i = 1, \ldots, d_M$.

Lemma 2.11 *Let v be a degree 2 vertex of G and M be a multicut of G strictly containing $\delta(v) = \{uv, vw\}$. If M is tight for a point of $P_k(G)$ with $k \geq 2$, then both $M \setminus uv$ and $M \setminus vw$ are multicuts of G of order $d_M - 1$.*

Proof It suffices to show that u and w belong to different classes of $M = \delta(v, V_2, \ldots, V_{d_M})$. Suppose that $u, w \in V_2$. Then M is the union of the two multicuts $\delta(v)$ and $M' = \delta(v \cup V_2, \ldots, V_{d_M})$. Since $d_{\delta(v)} + d_{M'} = d_M + 1$, the sum of the partition inequalities associated with $\delta(v)$ and M' implies that the partition inequality associated with M is tight for no point of $P_k(G)$ for every $k \geq 2$. ☐

Chopra [10] gave sufficient conditions for an inequality to be facet-defining for $P_k(G)$. The following proposition is a direct consequence of Theorems 2.4 and 2.6 of [10].

Lemma 2.12 *Let G be a connected graph having a K_4-minor. Then, there exist two disjoint nonempty subsets of edges of G, E' and E'', and a rational b such that*

$$x(E') + 2x(E'') \geq b, \tag{4}$$

is a facet-defining inequality of $P_{2h+1}(G)$.

Chen, Ding, and Zang [7] provided a box-TDI system for $P_2(G)$ for series-parallel graphs.

Theorem 2.13 ([7, Theorem 1.1]) *The system:*

$$\begin{cases} \frac{1}{2}x(D) \geq 1 \quad \text{for all cuts } D \text{ of } G, \\ x \geq \mathbf{0} \end{cases} \tag{5}$$

is box-TDI if and only if G is a series-parallel graph.

This result proves that the polyhedron $P_2(G)$ is box-TDI for all series-parallel graphs, and gives a TDI system describing this polyhedron in this case. However, Theorem 2.13 is not sufficient to state that $P_2(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

3 Box-TDIness of $P_k(G)$

In this section we show that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron for a connected graph G if and only if G is series-parallel. Since $P_k(G) = \emptyset$ when G is not connected, we assume from now on that G is connected.

When $k \geq 2$, $P_k(G)$ is not always box-TDI, as stated in Lemma 3.1. Indeed, by Theorem 2.4, if a polyhedron has a nonequimodular face-defining matrix, then it is not box-TDI. The proof of Lemma 3.1 exhibits such a matrix when G has a K_4-minor. This follows from the existence of a particular facet-defining inequality when k is odd, as shown by Chopra [10]. When k is even, we build a nonequimodular face-defining matrix based on the structure of cuts in a K_4-minor.

Lemma 3.1 *For $k \geq 2$, if $G = (V, E)$ has a K_4-minor, then $P_k(G)$ is not box-TDI.*

Proof When $k = 2h + 1$ is odd, Lemma 2.12 shows that there exists a facet-defining inequality that is described by a nonequimodular matrix as $P_k(G)$ is full-dimensional. Thus, $P_k(G)$ is not box-TDI by Statement (ii) of Theorem 2.4.

We now prove the case when k is even. Since G has a K_4-minor, there exists a partition $\{V_1, \dots, V_4\}$ of V such that $G[V_i]$ is connected and $\delta(V_i, V_j) \neq \emptyset$ for all $i < j \in \{1, \dots, 4\}$. We now prove that the matrix A whose three rows are $\chi^{\delta(V_i)}$ for $i = 1, 2, 3$ is a face-defining matrix of $P_k(G)$ which is not equimodular. This will end the proof by Statement (ii) of Theorem 2.4.

Let e_{ij} be an edge in $\delta(V_i, V_j)$ for all $i < j \in \{1, \dots, 4\}$. The submatrix of A formed by the columns associated with edges e_{ij} is the following:

$$\begin{array}{c c}
 & \begin{array}{cccccc} e_{12} & e_{13} & e_{23} & e_{14} & e_{24} & e_{34} \end{array} \\
\begin{array}{c} \chi^{\delta(V_1)} \\ \chi^{\delta(V_2)} \\ \chi^{\delta(V_3)} \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}$$

The matrix A is not equimodular as the first three columns form a matrix of determinant -2 whereas the last three ones give a matrix of determinant 1.

By Observation 2.7, to show that A is face-defining, it is enough to exhibit $|E| - 2$ affinely independent points of $P_k(G)$ satisfying $x(\delta(V_i)) = k$ for $i = 1, 2, 3$.

Let $D_1 = \{e_{12}, e_{14}, e_{23}, e_{34}\}$, $D_2 = \{e_{12}, e_{13}, e_{24}, e_{34}\}$, $D_3 = \{e_{13}, e_{14}, e_{23}, e_{24}\}$ and $D_4 = \{e_{14}, e_{24}, e_{34}\}$. First, we define the points $S_j = \sum_{i=1}^{4} k\chi^{E[V_i]} + \frac{k}{2}\chi^{D_j}$, for $j = 1, 2, 3$, and $S_4 = \sum_{i=1}^{4} k\chi^{E[V_i]} + k\chi^{D_4}$. Note that they are affinely independent.

Now, for each edge $e \notin \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, we construct the point S_e as follows. When $e \in E[V_i]$ for some $i = 1, \dots, 4$, we define $S_e = S_4 + \chi^e$. Adding the point S_e maintains affine independence as S_e is the only point not satisfying $x_e = k$. When $e \in \delta(V_i, V_j)$ for some i, j, we define $S_e = S_\ell - \chi^{e_{ij}} + \chi^e$, where S_ℓ is S_1 if $e \in \delta(V_1, V_4) \cup \delta(V_2, V_3)$ and S_2 otherwise. Affine independence comes because S_e is the only point involving e.

In total, we built $4 + |E| - 6 = |E| - 2$ affinely independent points. $\square$

The following theorem characterizes the class of graphs for which $P_k(G)$ is box-TDI. The case k even is obtained using the box-TDIness for $k = 2$ and the fact that integer dilations maintain box-TDIness. For the case k odd, on the contrary to what is generally done, the proof does not exhibit a box-TDI system describing $P_k(G)$. For this case, the proof is by induction on the number of edges of G. We prove that series-parallel operations preserve the box-TDIness of the polyhedron. The most technical part of the proof is the subdivision of an edge uw into two edges uv and vw. We proceed by contradiction: by Theorem 2.4, we suppose that there exists a face F of $P_k(G)$ defined by a nonequimodular matrix. We study the structure of the inequalities corresponding to this matrix. In particular, we show that they are all associated with multicuts, and that these multicuts contain either both uv and vw, or none of them—see Claims 3.1, 3.2, and 3.3. These last results allow us to build a nonequimodular face-defining matrix for the smaller graph, which contradicts the induction hypothesis.

Theorem 3.2 *For $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.*

Proof Necessity follows from Lemma 3.1. Let us now prove sufficiency. When $k = 2$, the box-TDIness of System (5) has been shown by Chen, Ding, and Zang [7]. This implies box-TDIness for all even k: multiplying the right-hand side of a box-TDI system by a positive rational preserves its box-TDIness [34, Section 22.5]. The system obtained by multiplying by $\frac{k}{2}$ the right-hand side of System (5) describes $P_k(G)$ when k is even. Hence, the latter is a box-TDI polyhedron.

The rest of the proof is devoted to the case where $k = 2h + 1$ for some $h \geq 1$. To this end, we prove that for every face of $P_{2h+1}(G)$ there exists an equimodular face-defining matrix. The characterization of box-TDIness given in Theorem 2.4 concludes. We proceed by induction on the number of edges of G.

If G is trivial, then $P_{2h+1}(G) = \{x \in \mathbb{R}_+ : x \geq 2h + 1\}$ is box-TDI. If G is the circuit $\{e, f\}$, then $P_{2h+1}(G) = \{x_e, x_f \in \mathbb{R}_+ : x_e + x_f \geq 2h + 1\}$ is also box-TDI.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G' = (W', E')$ and $G'' = (W'', E'')$. By induction, there exist two box-TDI systems $A'y \geq b'$ and $A''z \geq b''$ describing respectively $P_{2h+1}(G')$ and $P_{2h+1}(G'')$. If v is the vertex of G obtained by the identification, $G \setminus v$ is not connected, hence, by Statement (ii) of Theorem 2.10, a multicut M of G is tight for a face of $P_{2h+1}(G)$ only if $M \subseteq E'$ or $M \subseteq E''$. It follows that for every face F of $P_{2h+1}(G)$ there exist faces F' and F'' of $P_{2h+1}(G')$ and $P_{2h+1}(G'')$ respectively, such that $F = F' \times F''$. Then $P_{2h+1}(G) = \{(y, z) \in \mathbb{R}_+^{E'} \times \mathbb{R}_+^{E''} : A'y \geq b', A''z \geq b''\}$ and so it is box-TDI.

(Parallelization) Let $G = (V, E)$ be obtained from a series-parallel graph G' by adding an edge g parallel to an edge f of G' and suppose that $P_{2h+1}(G')$ is box-TDI. Let $A'x \geq b$ be a box-TDI system describing $P_{2h+1}(G')$. Note that $P_{2h+1}(G)$ is described by $Ax \geq b, x_f \geq 0, x_g \geq 0$, where A is the matrix obtained by duplicating f's column. By Theorem 22.10 of [34], the system $Ax \geq b$ is box-TDI, hence so is $Ax \geq b, x_f \geq 0, x_g \geq 0$. Thus, $P_{2h+1}(G)$ is a box-TDI polyhedron.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw. By contradiction, suppose there exists a nonempty face $F = \{x \in P_{2h+1}(G) : A_F x = b_F\}$ such that A_F is a face-defining matrix for F which is not equimodular. Take such a face with maximum dimension. Then, every submatrix of A_F which is face-defining for a face of $P_{2h+1}(G)$ is equimodular. We may assume that A_F is defined by

the partition constraints (2a) associated with the set of multicuts $\mathcal{M}_F$ and the nonnegativity constraints associated with the set of edges $\mathcal{E}_F$.

