# Substitutions on Stepped Surfaces

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# Introduction (1/3): Sturmian words

word: concatenation of letters (finite alphabet);

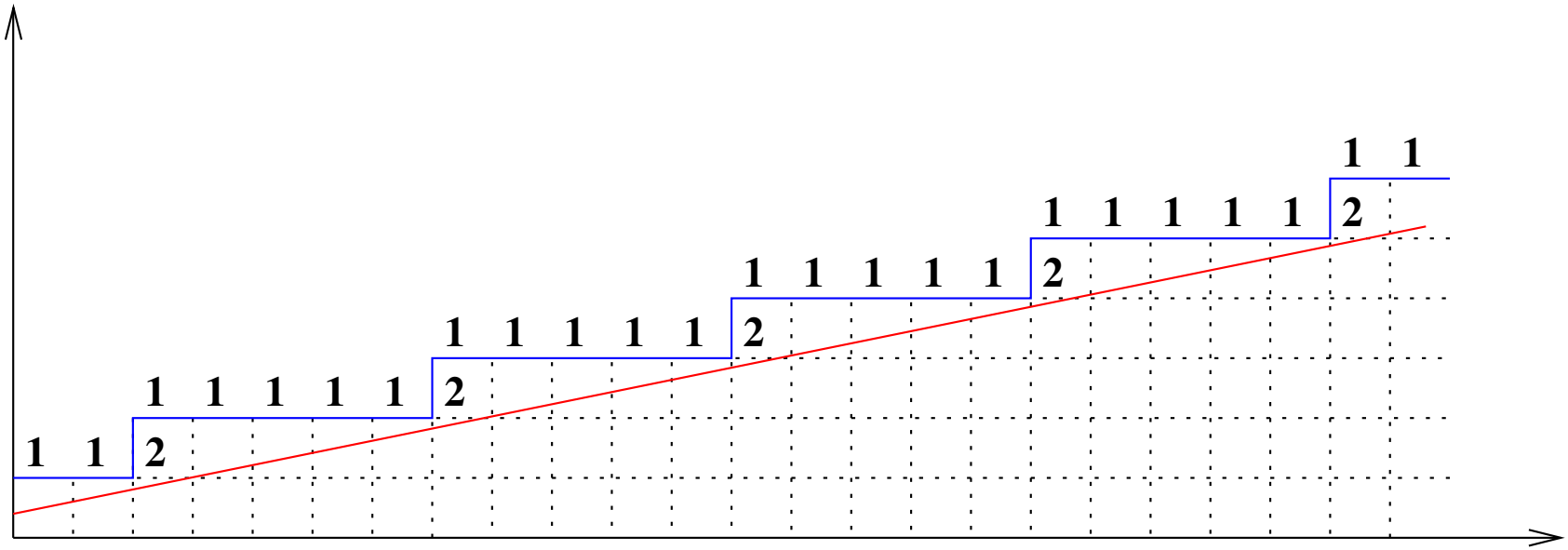
complexity: number $p(n)$ of factors of size $n$;

Sturmian words: aperiodic words of minimal complexity.