Claim 3.1 $\mathcal{E}_F = \emptyset$.

Proof Suppose there exists an edge $e \in \mathcal{E}_F$. Let $H = G \setminus e$ and let $A_{F_H}x = b_{F_H}$ be the system obtained from $A_F x = b_F$ by removing the column and the nonnegativity constraint associated with e. Since the matrix A_F is of full row rank, so is A_{F_H}. Since $e \in \mathcal{E}_F$, for all multicuts M tight for F not containing e, $M \cup e$ is not a multicut. Hence $M \setminus e$ is a multicut of H of order d_M, for all M in $\mathcal{M}_F$. Hence, the set $F_H = \{x \in P_{2h+1}(H) : A_{F_H}x = b_{F_H}\}$ is a face of $P_{2h+1}(H)$. Moreover, deleting e's coordinate of $\mathrm{aff}(F)$ gives $\mathrm{aff}(F_H)$ so A_{F_H} is face-defining for F_H. By the induction hypothesis, A_{F_H} is equimodular. Since maximal invertible square submatrices of A_F are in bijection with those of A_{F_H} and have the same determinant in absolute value, A_F is equimodular, a contradiction. □

Claim 3.2 *For $e \in \{uv, vw\}$, at least one multicut of $\mathcal{M}_F$ different from $\delta(v)$ contains e.*

Proof By contradiction, suppose for instance that uv belongs to no multicut of $\mathcal{M}_F$ different from $\delta(v)$.

First, suppose that $\delta(v)$ does not belong to $\mathcal{M}_F$. Then, the column of A_F associated with uv is zero. Let A'_F be the matrix obtained from A_F by removing this column. Every multicut of G not containing uv is a multicut of G' (relabelling vw by uw), so the rows of A'_F are associated with multicuts of G'. Thus, $F' = \{x \in P_k(G') : A'_F x = b_F\}$ is a face of $P_{2h+1}(G')$. Removing uv's coordinate from the points of F gives a set of points of F' of affine dimension at least $\dim(F) - 1$. Since A'_F has the same rank as A_F and has one column fewer than A_F, then A'_F is face-defining for F' by Observation 2.7. By the induction hypothesis, A'_F is equimodular. Since adding a column of zeros preserves equimodularity, A_F is also equimodular.

Suppose now that $\delta(v)$ belongs to $\mathcal{M}_F$. Then, the column of A_F associated with uv has zeros in each row but $\chi^{\delta(v)}$. Let $A^\star_F x = b^\star_F$ be the system obtained from $A_F x = b_F$ by removing the equation associated with $\delta(v)$. Then $F^\star = \{x \in P_k(G) : A^\star_F x = b^\star_F\}$ is a face of $P_k(G)$ of dimension $\dim(F)+1$. Indeed, it contains F and $z + \alpha\chi^{uv} \notin F$ for every point z of F and $\alpha > 0$. Hence, $A^\star_F$ is face-defining for $F^\star$. This matrix is equimodular by the maximality assumption on F, and so is A_F by Statement (ii) of Observation 2.6. □

Claim 3.3 $|M \cap \delta(v)| \neq 1$ *for every multicut $M \in \mathcal{M}_F$.*

Proof Suppose there exists a multicut M tight for F such that $|M \cap \delta(v)| = 1$. Without loss of generality, suppose that M contains uv but not vw. Then, $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$ because of the partition inequality (2a) associated with the multicut $M \triangle \delta(v)$. Moreover, the partition inequality associated with $\delta(v)$ and the integrality of $P_{2h+1}(G)$ imply $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq h+1\}$. The proof is divided into two cases.

Case 1: $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. We prove this case by exhibiting an equimodular face-defining matrix for F. By Observation 2.5, this implies that A_F is equimodular, which contradicts the assumption on F.

Equality $x_{vw} = h+1$ can be expressed as a linear combination of equations of $A_F x = b_F$. Let $A'_F x = b'_F$ denote the system obtained by replacing an equation of $A_F x = b_F$ by $x_{vw} = h+1$ in such a way that the underlying affine space remains unchanged. Denote by $\mathcal{N}$ the set of multicuts of $\mathcal{M}_F$ containing vw but not uv. If $\mathcal{N} \neq \emptyset$, then let N be in $\mathcal{N}$. We now modify the system $A'_F x = b'_F$ by performing the following operations.

1. For all $M \in \mathcal{M}_F$ strictly containing $\delta(v)$, replace the equation associated with M by the partition constraint (2a) associated with $M \setminus vw$ set to equality, that is, $x(M \setminus vw) = (h+1)d_{M\setminus vw} - 1$.
2. If $\delta(v) \in \mathcal{M}_F$, then replace the equation associated with $\delta(v)$ by the box constraint $x_{uv} = h$.
3. If $\mathcal{N} \neq \emptyset$, then replace the equation associated with N by the box constraint $x_{uv} = h+1$.
4. For all $M \in \mathcal{N} \setminus N$, replace the equation associated with M by the partition constraint (2a) associated with $M \triangle \delta(v)$ set to equality, that is, $x(M \triangle \delta(v)) = (h+1)d_{M\triangle\delta(v)} - 1$.

These operations do not change the underlying affine space. Indeed, for every multicut M strictly containing $\delta(v)$ and tight for F, the set $M \setminus vw$ is a multicut tight for F by Lemma 2.11 and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. If $\delta(v)$ is tight for F, then $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h\}$ because $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. For $M \in \mathcal{N}$, by Observation 2.2, the set $M \triangle \delta(v)$ is a multicut of order d_M. The tightness of the constraint (2a) associated with N and the constraint (2a) associated with $M \triangle \delta(v)$ imply that $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \leq x_{uv}\}$. Since $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$, we have $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h+1\}$ and $M \triangle \delta(v)$ is tight for F. It follows that, if $\delta(v) \in \mathcal{M}_F$, then $\mathcal{N} = \emptyset$. Therefore, at most one among Operations 2 and 3 is applied so the rank of the matrix remains unchanged.

Let $A''_F x = b''_F$ be the system obtained by removing the equation $x_{vw} = h+1$ from $A'_F x = b'_F$. By construction, $A''_F x = b''_F$ is composed of constraints (2a) set to equality and possibly $x_{uv} = h$ or $x_{uv} = h+1$. Moreover, the column of A''_F associated with vw is zero. Let $F'' = \{x \in P_{2h+1}(G) : A''_F x = b''_F\}$. For every point z of F and $\alpha \geq 0$, $z + \alpha\chi^{vw}$ belongs to F'' because the column of A''_F associated with vw is zero, and $z + \alpha\chi^{vw} \in P_{2h+1}(G)$. This implies that $\dim(F'') \geq \dim(F) + 1$.

If F'' is a face of $P_{2h+1}(G)$, then A''_F is face-defining for F'' by Observation 2.7 and because A'_F is face-defining for F. By the maximality assumption on F, A''_F is equimodular, and hence so is A'_F by Statement (i) of Observation 2.6.

Otherwise, by construction, $F'' = F^\star \cap \{x \in \mathbb{R}^E : x_{uv} = t\}$ where $F^\star$ is a face of $P_{2h+1}(G)$ strictly containing F and $t \in \{h, h+1\}$. Therefore, there exists a face-defining matrix for F'' given by a face-defining matrix for $F^\star$ and the row χ^{uv}. Such a matrix is equimodular by the maximality assumption of F and Statement (i) of Observation 2.6. Hence, A''_F is equimodular by Observation 2.5, and so is A'_F by Statement (i) of Observation 2.6.

Case 2: $F \not\subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. Thus, there exists $z \in F$ such that $z_{vw} > h+1$. By Claim 3.2, there exists a multicut $N \neq \delta(v)$ containing vw which is tight for F. By Statement (i) of Theorem 2.10, the existence of z implies that N is a bond, hence it does not contain uv. The set $L = N \triangle \delta(v)$ is a bond of G. The partition inequality (2a) associated with L implies that $F \subseteq$

$\{x \in P_{2h+1}(G) : x_{vw} = x_{uv}\}$ and L is tight for F. Moreover, N is the unique multicut tight for F containing vw. Suppose indeed that there exists a multicut B containing vw tight for F. Then, B is a bond by Statement (i) of Theorem 2.10 and the existence of z. Moreover, $B \triangle N$ is a multicut not containing vw. This implies that no point x of F satisfies the partition constraint associated with $B \triangle N$ because $x(B \triangle N) = x(B)+x(N)-2x(B \cap N) = 2(2h+1)-2x(B \cap N) \leq 4h+2-2x_{vw} \leq 2h$, a contradiction.

Consider the matrix $A^\star_F$ obtained from A_F by removing the row associated with N. Matrix $A^\star_F$ is a face-defining matrix for a face $F^\star \supseteq F$ of $P_{2h+1}(G)$ because $F^\star$ contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. By the maximality assumption, the matrix $A^\star_F$ is equimodular. Let B_F be the matrix obtained from A_F by replacing the row χ^N by the row $\chi^N - \chi^L$. Then, B_F is face-defining for F. Moreover, B_F is equimodular by Statement (ii) of Observation 2.6—a contradiction. ◻

Let $A'_F x = b'_F$ be the system obtained from $A_F x = b_F$ by removing uv's column from A_F and subtracting $h+1$ times this column to b_F. We now show that $\{x \in P_{2h+1}(G') : A'_F x = b'_F\}$ is a face of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and of $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise. Indeed, consider a multicut M in $\mathcal{M}_F$. If $M = \delta(v)$, then the equation of $A'_F x = b'_F$ induced by M is nothing but $x_{uw} = h$. Otherwise, by Lemma 2.11 and Claim 3.3, the set $M \setminus uv$ is a multicut of G' (relabelling vw by uw) of order d_M if $uv \notin M$ and $d_M - 1$ otherwise. Thus, the equation of $A'_F x = b'_F$ induced by M is the partition constraint (2a) associated with $M \setminus uv$ set to equality.

By construction and Claim 3.3, A'_F has full row rank and one column less than A_F. We prove that A'_F is face-defining by exhibiting $\dim(F)$ affinely independent points of $P_{2h+1}(G')$ satisfying $A'_F x = b'_F$. Because of the integrality of $P_{2h+1}(G)$, there exist $n = \dim(F) + 1$ affinely independent integer points $z^1, \ldots, z^n$ of F. By Claims 3.2 and 3.3, there exists a multicut strictly containing $\delta(v)$. Then, Statement (i) of Theorem 2.10 implies that $F \subseteq \{x \in \mathbb{R}^E : x_{uv} \leq h+1, x_{vw} \leq h+1\}$. Combined with the partition inequality $x_{uv} + x_{vw} \geq 2h+1$ associated with $\delta(v)$, this implies that at least one of z^i_{uv} and z^i_{vw} is equal to $h+1$ for $i = 1, \ldots, n$. Since exchanging the uv and vw coordinates of any point of F gives a point of F by Claim 3.3, the hypotheses on $z^1, \ldots, z^n$ are preserved under the assumption that $z^i_{uv} = h+1$ for $i = 1, \ldots, n-1$. Let $y^1, \ldots, y^{n-1}$ be the points obtained from $z^1, \ldots, z^{n-1}$ by removing uv's coordinate. Since every multicut of G' is a multicut of G with the same order, $y^1, \ldots, y^{n-1}$ belong to $P_{2h+1}(G')$. By construction, they satisfy $A'_F x = b'_F$ so they belong to a face of $P_{2h+1}(G')$ or $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. This implies that A'_F is a face-defining matrix of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and of $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise.

By induction, $P_{2h+1}(G')$ is a box-TDI polyhedron and hence so is $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. Hence, A'_F is equimodular by Theorem 2.4. Since A_F is obtained from A'_F by copying a column, then also A_F is equimodular—a contradiction. ◻

By definition of box-TDIness and $Q_k(G)$, Theorem 3.2 implies that $Q_k(G)$ is box-TDI when G is series-parallel. The converse does not hold. Indeed, for instance, when $G = (V, E)$ is a minimal k-edge-connected graph, $Q_k(G)$ is nothing but the single point χ^E so it is a box-TDI polyhedron.

4 An Integer TDI System for $P_{2h}(G)$

Let G be a series-parallel graph. In this section we provide an integer TDI system for $P_{2h}(G)$ with h positive and integer.

The proof of the main result of the section is based on the characterization of TDIness by means of Hilbert bases. A set of vectors $\{v^1, \ldots, v^k\}$ is a *Hilbert basis* if each integer vector that is a nonnegative combination of $v^1, \ldots, v^k$ can be expressed as a nonnegative integer combination of them. The link between Hilbert basis and TDIness is stated in the following theorem.

Theorem 4.1 (Theorem 22.5 of [34]) *A system $Ax \geq b$ is TDI if and only if for every face F of $P = \{x : Ax \geq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

In the previous theorem, we could restrict to minimal faces: indeed, the cone generated by the constraints tight for a face F is a face of the cone generated by the constraints active for a face $F' \subseteq F$ [34].

Remark 4.2 A system $Ax \geq b$ is TDI if and only if, for each minimal face F of $P = \{x : Ax \geq b\}$, the rows of A associated with constraints tight for F form a Hilbert basis.

The rest of the section is devoted to prove that the system given by the partition constraints and the nonnegativity constraints, which describes $P_k(G)$ when k is even, is TDI when G is series-parallel.

The proof is based on the TDIness of System (5) and the structure of inequalities (3a). Their right-hand sides are proportional to k, hence it is enough to prove the case $k = 2$. This allows us to use Theorem 2.13 to obtain a TDI system for $P_2(G)$. In terms of Hilbert bases, the TDIness of this system implies that, given a face F of $P_2(G)$, the integer points of the associated cone are the half sum of the cuts tight for F. The technical part of the proof is to show that each integer point of this cone is also the sum of incidence vectors of the multicuts tight for F.

Theorem 4.3 *For a series-parallel graph G and a positive integer h, System* (3) *is TDI.*

Proof We only prove the case $h = 1$ since multiplying the right hand side of a system by a positive constant preserves its TDIness [34, Section 22.5].

The proof is done by induction on the number of edges of the graph $G = (V, E)$. When G consists of two vertices connected by a single edge ℓ, System (3) is $x_\ell \geq 2, x_\ell \geq 0$ and is TDI. If G is the circuit $\{e, f\}$, System (3) is $x_e + x_f \geq 2, x \geq \mathbf{0}$ and is TDI.

(Parallelization) Let now G be obtained from a series-parallel graph H by adding an edge g parallel to an edge f of H. System (3) associated with G is obtained from that associated with H by duplicating f's column in constraints (3a) and adding the nonnegativity constraint $x_g \geq 0$. By Lemma 3.1 of [7], System (3) is TDI.

For the other cases, we prove the TDIness of System (3) associated with G using Remark 4.2. More precisely, we prove that for any extreme point z of $P_2(G)$, the set of vectors $\{\chi^M : M \in \mathcal{T}_z\} \cup \{\chi^e : e \in E, z_e = 0\}$ is a Hilbert basis, where $\mathcal{T}_z$ is the set of multicuts tight for z.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G^1 = (W^1, E^1)$ and $G^2 = (W^2, E^2)$ and let z be an extreme point of $P_2(G)$. By construction, we have $z = (z^1, z^2)$ where $z^i \in P_2(G^i)$ for $i = 1, 2$. Moreover, for each multicut $M \in \mathcal{T}_z$, the graph obtained from $G(z)$ by contracting the edges of $E \setminus M$ is a circuit. Indeed, it is 2-edge-connected since $G(z)$ is, and it has $z(M) = d_M$ edges and d_M vertices. Therefore M is either a multicut of G^1 tight for z^1 or one of G^2 tight for z^2.

By induction, Systems (3) associated with G^1 and G^2 are TDI. Thus, $\{\chi^M : M \in \mathcal{T}_z \cap \mathcal{M}(G^i)\} \cup \{\chi^e : e \in E^i, z_e = 0\}$ is a Hilbert basis for $i = 1, 2$ by Theorem 4.1. Since they belong to disjoint spaces, their union is a Hilbert basis. By Theorem 4.1, System (3) is TDI.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw, and let z be an extreme point of $P_2(G)$.

Without loss of generality, suppose $z_{uv} \geq z_{vw}$. Define $z' \in \mathbb{Z}^{E'}$ by $z'_{uw} = z_{vw}$ and $z'_e = z_e$ for all edges e in $E' \setminus uw$. Note that z' belongs to $P_2(G')$ since $G'(z')$ is obtained by contracting the edge uv in $G(z)$, and this contraction preserves 2-edge-connectivity.

Note that for all $e \in E$, $z_e \in \{0, 1, 2\}$. Indeed, since z is an extreme point of $P_2(G)$ which is also described by System (1), if $z_e > 0$, then e belongs to a cut D tight for z. Moreover, as $z_{uv} \geq z_{vw}$, the partition constraint (3a) associated with $\delta(v)$ implies that $z_{uv} \in \{1, 2\}$. We now consider two different cases depending on the value of z_{uv}.

Case 1: $z_{uv} = 2$.

We first show that every multicut of $\mathcal{T}_z$ containing uv is a bond. Indeed, note that every multicut M with $d_M = 2$ is a bond. If a multicut $M = \delta(V_1, \ldots, V_{d_M}) \in \mathcal{T}_z$ satisfies $d_M \geq 3$ and $uv \in \delta(V_1, V_2)$, then $M' = \delta(V_1 \cup V_2, V_3, \ldots, V_{d_M})$ is a multicut and satisfies

$$z(M') \leq z(M) - 2 < d_M - 1 = d_{M'}.$$

Hence, the partition constraint (3a) associated with M' is violated, a contradiction.

Moreover, there exists at most one bond of $\mathcal{T}_z$, say N, containing uv. As otherwise suppose there exist two bonds B_1 and B_2 in $\mathcal{T}_z$ containing uv. Then, $z(B_1 \triangle B_2) \leq z(B_1) + z(B_2) - 2z_{uv} = 0$, which contradicts the constraint (3a) associated with the multicut $B_1 \triangle B_2$. For a multicut M not containing uv, $M \in \mathcal{T}_z$ if and only if $M \in \mathcal{T}_{z'}$. This implies that $\mathcal{T}_z = \mathcal{T}_{z'} \cup N$. By induction and Theorem 4.1, $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis. As $\mathcal{E}_z = \mathcal{E}_{z'}$ (identifying uv and vw) and N is the only member of $\mathcal{T}_z \cup \mathcal{E}_z$ containing uv, $\mathcal{T}_z \cup \mathcal{E}_z$ is also a Hilbert basis.

Case 2: $z_{uv} = 1$.

Let $\mathbf{v}$ be an integer point of the cone generated by $\mathcal{T}_z \cup \mathcal{E}_z$. We prove that $\mathbf{v}$ can be expressed as an integer nonnegative combination of the vectors of $\mathcal{T}_z \cup \mathcal{E}_z$. This implies that $\mathcal{T}_z \cup \mathcal{E}_z$ is a Hilbert basis.

Let $\mathcal{B}_z$ be the set of bonds of $\mathcal{T}_z$. Since System (5) is a TDI system describing $P_2(G)$ in series-parallel graphs, the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_z\} \cup \mathcal{E}_z$ forms a Hilbert basis by Theorem 4.1. Then, there exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_z$ and $\mu_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_z$ such that $\mathbf{v} = \sum_{B\in\mathcal{B}_z} \lambda_B \chi^B + \sum_{e\in\mathcal{E}_z} \mu_e \chi^e$.