$$u = 121121121121121121121121\ldots \rightsquigarrow p(n) = n + 1.$$

# Introduction (2/3): Stepped lines

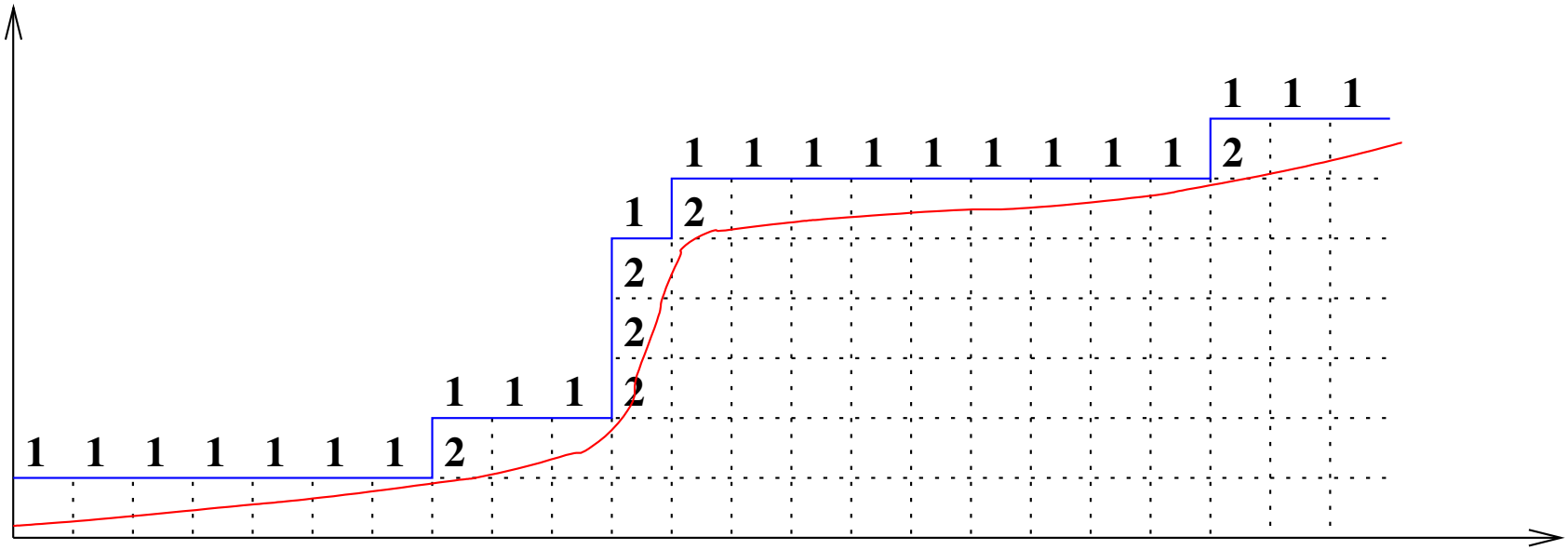
Straight half-line (red) $\rightsquigarrow$ stepped line (blue) $\rightsquigarrow$ 2-letter word:




Morse&Hedlund: Sturmian words $\equiv$ irrational slopes

# Introduction (3/3): Stepped curves

funct. curve (red) $\rightsquigarrow$ stepped curve (blue) $\equiv$ 2-letter word:




Sturmian words: aperiodic stepped curves of minimal complexity

# Outline

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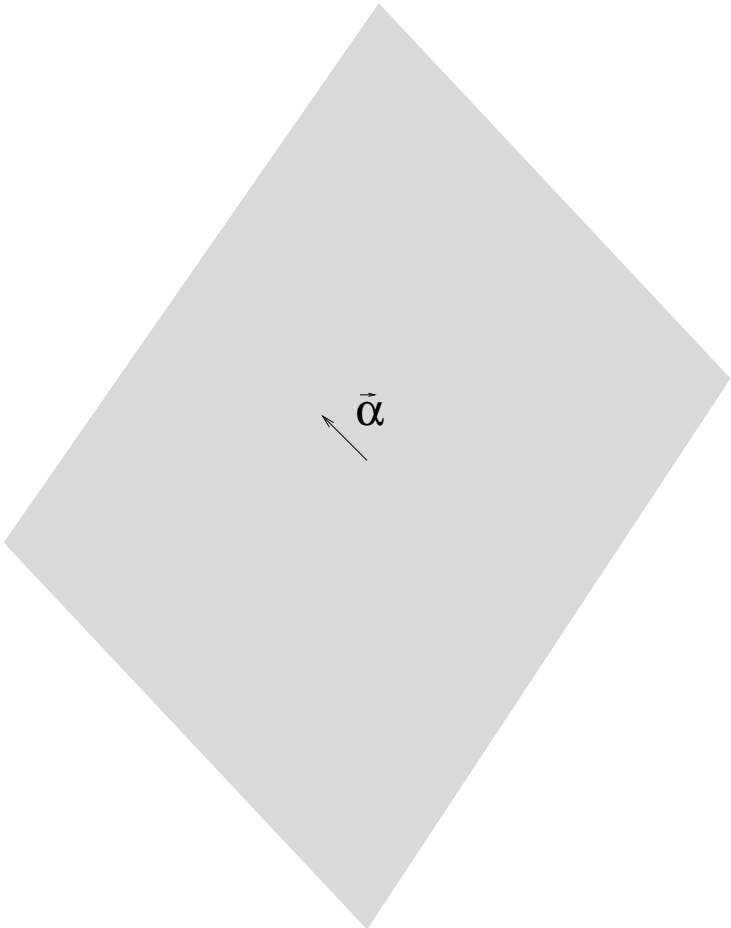
- real plane normal to $\vec{\alpha}$
- *stepped plane*
- 3-letter 2-dim. word