Since $z_{uv} \geq z_{vw}$, the partition inequality (3a) associated with $\delta(v)$ implies that $z_{vw} = 1$ and $\delta(v) \in \mathcal{T}_z$. In particular, $vw \notin \mathcal{E}_z$. The vector $\mathbf{v}$ is an integer combination of vectors of $\mathcal{T}_z \cup \mathcal{E}_z$ if and only if $\mathbf{v} - \lfloor \lambda_{\delta(v)} \rfloor \chi^{\delta(v)}$ is, thus we may assume that $\lambda_{\delta(v)} \in \{0, \frac{1}{2}\}$. Define $\mathbf{w} \in \mathbb{Z}^{E'}$ by:

$$\mathbf{w}_e = \begin{cases} \mathbf{v}_{uv} + \mathbf{v}_{vw} - 2\lambda_{\delta(v)} & \text{if } e = uw, \\ \mathbf{v}_e & \text{otherwise.} \end{cases}$$

Note that $(B \setminus uw) \cup uv$ and $(B \setminus uw) \cup vw$ are bonds of $\mathcal{T}_z$ whenever B is a bond of $\mathcal{T}_{z'}$ containing uw because $z'_{uw} = z_{uv} = z_{vw} = 1$. Moreover, a bond B of $\mathcal{T}_{z'}$ which does not contain uw is a bond of $\mathcal{T}_z$. Since $\delta(v)$ is the unique bond of G containing both uv and vw and $\mathcal{E}_z = \mathcal{E}_{z'}$, we have:

$$\mathbf{w} = \sum_{B\in\mathcal{B}_{z'}:uw\in B} (\lambda_{(B\setminus uw)\cup uv} + \lambda_{(B\setminus uw)\cup vw})\chi^B + \sum_{B\in\mathcal{B}_{z'}:uw\notin B} \lambda_B \chi^B + \sum_{e\in\mathcal{E}_{z'}} \mu_e \chi^e.$$

Thus, $\mathbf{w}$ belongs to the cone generated by $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$. By the induction hypothesis, $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis, hence there exist $\lambda'_M \in \mathbb{Z}_+$ for all $M \in \mathcal{T}_{z'}$ and $\mu'_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_{z'}$ such that $\mathbf{w} = \sum_{M\in\mathcal{T}_{z'}} \lambda'_M \chi^M + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e$.

Consider the family $\mathcal{N}$ of multicuts of $\mathcal{T}_{z'}$ where each multicut M of $\mathcal{T}_{z'}$ appears λ'_M times. Suppose first that $\lambda_{\delta(v)} = 0$. Then, $\mathbf{v}_{uv}+\mathbf{v}_{vw}$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv}$ multicuts of $\mathcal{N}$ containing uw and $\mathcal{Q} = \{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:uw\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus uw)\cup uv} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus uw)\cup vw} + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e. \tag{6}$$

Suppose now that $\lambda_{\delta(v)} = \frac{1}{2}$. Then, $\mathbf{w}_{uw} = \mathbf{v}_{uv} + \mathbf{v}_{vw} - 1$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv} - 1$ multicuts of $\mathcal{N}$ containing uw, let $\mathcal{Q}$ be a family of $\mathbf{v}_{vw} - 1$ multicuts in $\{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing uw which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:uw\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus uw)\cup uv} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus uw)\cup vw} + \chi^{(N\setminus uw)\cup\delta(v)} + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e. \tag{7}$$

Every $M \in \mathcal{T}_{z'}$ not containing uw is in $\mathcal{T}_z$. For every $M \in \mathcal{T}_{z'}$ containing uw, $(M \setminus uw) \cup uv$, $(M \setminus uw) \cup vw$ and $(M \setminus uw) \cup \delta(v)$ belong to $\mathcal{T}_z$ since $z'_{uw} = z_{uv} = z_{vw} = 1$. Since $\mathcal{E}_z = \mathcal{E}_{z'}$, then $\mathbf{v}$ is a nonnegative integer combination of vectors of $\mathcal{T}_z \cup \mathcal{E}_z$ in both (6) and (7). This proves that $\mathcal{T}_z \cup \mathcal{E}_z$ is a Hilbert basis. Therefore by Remark 4.2, System (3) is TDI. □

The box-TDIness of $P_k(G)$ and the TDIness of System (3) give the following result.

Corollary 4.4 *System* (3) *is box-TDI if and only if* G *is series-parallel.*

Proof If G is series-parallel, then System (3) is box-TDI by Theorems 3.2 and 4.3, since a TDI system describing a box-TDI polyhedron is box-TDI [12]. If G is not series-parallel, Theorem 3.2 ensures that $P_k(G)$ is not box-TDI, therefore System (3) is not box-TDI. $\square$

Theorem 4.3 leaves open the following problem:

Open Problem 4.5 *Characterize the classes of graphs such that System* (3) *is TDI.*

5 An Integer TDI System for $P_{2h+1}(G)$

In this section, we prove that System (2) is TDI if and only if G is a series-parallel graph—see Theorem 5.1. Proving the TDIness for k odd is considerably more involved than for k even. The first difference with the even case is the lack of a known TDI system describing $P_k(G)$ when k is odd, even a noninteger one. Thus, no property of the Hilbert bases associated with $P_k(G)$ is known, and the approach used to prove Theorem 4.3 cannot be applied. Instead, following the definition of TDIness, we prove the existence of an integer optimal solution to each feasible dual problem.

Another difference with the case k even follows from the structure of the partition inequalities (2a). In particular, the presence of the constant "-1" in the right-hand sides perturbs the structure of tight multicuts. Indeed, when k is odd, the tightness of $\delta(V_1, \ldots, V_n)$ does not imply that of $\delta(V_1), \ldots, \delta(V_n)$. Consequently, it is not clear how the contraction of an edge impacts the tightness of a multicut $\delta(V_1, \ldots, V_n)$: merging adjacent V_i's is not sufficient to obtain new tight multicuts. Due to the link between tight multicuts and positive dual variables, the structure of the optimal solutions to the dual problem is completely modified when subdividing an edge. Proving directly that subdivision preserves TDIness turned out to be challenging, and we overcome this difficulty by deriving new properties of series-parallel graphs—see Lemma 2.3.

The proof of Theorem 5.1 focuses on properties of vertices of degree 2 in a minimal counterexample to the TDIness of System (2). In particular, we prove that no two vertices of degree 2 are adjacent (Claim 5.9), or in the same circuit of length 4 (Claim 5.11). Moreover, no triangle contains vertices of degree 2 (Claim 5.10). By Lemma 2.3, this implies that the graph is not series-parallel. To derive these properties, we study the interplay between cuts associated with degree 2 vertices and dual optimal solutions—see Claims 5.3-5.8.

Theorem 5.1 *For h positive and integer, System* (2) *is TDI if and only if G is series-parallel.*

Proof If G is not series-parallel, then System (2) is not TDI because every TDI system with integer right-hand side describes an integer polyhedron [22], but when G has a K_4-minor, System (2) describes a noninteger polyhedron [10].

We now prove that, if G is series-parallel, then System (2) is TDI. We prove the result by contradiction. Let $G = (V, E)$ be a series-parallel graph such that

System (2) is not TDI. By definition of TDIness, there exists $c \in \mathbb{Z}^E$ such that $\mathcal{D}_{(G,c)}$:

$$\max \sum_{M \in \mathcal{M}_G} b_M y_M$$

$$\text{s.t.} \begin{cases} \displaystyle\sum_{M \in \mathcal{M}_G : e \in M} y_M \leq c_e & \text{for all } e \in E, \quad (8a) \\ y_M \geq 0 & \text{for all } M \in \mathcal{M}_G, \quad (8b) \end{cases}$$

is feasible, bounded, but admits no integer optimal solution, where $b_M = (h+1)d_M - 1$ for all $M \in \mathcal{M}_G$. Without loss of generality, we assume that:

(i) G has a minimum number of edges,
(ii) $\sum_{e \in E} c_e$ is minimum with respect to (i).

By definition, $\mathcal{D}_{(G,c)}$ is feasible if and only if $c \geq \mathbf{0}$. Hence, by minimality assumption (ii), $\mathcal{D}_{(G,c')}$ has an optimal integer solution for every integer $c' \neq c$ such that $0 \leq c' \leq c$.

Let M be a multicut of G. We denote by ξ_M the vector of $\{0,1\}^{\mathcal{M}_G}$ whose only nonzero coordinate is the one associated with M. We say that M is *active* for a solution y to $\mathcal{D}_{(G,c)}$ if $y_M > 0$. Note that, by complementary slackness, a multicut is active for an optimal solution to $\mathcal{D}_{(G,c)}$ only if it is tight for an optimal solution to the primal problem. In particular, if a multicut is tight for no point of $P_{2h+1}(G)$, then it is active for no optimal solution to $\mathcal{D}_{(G,c)}$. Thus, we will use Lemma 2.11 and Theorem 2.10 to deduce properties on the optimal solutions to $\mathcal{D}_{(G,c)}$.

Claim 5.1 *G is simple, 2-connected, and nontrivial.*

Proof Suppose for a contradiction that there exist two parallel edges e_1 and e_2 and $c_{e_1} \leq c_{e_2}$. Since a multicut contains either both e_1 and e_2 or none of them, the inequality (8a) associated with e_2 is redundant because $c_{e_1} \leq c_{e_2}$. This contradicts minimality assumption (i), so G is simple.

Assume for a contradiction that G is not 2-connected. Then G is the 1-sum of two distinct graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. By Statement (ii) of Theorem 2.10, the multicuts of G that intersect both E_1 and E_2 are not tight for the points of $P_{2h+1}(G)$, by complementary slackness, these multicuts are not active for the optimal solutions to $\mathcal{D}_{(G,c)}$. Hence, every optimal solution y to $\mathcal{D}_{(G,c)}$ is of the form:

$$y_M = \begin{cases} y^1_M & \text{if } M \in \mathcal{M}_{G_1}, \\ y^2_M & \text{if } M \in \mathcal{M}_{G_2}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

where y^i is an optimal solution to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1, 2$. By minimality assumption (i), there exists an integer optimal solution $\bar{y}^i$ to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1, 2$, implying that $(\bar{y}^1, \bar{y}^2)$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Finally, if $G = K_2$, $\mathcal{M}_G$ contains only one multicut, say $\{e\}$, and the optimal solution to $\mathcal{D}_{(G,c)}$ is $y^*_{\{e\}} = c_e$ which is integer. □

Claim 5.2 *For all edges $e \in E$, $c_e \geq 1$.*

Proof By hypothesis, $c \geq \mathbf{0}$ is integer and $\mathcal{D}_{(G,c)}$ has an optimal solution, say y^*. Suppose for a contradiction that there exists an edge $e \in E$ with $c_e = 0$. Set $G' = G/e$ and $c' = c_{|(E \setminus e)}$. The active multicuts for y^* do not contain the edge e so they are multicuts of G' since $\mathcal{M}_{G'} = \{M \in \mathcal{M}_G | e \notin M\}$. Hence, the point $y' \in \mathbb{R}^{\mathcal{M}_{G'}}$ defined by $y'_M = y^*_M$ for all $M \in \mathcal{M}_{G'}$ is a solution to $\mathcal{D}_{(G',c')}$.

By minimality assumption (i), there exists an integer optimal solution $\tilde{y}$ to $\mathcal{D}_{(G',c')}$. Extending $\tilde{y}$ to a point of $\mathbb{Z}^{\mathcal{M}_G}$ by setting to 0 the missing component gives an integer solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top \tilde{y} \geq b^\top y' = b^\top y^*$. This is an integer optimal solution to $\mathcal{D}_{(G,c)}$ since y^* is optimal, a contradiction to the hypothesis that $\mathcal{D}_{(G,c)}$ has no integer optimal solution. □

Claim 5.3 *Every optimal solution y to $\mathcal{D}_{(G,c)}$ satisfies $0 \leq y_M < 1$ for all $M \in \mathcal{M}_G$.*

Proof By contradiction, suppose that y^* is an optimal solution to $\mathcal{D}_{(G,c)}$ such that there exists a multicut M such that $y^*_M \geq 1$. Therefore, the point y' defined by $y' = y^* - \xi_M$ is a solution to $\mathcal{D}_{(G,c')}$ where $c' = c - \chi^M$. By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution y''. The point $\tilde{y}$ defined by $\tilde{y} = y'' + \xi_M$ is an integer solution to $\mathcal{D}_{(G,c)}$ and we have:

$$b^\top \tilde{y} = b^\top y'' + b_M \geq b^\top y' + b_M = b^\top y^*.$$

Therefore $\tilde{y}$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. □

From the definition of series-parallel graphs, Claim 5.1 implies that G contains at least one degree 2 vertex. Let $\widehat{V}$ be the set of degree 2 vertices of G.

Claim 5.4 *Let $v \in \widehat{V}$, $\delta(v) = \{e_1, e_2\}$, y be an optimal solution to $\mathcal{D}_{(G,c)}$, and M_1 be an active multicut for y such that $M_1 \cap \delta(v) = e_1$. If $\delta(v)$ is active for y, then no multicut whose intersection with $\delta(v)$ is e_2 is active for y.*

Proof We prove the result by contradiction. Assume that M_1 and $\delta(v)$ are active for y and that there exists a M_2 active for y with $M_2 \cap \delta(v) = e_2$. By Observation 2.2, $M'_i = M_i \cup \delta(v)$ is a multicut of G such that $d_{M'_i} = d_{M_i} + 1$ for $i = 1, 2$. Let $0 < \varepsilon \leq \min\{y_{M_1}, y_{M_2}, y_{\delta(v)}\}$. Then, the point:

$$y' = y - \varepsilon \left(\chi^{M_1} + \chi^{M_2} + \chi^{\delta(v)} \right) + \varepsilon \left(\chi^{M'_1} + \chi^{M'_2} \right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and we have $b^\top y' = b^\top y + \varepsilon$, implying that y is not optimal, a contradiction. □

Claim 5.5 *For every optimal solution to $\mathcal{D}_{(G,c)}$, the constraints* (8a) *associated with the edges incident to a degree 2 vertex are tight.*

Proof We prove the result by contradiction. Suppose that there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v with $\delta(v) = \{e_1, e_2\}$ such that the inequality (8a) associated with e_1 is not tight. For $i = 1, 2$, let s_i be the slack of the constraint associated with e_i, that is,

$$s_i = c_{e_i} - \sum_{M \in \mathcal{M}_G : e_i \in M} y^*_M.$$

Inequality (8a) associated with e_2 is tight, as otherwise there exists $0 < \eta \leq \min\{s_1, s_2\}$, such that $y^* + \eta\xi_{\delta(v)}$ is a solution to $\mathcal{D}_{(G,c)}$, a contradiction to the optimality of y^*. Hence, Claims 5.2 and 5.3 imply that there are at least two distinct multicuts M_1 and M_2 active for y^* and containing e_2. Let $0 < \varepsilon \leq \min\{y^*_{M_1}, y^*_{M_2}, s_1\}$. For $i = 1, 2$, $e_1 \in M_i$, as otherwise $y' = y^* + \varepsilon(\xi_{M_i \cup e_1} - \xi_{M_i})$ is a solution to $\mathcal{D}_{(G,c)}$. This solution is such that $b^\top y' = b^\top y^* + \varepsilon(h+1) > b^\top y^*$, a contradiction to the optimality of y^*. Thus, both M_1 and M_2 contain $\delta(v)$. Since they are distinct, at least one of them, say M_1, strictly contains $\delta(v)$. Then, $y'' = y^* + \varepsilon(-\xi_{M_1} + \xi_{M_1 \setminus e_2} + \xi_{\delta(v)})$ is a solution to $\mathcal{D}_{(G,c)}$ because $M_1 \setminus e_2$ belongs to $\mathcal{M}_G$ by Lemma 2.11. Then, $b^\top y'' = b^\top y^* + \varepsilon(-b_{M_1} + b_{M_1} - (h+1) + 2h+1) > b^\top y^*$, a contradiction. □

Given a solution y to $\mathcal{D}_{(G,c)}$, we define for each vertex $v \in \widehat{V}$ the set $\mathcal{A}^y_v$ as the set of multicuts active for y that strictly contain $\delta(v)$. Moreover we define the value α^y_v as:

$$\alpha^y_v = \sum_{M \in \mathcal{A}^y_v} y_M. \tag{9}$$

Claim 5.6 *Every optimal solution y to $\mathcal{D}_{(G,c)}$ satisfies $0 < \alpha^y_v < 1$ for all $v \in \widehat{V}$.*

Proof Suppose for a contradiction that there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v of $\widehat{V}$ such that either $\alpha^{y^*}_v \geq 1$ or $\alpha^{y^*}_v = 0$. Denote the two edges incident to v by e_1 and e_2 in such a way that $c_{e_1} \leq c_{e_2}$.

Suppose first that $\alpha^{y^*}_v \geq 1$. By Claim 5.3, there exist at least two multicuts in $\mathcal{A}^{y^*}_v$. Let $\mathcal{A}^{y^*}_v = \{M_1, \dots, M_n\}$. By Lemma 2.11, for all $i = 1, \dots, n$, $M'_i = M_i \setminus e_1$ is a multicut of G with $d_{M'_i} = d_{M_i} - 1$. Let $c' = c - \chi^{e_1}$. By $\alpha^{y^*}_v \geq 1$, there exist ϵ_i for all $i = 1, \dots, n$, such that $0 \leq \epsilon_i \leq y^*_{M_i}$ and $\sum_{i=1}^n \epsilon_i = 1$. The point y^1 defined by:

$$y^1 = y^* + \sum_{i=1}^{n} \left(-\epsilon_i \xi_{M_i} + \epsilon_i \xi_{M'_i}\right)$$

is a solution to $\mathcal{D}_{(G,c')}$. By definition of b, we have:

$$b^\top y^1 = b^\top y^* - h - 1. \tag{10}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. This solution satisfies with equality the constraint (8a) associated with e_2 as otherwise $y^2 + \xi_{\delta(v)}$ would be a solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top y^2 + b_{\delta(v)} \geq b^\top y^1 + 2h + 1$, contradicting the assumption that y^* is optimal by (10) and $h \geq 1$. Hence, there exists a multicut $\bar{M}$ active for y^2 containing e_2 but not e_1 since $c'_{e_1} + 1 \leq c'_{e_2}$. By definition, $\bar{M} \cup e_1$ is a multicut of G of order $d_{\bar{M}} + 1$. Define $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ by:

$$y^3_M = y^2 - \chi^{\bar{M}} + \chi^{\bar{M} \cup e_1}$$

By definition of c' and y^2, the point y^3 is an integer solution to $\mathcal{D}_{(G,c)}$. Therefore, by (10), since y^2 is optimal in $\mathcal{D}_{(G,c')}$ and by definition of y^3, we have:

$$b^\top y^* = b^\top y^1 + h + 1 \leq b^\top y^2 + h + 1 \leq b^\top y^3.$$

Thus, y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Suppose now that $\alpha_v^{y^*} = 0$. First, note that $\delta(v)$ is not an active multicut for y^*. Otherwise by Claims 5.2, 5.3 and 5.5, there would be a multicut containing e_1 and not e_2, say N_1, and a multicut containing e_2 and not e_1, say N_2, which are both active for y^*. This contradicts Claim 5.4. Hence, by definition of $\alpha_v^{y^*}$, no active multicut contains $\delta(v)$.