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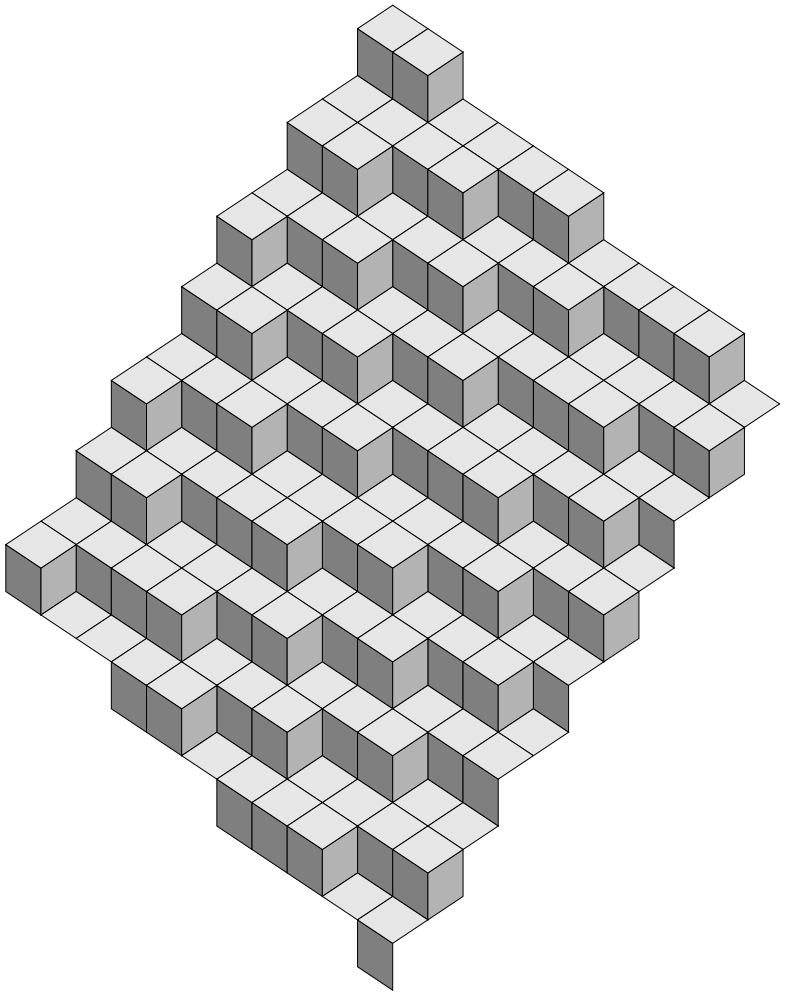

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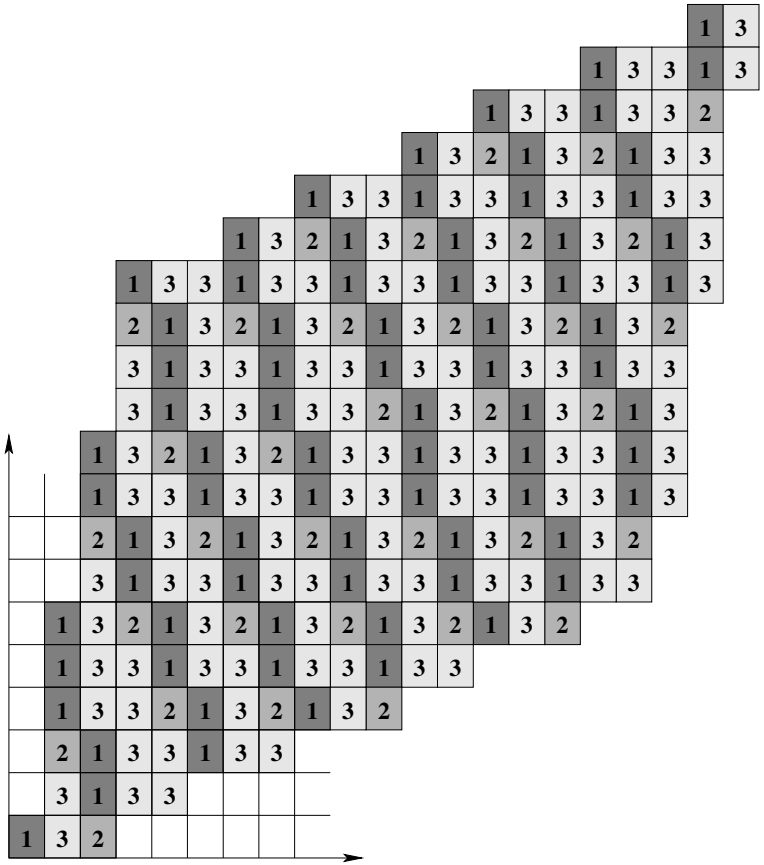