By Observation 2.2, if a multicut M contains e_2 but not e_1, then $M \triangle \delta(v)$ is a multicut with the same order and $b_M = b_{M\triangle\delta(v)}$. Hence, we can define the point $y^4 \in \mathbb{Q}^{\mathcal{M}_G}$:

$$y^4_M = \begin{cases} 0 & \text{if } e_1 \in M, \\ y^*_M + y^*_{M\triangle\delta(v)} & \text{if } e_1 \notin M \text{ and } e_2 \in M, \\ y^*_M & \text{otherwise,} \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

which is a solution to $\mathcal{D}_{(G,\hat{c})}$, where $\hat{c}$ is defined by:

$$\hat{c}_e = \begin{cases} c_{e_1} + c_{e_2} & \text{if } e = e_2, \\ 0 & \text{if } e = e_1, \\ c_e & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E.$$

By construction, we have:

$$b^\top y^4 = b^\top y^*. \tag{11}$$

Using the argument given in the proof of Claim 5.2, we deduce that $\mathcal{D}_{(G,\hat{c})}$ admits an integer optimal solution, say y^5. Let $\mathcal{S}$ be the family of active multicuts for y^5 containing e_2, where each multicut M appears y^5_M times in $\mathcal{S}$. We have $|\mathcal{S}| > c_{e_2}$ as otherwise y^5 would be an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

We now construct from y^5 an integer solution y^6 to $\mathcal{D}_{(G,c)}$ with the same cost by replacing e_2 by e_1 in some active multicuts for y^5. More formally, since $|\mathcal{S}| \geq c_{e_1}$, there exists $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = c_{e_1}$. By Observation 2.2, $M \triangle \delta(v)$ is a multicut of G for all $M \in \mathcal{S}'$ and $b_M = b_{M\triangle\delta(v)}$. Let $y^6 \in \mathbb{Z}^{\mathcal{M}_G}$ be the point defined by:

$$y^6 = y^5 + \sum_{M \in \mathcal{S}'} \left(\xi_{M\triangle\delta(v)} - \xi_M\right) \tag{12}$$

By construction, we have:

$$b^\top y^6 = b^\top y^5. \tag{13}$$

Note that for each $M \in \mathcal{S}'$, adding $\xi_{M\triangle\delta(v)} - \xi_M$ to a point of $\mathbb{R}^{\mathcal{M}_G}$ increases (resp. decreases) by 1 the left-hand side of the inequality (8a) associated with e_1 (resp. e_2) while not changing the left-hand side of the inequalities (8a) associated with the edges of $E \setminus \{e_1, e_2\}$. Therefore, by definition of $\hat{c}$, y^6 is a solution to $\mathcal{D}_{(G,c)}$. By (13), the optimality of y^5, and (11), we have:

$$b^\top y^6 = b^\top y^5 \geq b^\top y^4 = b^\top y^*.$$

Therefore y^6 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. $\square$

Claim 5.6 implies that for each optimal solution y and for each $v \in \widehat{V}$ there exists at least one multicut strictly containing $\delta(v)$ that is active for y. For the following claims we need to define a subset of optimal solutions to $\mathcal{D}_{(G,c)}$: let $\mathfrak{D}_v$ be the set of optimal solutions to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Note that if $\mathfrak{D}_v$ is not empty, then there exists a solution y in $\mathfrak{D}_v$ maximizing α_v^z over all $z \in \mathfrak{D}_v$.

The following claim presents the structure of a specific optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active.

Claim 5.7 *Let $v \in \widehat{V}$ with $\delta(v) = \{e_1, e_2\}$ and let $y^* \in \mathfrak{D}_v$ maximize α_v^z over all $z \in \mathfrak{D}_v$. Then, there are exactly 3 multicuts active for y^* intersecting $\delta(v)$: two bonds $F \cup e_1$ and $F \cup e_2$ and a multicut $F \cup \{e_1, e_2\}$ of order 3, for some $F \subseteq E$.*

Proof By Claim 5.6, there exists at least one multicut strictly containing $\delta(v)$ which is active for y^*, say M_0. By definition of $\mathfrak{D}_v$, $\delta(v)$ is not active for y^*. Hence, by Claim 5.5, there exists at least one multicut active for y^* which contains e_i and not $\delta(v) \setminus e_i$, for $i = 1, 2$. Let M_i be such a multicut with maximum order.

First, we prove that $d_{M_0} = 3$. By definition, $M_0 = \delta(v, V_2, V_3, \dots, V_{d_{M_0}})$. Moreover, by Lemma 2.11 and complementary slackness, the two vertices adjacent to v belong to two different classes, say V_2 and V_3. By contradiction, suppose that $d_{M_0} \geq 4$. Then, $M_0' = \delta(v \cup V_2 \cup V_3, \dots, V_{d_{M_0}})$ is a multicut of order $d_{M_0} - 2$. For $i = 1, 2$, $M_i' = M_i \cup \delta(v)$ is a multicut of order $d_{M_i} + 1$. Let $0 < \varepsilon \leq \min\{y^*_{M_0}, y^*_{M_1}, y^*_{M_2}\}$. Then, let $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_0} + \varepsilon\xi_{M_0'} + \varepsilon \sum_{i=1,2} \left(-\xi_{M_i} + \xi_{M_i'}\right).$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_i'$ for $i = 1, 2$. This contradicts the maximality of $\alpha_v^{y^*}$. Therefore $d_{M_0} = 3$.

Now, we show that M_1 is a bond. The result for M_2 holds by symmetry. By contradiction, suppose that $M_1 = \delta(V_1, \dots, V_{d_{M_1}})$ with $d_{M_1} \geq 3$. Without loss of generality, we suppose that $e \in \delta(V_1) \cap \delta(V_2)$. Then, $M_1' = \delta(V_1 \cup V_2, \dots, V_{d_{M_1}})$ is a multicut of order $d_{M_1} - 1$. Moreover, $M_2' = M_2 \cup \delta(v)$ is a multicut of order $d_{M_2} + 1$. Let $0 < \varepsilon \leq \min\{y^*_{M_1}, y^*_{M_2}\}$ and $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_1} + \varepsilon\xi_{M_1'} - \varepsilon\xi_{M_2} + \varepsilon\xi_{M_2'}.$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_2'$. This contradicts the maximality of $\alpha_v^{y^*}$. Therefore, $d_{M_1} = d_{M_2} = 2$.

We now prove that there exists a set F such that $M_0 = F \cup \delta(v)$, and $M_i = F \cup e_i$ for $i = 1, 2$. Note that $M_1 \cup M_2$ is a multicut so $y'' = y^* + \varepsilon(\xi_{M_1 \cup M_2} - \xi_{M_1} - \xi_{M_2})$ is a solution to $\mathcal{D}_{(G,c)}$. The optimality of y^* implies $d_{M_1 \cup M_2} \leq 3$. Since M_1 and M_2 are distinct bonds, there exists $F \subseteq E \setminus \delta(v)$ such that $M_i = F \cup e_i$, for $i = 1, 2$. Finally, let $N_0 = M_0 \setminus e_2$ and $N_1 = M_1 \cup e_2$. Note that $\tilde{y} = y^* + \varepsilon(\xi_{N_0} - \xi_{M_0} + \xi_{N_1} - \xi_{M_1})$ is an optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Moreover, N_0 and M_2 are bonds active for $\tilde{y}$ since $d_{M_0} = 3$. This implies that $N_0 = F \cup e_1$, and hence $M_0 = F \cup \delta(v)$.

This implies that M_0, M_1, and M_2 are the only multicuts active for y^* intersecting $\delta(v)$. Indeed, if M is a multicut active for y^* strictly containing $\delta(v)$, then repeating the proof above with M, M_1, and M_2 shows that there exists F' such that $M = F' \cup \delta(v)$, and $M_i = F' \cup e_i$ for $i = 1, 2$. Since $M_i = F \cup e_i$ for $i = 1, 2$, we have $F' = F$ and hence $M = M_0$. A similar argument holds for any multicut active for y^* and intersecting $\delta(v)$. □

Claim 5.8 *Let $v \in \widehat{V}$ and y^* be an optimal solution to $\mathcal{D}_{(G,c)}$. Then,*

(i) if $y^*_{\delta(v)} = 0$, then $c_e = 1$ for all $e \in \delta(v)$,

(ii) if $y^*_{\delta(v)} > 0$, then $\alpha^{y^*}_v + y^*_{\delta(v)} = 1$, and there exists $e \in \delta(v)$ such that $c_e = 1$.

Proof (*i.*) First suppose that $y^*_{\delta(v)} = 0$, then $\mathfrak{D}_v \neq \emptyset$. Let $y' \in \mathfrak{D}_v$ maximize α^z_v over all $z \in \mathfrak{D}_v$. Then, by Claim 5.7, there exist exactly two active multicuts for y' containing e_i for $i = 1, 2$. Combining Claims 5.3 and 5.5, and the integrality of c, we obtain that $c_{e_i} = 1$ for $i = 1, 2$.

(*ii.*) Let now $y^*_{\delta(v)} > 0$. By Claim 5.4, there exists an edge $e \in \delta(v)$ such that all multicuts containing e that are active for y contain $\delta(v)$. Hence, the constraint (8a) associated with e is:

$$c_e \geq \sum_{M : e \in M} y^*_M = y^*_{\delta(v)} + \sum_{M \in \mathcal{A}^{y^*}_v} y^*_M = y^*_{\delta(v)} + \alpha^{y^*}_v . \tag{14}$$

By Claim 5.5, the constraint (8a) associated with e is tight. Thus, $y^*_{\delta(v)} + \alpha^{y^*}_v = c_e$. By Claims 5.3 and 5.6 and since c_e is integer, we have $c_e = 1$. □

The last three claims of the proof give some structural property of the graph G. In particular we focus our attention on the vertices of $\widehat{V}$.

Claim 5.9 *Vertices of degree 2 are pairwise nonadjacent.*

Proof Assume for a contradiction that there exist two adjacent vertices v_1 and v_2 in $\widehat{V}$, and denote $\delta(v_i) = \{e_0, e_i\}$ for $i = 1, 2$.

We prove that $\delta(v_1)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$, the result for $\delta(v_2)$ is obtained by symmetry. By contradiction, suppose that $\mathfrak{D}_{v_1} \neq \emptyset$. Among all the solutions $y \in \mathfrak{D}_{v_1}$, let y^1 be one having $\alpha^y_{v_1}$ maximum. Then, by Claim 5.7, the three multicuts active for y^1 intersecting $\delta(v_1)$ are $M_0 = F \cup \delta(v_1)$, $B_0 = F \cup e_0$, and $B_1 = F \cup e_1$, where B_i are bonds for $i = 0, 1$, and $F \subseteq E \setminus \delta(v_1)$ contains no nonempty multicut. By Claim 5.6 on v_2, there exists a multicut M active for y^1 strictly containing $\delta(v_2)$. By $\delta(v_1) \cap \delta(v_2) = e_0$, M intersects $\delta(v_1)$. Since $d_M \geq 3$, Claim 5.7 for v_1 implies $M = M_0$, $F = \{e_2\}$, and $B_0 = \delta(v_2)$.

As $y^1_{\delta(v_1)} = 0$, by Statement (i) of Claim 5.8, $c_{e_0} = c_{e_1} = 1$. By Claim 5.5, the constraints associated with e_0 and e_1 are tight. Since $\mathcal{A}^{y^1}_{v_1} = \{M_0\}$ by Claim 5.7, we have:

$$c_{e_i} = y^1_{M_0} + y^1_{B_i} = 1 \quad \text{for } i = 0, 1. \tag{15}$$

Let $\{M_1, \ldots, M_n\}$ be the set of active multicuts for y^1 such that $M_i \cap \{e_0, e_1, e_2\} = e_2$, for $i = 1, \ldots, n$. By Claim 5.5, the constraint (8a) associated with e_2 is tight, hence, using (15):

$$c_{e_2} = y^1_{M_0} + y^1_{B_0} + y^1_{B_1} + \sum_{i=1}^{n} y^1_{M_i} = 1 + y^1_{B_0} + \sum_{i=1}^{n} y^1_{M_i}. \tag{16}$$

By Claim 5.3, B_0 active for y^1, and $c_{e_2} \in \mathbb{Z}$, we have $\{M_1, \dots, M_n\} \neq \emptyset$ and $c_{e_2} \geq 2$. Thus, combining (15) and (16), we have:

$$\sum_{i=1}^{n} y^1_{M_i} = c_{e_2} - 1 - y^1_{B_0} \geq y^1_{M_0}. \tag{17}$$

Then, there exist $\epsilon_1, \dots, \epsilon_n$ such that $0 \leq \epsilon_i \leq y^1_{M_i}$ for $i = 1, \dots, n$, and

$$\sum_{i=1}^{n} \epsilon_i = y^1_{M_0}.$$

For $i = 1, \dots, n$, $M_i \cup e_0$ is a multicut with order $d_{M_i} + 1$, hence we can consider the following solution to $\mathcal{D}_{(G,c)}$:

$$y^2 = y^1 - \left(y^1_{M_0} \xi_{M_0} + \sum_{i=1}^{n} \epsilon_i \xi_{M_i} \right) + \left(y^1_{M_0} \xi_{M_0 \setminus e_0} + \sum_{i=1}^{n} \epsilon_i \xi_{M_i \cup e_0} \right). \tag{18}$$

We have $b^\top y^1 = b^\top y^2$, but $\alpha^{y^2}_{v_1} = 0$, a contradiction to Claim 5.6. Therefore $\mathfrak{D}_v \neq \emptyset$, and by symmetry we deduce that both $\delta(v_1)$ and $\delta(v_2)$ are active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

By Claim 5.4, for every optimal solution y to $\mathcal{D}_{(G,c)}$ and every multicut M of G, if M is active for y and contains e_i for some $i \in \{1, 2\}$, then $e_0 \in M$.

Let y^* be the optimal solution to $\mathcal{D}_{(G,c)}$ maximizing $\alpha^y_{v_1}$ over all y solutions to $\mathcal{D}_{(G,c)}$. We have $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$ and all the multicuts in $\mathcal{A}^{y^*}_{v_2}$ have order at most 3. Otherwise, let $M \in \mathcal{A}^{y^*}_{v_2} \setminus \mathcal{A}^{y^*}_{v_1}$ (resp. $M \in \mathcal{A}^{y^*}_{v_2}$ such that $d_M \geq 4$), and $0 < \varepsilon \leq \min\{y^*_M, y^*_{\delta(v_1)}\}$. The solution

$$y^3 = y^* - \varepsilon(\xi_M + \xi_{\delta(v_1)}) + \varepsilon(\xi_{M \setminus e_2} + \xi_{\delta(v_1) \cup e_2})$$

is optimal, but $\alpha^{y^3}_{v_1} = \alpha^{y^*}_{v_1} + \varepsilon$ by the choice of M, a contradiction to the maximality of $\alpha^{y^*}_{v_1}$. Thus, $\bar{M} = \{e_0, e_1, e_2\}$ is the only multicut in $\mathcal{A}^{y^*}_{v_2}$.

Let $\{N_1, \dots, N_m\}$ be the set of active multicuts for y^* such that $N_i \cap \{e_0, e_1, e_2\} = e_0$ for $i = 1, \dots, m$. The constraint associated with e_0 is tight by Claim 5.5, hence, by $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$, we have:

$$c_{e_0} = \alpha^{y^*}_{v_1} + y^*_{\delta(v_1)} + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{19}$$

By Statement (ii) of Claim 5.8 applied to v_1, we have $y^*_{\delta(v_1)} + \alpha^{y^*}_{v_1} = 1$, and so:

$$c_{e_0} = 1 + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{20}$$

By $\mathcal{A}^{y^*}_{v_2} = \{\bar{M}\}$ and Statement (ii) of Claim 5.8 applied to v_2, we have $y^*_{\delta(v_2)} + y^*_{\bar{M}} = 1$, hence:

$$c_{e_0} = 2 - y^*_{\bar{M}} + \sum_{i=1}^{m} y^*_{N_i}. \tag{21}$$

Since c_{e_0} is integer and since $y^*_{\bar{M}} < 1$ by Claim 5.3, by (21), we have:

$$\sum_{i=1}^{m} y^*_{N_i} \geq y^*_{\bar{M}}. \tag{22}$$

Hence, let $\lambda_1, \ldots, \lambda_m$ be such that $0 \leq \lambda_i \leq y^*_{N_i}$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \lambda_i = y^*_{\bar{M}}$. Note that $\delta(v_2) = \bar{M} \setminus e_1$. Then, the point

$$y^5 = y^* - \left(y^*_{\bar{M}} \xi_{\bar{M}} + \sum_{i=1}^{m} \lambda_i \xi_{N_i} \right) + \left(y^*_{\bar{M}} \xi_{\delta(v_2)} + \sum_{i=1}^{m} \lambda_i \xi_{N_i \cup e_1} \right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and it is optimal by definition of b. Moreover,

$$y^5_{\delta(v_2)} = y^*_{\bar{M}} + y^*_{\delta(v_2)} = 1,$$

a contradiction to Claim 5.3. □

The following claim forbids a circuit of length 3 to contain a vertex of $\widehat{V}$.

Claim 5.10 *No circuit of length 3 contains a vertex of degree 2.*

Proof Assume for a contradiction that in G there exist a vertex $v \in \widehat{V}$ and a circuit $\{e_1, e_2, e_3\}$ such that $\delta(v) = \{e_1, e_2\}$. By Lemma 2.1, a multicut contains e_3 only if it intersects $\delta(v)$. On the other hand, by Lemma 2.11 and complementary slackness, each multicut strictly containing $\delta(v)$ and active for an optimal solution contains e_3. Thus, for every optimal solution y to $\mathcal{D}_{(G,c)}$, we have:

$$\sum_{M: e_3 \in M} y_M = \sum_{M: e_1 \in M, M \neq \delta(v)} y_M + \sum_{M: e_2 \in M, M \neq \delta(v)} y_M - \alpha^y_v. \tag{23}$$

Let y^* be an optimal solution to $\mathcal{D}_{(G,c)}$. By the constraint (8a) associated with e_3, (23), and Claim 5.5, we have:

$$c_{e_3} \geq \sum_{M: e_3 \in M} y^*_M = c_{e_1} + c_{e_2} - 2y^*_{\delta(v)} - \alpha^{y^*}_v. \tag{24}$$

By Claim 5.6 and Statement (ii) of Claim 5.8, we have $2y^*_{\delta(v)} + \alpha^{y^*}_v < 2$. Thus, by (24) and $c_{e_3} \in \mathbb{Z}$, we have $c_{e_3} \geq c_{e_1} + c_{e_2} - 1$.

Define $G' = G \setminus e_3$ and $c' = c_{|(E \setminus e_3)}$. Note that for each multicut $M \in \mathcal{M}_G$, $M \setminus e_3$ is a multicut of G' with order at least d_M. Hence, y^* induces a solution to $\mathcal{D}_{(G',c')}$ of cost at least $b^\top y^*$. By minimality assumption (i), there exists an integer optimal solution y' to $\mathcal{D}_{(G',c')}$, and we have $b^\top y' \geq b^\top y^*$.

Let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of multicuts $M = \delta(V_1, \ldots, V_{d_M})$ of G' active for y' such that the endpoints of e_3 belong (resp. do not belong) to a same V_i for some $i \in \{1, \ldots, d_M\}$. For each $M \in \mathcal{M}_1$ (resp. $M \in \mathcal{M}_2$), M (resp. $M \cup e_3$) is a multicut of G with the same order. Hence,

$$y'' = \sum_{M \in \mathcal{M}_1} y'_M \xi_M + \sum_{M \in \mathcal{M}_2} y'_M \xi_{M \cup e_3}$$

is a point of $\mathbb{Z}_+^{\mathcal{M}_G}$ with $b^\top y'' = b^\top y'$. Thus, $b^\top y'' \geq b^\top y^*$, and y'' is not a solution to $\mathcal{D}_{(G,c)}$. By definition, y'' respects every constraint of $\mathcal{D}_{(G,c)}$ but the constraint (8a) associated with e_3.