# Outline

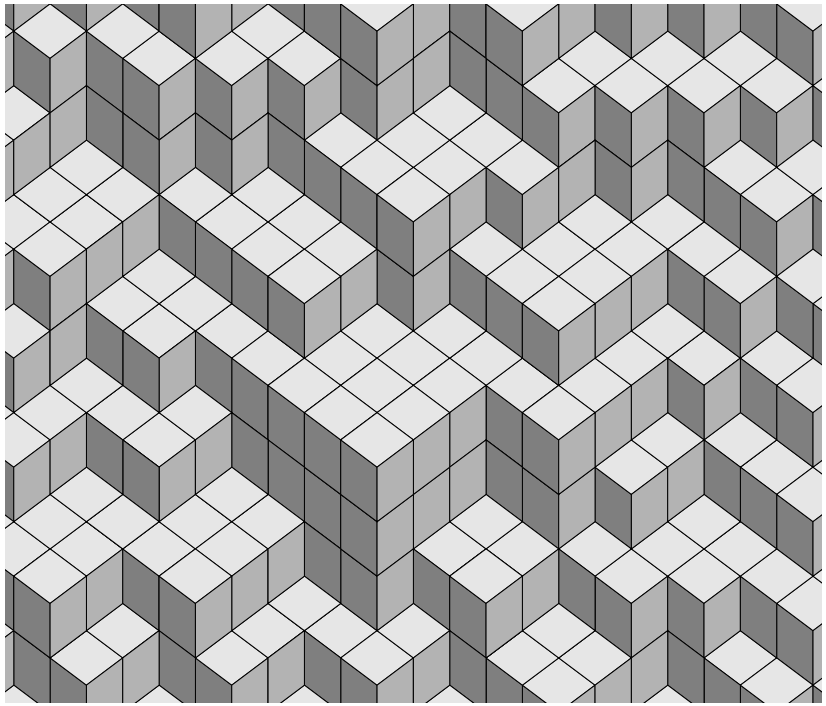
Recall: aperiodic digitizations of lines $\equiv$ Sturmian words.

**Definition (Vuillon,98)**

Sturmian 2-dim. words $\equiv$ aperiodic digitizations of planes

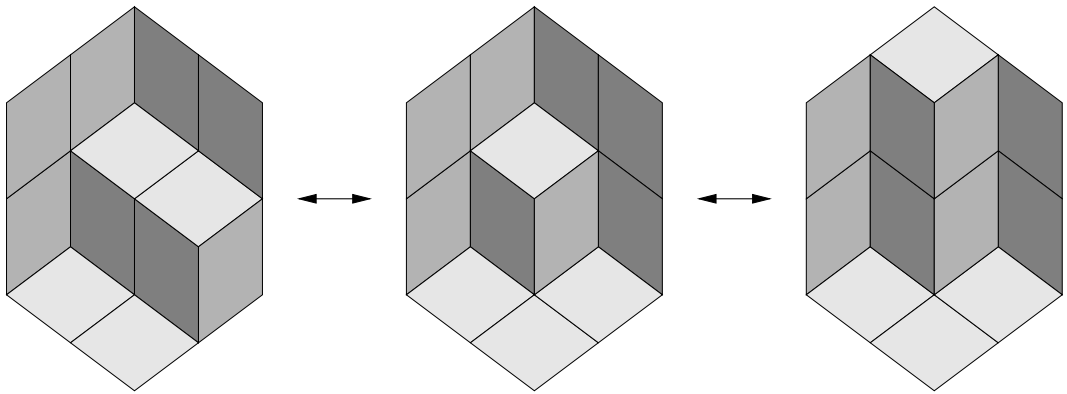
# Outline

funct. surface $\rightsquigarrow$ stepped surface $\rightsquigarrow$ 3-letter 2-dim. word (not all):