By definition of y'', we have:

$$\sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M, M\neq\delta(v)} y''_M + \sum_{M:e_2\in M, M\neq\delta(v)} y''_M - \alpha_v^{y''}. \tag{25}$$

Therefore, by y'' violating the constraint (8a) associated with e_3, (25), Statement (*ii*) of Claim 5.8, and the inequalities (8a) associated with e_1 and e_2, we have:

$$c_{e_3} < \sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M} y''_M + \sum_{M:e_2\in M} y''_M - \alpha_v^{y''} - 2y''_{\delta(v)} \leq c_{e_1} + c_{e_2} - \alpha_v^{y''} - 2y''_{\delta(v)}. \tag{26}$$

Thus, by (24), we have $\alpha_v^{y''} + 2y''_{\delta(v)} < \alpha_v^{y^*} + 2y^*_{\delta(v)} < 2$. By $c_{e_3} \geq c_{e_2} + c_{e_1} - 1$, the integrality of y'', and (26), we have $\alpha_v^{y''} = y''_{\delta(v)} = 0$, and so $c_{e_3} = c_{e_1} + c_{e_2} - 1$. Hence, by the integrality of y'' and equation (25):

$$c_{e_3} + 1 = \sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M} y''_M + \sum_{M:e_2\in M} y''_M = c_{e_1} + c_{e_2}. \tag{27}$$

For $i = 1, 2$, since $c_{e_i} \geq 1$, there exists a multicut M_i active for y'' such that $M_i \cap \delta(v) = e_i$.

We claim that the constraint (8a) associated with e_3 is not tight for y^*. By $c_{e_3} = c_{e_1} + c_{e_2} - 1$, (24), and Claim 5.6, $\delta(v)$ is active for y^*. Hence, by Statement (ii) of Claim 5.8, we have:

$$\alpha_v^{y^*} + y^*_{\delta(v)} = 1. \tag{28}$$

Hence, by (23) and Claim 5.5, (28), (27), and $\delta(v)$ active for y^*, we have:

$$\sum_{M:e_3\in M} y^*_M = c_{e_1} + c_{e_2} - 1 - y^*_{\delta(v)} = c_{e_3} - y^*_{\delta(v)} < c_{e_3}. \tag{29}$$

The point y'' respects all the constraints of $\mathcal{D}_{(G,c)}$ except the one associated with e_3, and this constraint is not tight for y^*. Therefore, there exists $0 < \lambda < 1$ such that

$$\tilde{y} = \lambda y^* + (1-\lambda) y''$$

is a solution to $\mathcal{D}_{(G,c)}$. Moreover, $\tilde{y}$ is optimal because $b^\top y^* \leq b^\top y''$.

All multicuts active for at least one between y^* and y'' are active for $\tilde{y}$. Since $\delta(v)$ is active for y^* and M_1, M_2 are active for y'', the three multicuts M_1, M_2, and $\delta(v)$ are active for $\tilde{y}$, a contradiction to Claim 5.4. $\square$

Claim 5.11 *Each circuit of length 4 contains at most one vertex of degree 2.*

Proof Assume for a contradiction that there exists a circuit $C = \{e_1, \ldots, e_4\}$ in G covering two vertices of $\widehat{V}$, say v_1, v_2. By Claim 5.9, v_1 and v_2 are not adjacent, hence we assume that $\delta(v_1) = \{e_1, e_2\}$ and $\delta(v_2) = \{e_3, e_4\}$. Let v_3 and v_4 be the remaining vertices of C.

We prove that $\delta(v_1)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$. Indeed, if $\mathfrak{D}_{v_1} \neq \emptyset$, then let $y' \in \mathfrak{D}_{v_1}$ maximize $\alpha_{v_1}^z$ over all $z \in \mathfrak{D}_{v_1}$. By Statement (ii) of

Theorem 2.10, for every multicut M in $\mathcal{A}^{y'}_{v_2}$, we have $M = \delta\{v_2, V_2, \ldots, V_{d_M}\}$, with v_3 and v_4 belonging to different V_i's, hence $M \cap \delta(v_1) \neq \emptyset$. However, $M \setminus \delta(v_1)$ contains $\delta(v_2)$, a contradiction to Claim 5.7 applied to v_1. Exchanging the role of v_1 and v_2, we deduce that $\delta(v_2)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

Without loss of generality, there exists an optimal solution y such that $\alpha^y_{v_1} \geq \alpha^y_{v_2}$. Then, we can build from y an optimal solution y^* to $\mathcal{D}_{(G,c)}$ such that $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$. Indeed, suppose $\mathcal{A}^y_{v_2} \setminus \mathcal{A}^y_{v_1} = \{M_1, \ldots, M_n\}$. Then, since $\alpha^y_{v_1} \geq \alpha^y_{v_2}$, there exist $N_1, \ldots, N_m \in \mathcal{A}^y_{v_1} \setminus \mathcal{A}^y_{v_2}$ such that:

$$\sum_{i=1}^{n} y_{M_i} \leq \sum_{j=1}^{m} y_{N_j}. \tag{30}$$

Hence, there exist $\epsilon_1, \ldots, \epsilon_m$ such that $0 \leq \epsilon_j \leq y_{N_j}$, for $j = 1, \ldots, m$, and

$$\sum_{j=1}^{m} \epsilon_j = \sum_{i=1}^{n} y_{M_i}. \tag{31}$$

By Statement (ii) of Theorem 2.10 and complementary slackness, v_3 and v_4 belong to different classes of N_j for each $j = 1, \ldots, m$, implying that $N_j \cap \delta(v_2) \neq \emptyset$. Moreover, since $N_j \notin \mathcal{A}^y_{v_2}$, we have $|N_j \cap \delta(v_2)| = 1$, for all $j = 1, \ldots, m$. Furthermore, since $\delta(v_2)$ is active for y, by Claim 5.4, there exists an edge in $\delta(v_2)$, say e_3, such that $N_j \cap \delta(v_2) = e_3$ for all $j = 1, \ldots, m$. Therefore, the point

$$y^* = y - \left(\sum_{i=1}^{n} y_{M_i}\xi_{M_i} - \sum_{i=1}^{n} y_{M_i}\xi_{M_i\setminus e_4}\right) + \left(\sum_{j=1}^{m} \epsilon_j\xi_{N_j\cup e_4} - \sum_{j=1}^{m} \epsilon_j\xi_{N_j}\right) \tag{32}$$

is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y$ and $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$. Let $\mathcal{A}^{y^*}_{v_2} = \{M'_1 \ldots, M'_p\}$. For $i = 1, \ldots, p$, since $M'_i \in \mathcal{A}^{y^*}_{v_1}$, Statement (ii) of Theorem 2.10 implies $M'_i = \delta(v_1, v_2, V^i_3, V^i_4, \ldots, V^i_{d_{M'_i}})$, where V^i_3 and V^i_4 contain respectively v_3 and v_4. Then, $M''_i = \delta(v_1, v_2 \cup V^i_3 \cup V^i_4, \ldots, V^i_{d_{M'_i}})$ is a multicut of order $d_{M'_i} - 2$ for $i = 1, \ldots, p$. Since $\delta(v_2)$ is active for y^*, by Statement (ii) of Claim 5.8, we have $\alpha^{y^*}_{v_2} + y^*_{\delta(v_2)} = 1$. Then, the point $y^1 \in \mathbb{Q}^{\mathcal{M}_G}$ defined by:

$$y^1 = y^* - \left(y^*_{\delta(v_2)}\xi_{\delta(v_2)} + \sum_{i=1}^{p} y^*_{M'_i}\xi_{M'_i}\right) + \left(\sum_{i=1}^{p} y^*_{M'_i}\xi_{M''_i}\right),$$

is a solution to $\mathcal{D}_{(G,c')}$, where $c' = c - \chi^{\delta(v_2)}$.

By $d_{M''_i} = d_{M'_i} - 2$ for all $i = 1, \ldots, p$, and $\alpha^{y^*}_{v_2} + y^*_{\delta(v_2)} = 1$, we have:

$$b^\top y^1 = b^\top y^* - \alpha^{y^*}_{v_2}(2h+2) - y^*_{\delta(v_2)}(2h+1) = b^\top y^* - (2h+1) - \alpha^{y^*}_{v_2}. \tag{33}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. The point $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ defined by $y^3 = y^2 + \xi_{\delta(v_2)}$ is a solution to $\mathcal{D}_{(G,c)}$ such that:

$$b^\top y^3 = b^\top y^2 + 2h + 1. \tag{34}$$

Therefore, by (33), the optimality of y^2, and (34), we have:

$$b^\top y^* = b^\top y^1 + 2h + 1 + \alpha_{v_2}^{y^*} \leq b^\top y^2 + 2h + 1 + \alpha_{v_2}^{y^*} = b^\top y^3 + \alpha_{v_2}^{y^*}. \tag{35}$$

By integrality of $P_{2h+1}(G)$ and duality, we have $b^\top y^* \in \mathbb{Z}$. Furthermore, y^3 is integer by construction, so $b^\top y^3 \in \mathbb{Z}$. Then, by (35) and Claim 5.6, we have $b^\top y^* \leq b^\top y^3$, and so y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. □

Claims 5.1, 5.9, 5.10, 5.11 and Lemma 2.3 imply that G is not series-parallel, a contradiction. □

The box-TDIness of $P_k(G)$ and the TDIness of System (2) give the following result.

Corollary 5.2 *System* (2) *is box-TDI if and only if* G *is series-parallel.*

Proof By Theorem 5.1, when G is not series-parallel, System (2) is not TDI. Whenever G is series-parallel, $P_k(G)$ is box-TDI by Theorem 3.2 and System (2) is TDI by Theorem 5.1. □

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