# Outline

Flip: adding/removing a unit cube



**Theorem**

*Stepped surface* $\equiv$ *stepped plane* $+$ *flips.*

# Outline

substitution: non-erasing morphism: $\sigma(u\cdot v) = \sigma(u)\cdot\sigma(v)$;

Sturmian substitution: maps Sturmian words to Sturmian words.

$$\sigma : \begin{array}{l} 1 \to 12 \\ 2 \to 1 \end{array} \quad \rightsquigarrow \quad \sigma(12112\ldots) = 12112121\ldots$$

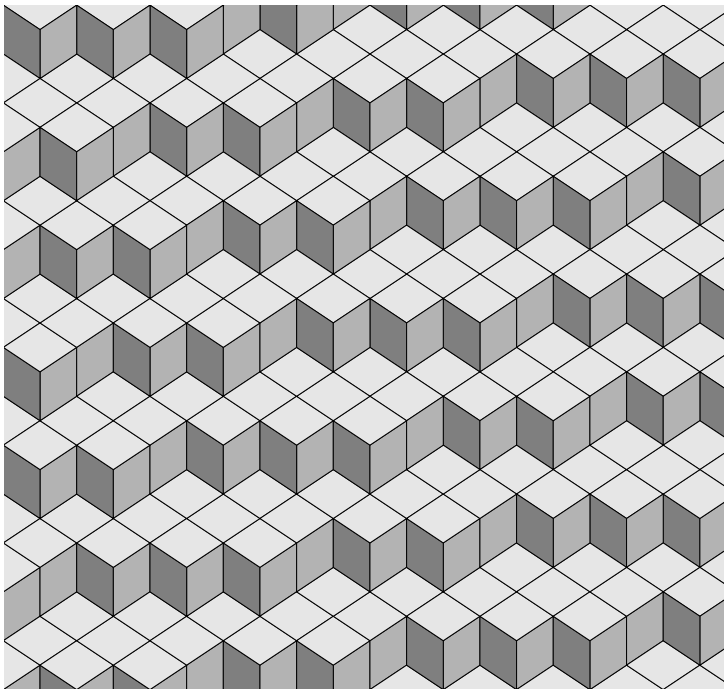
$\rightsquigarrow$ useful for *generating* and *classifying*

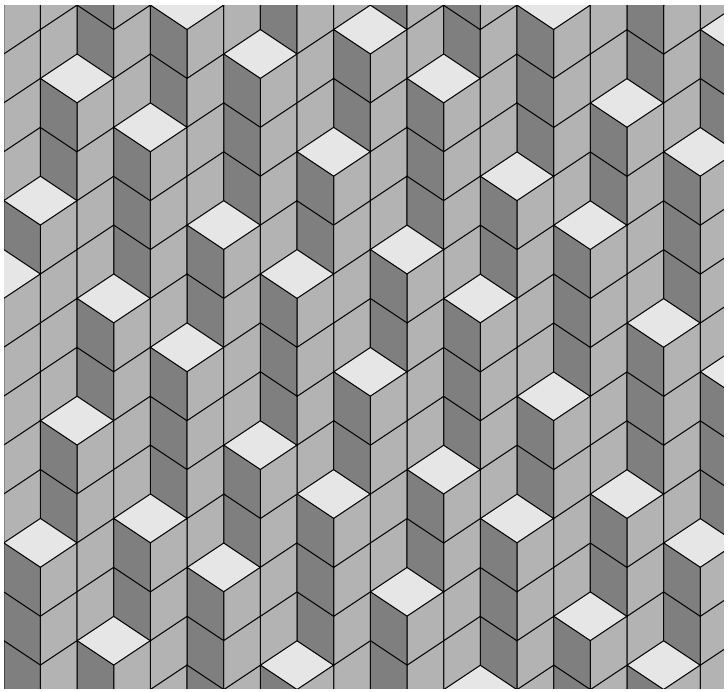
# Outline

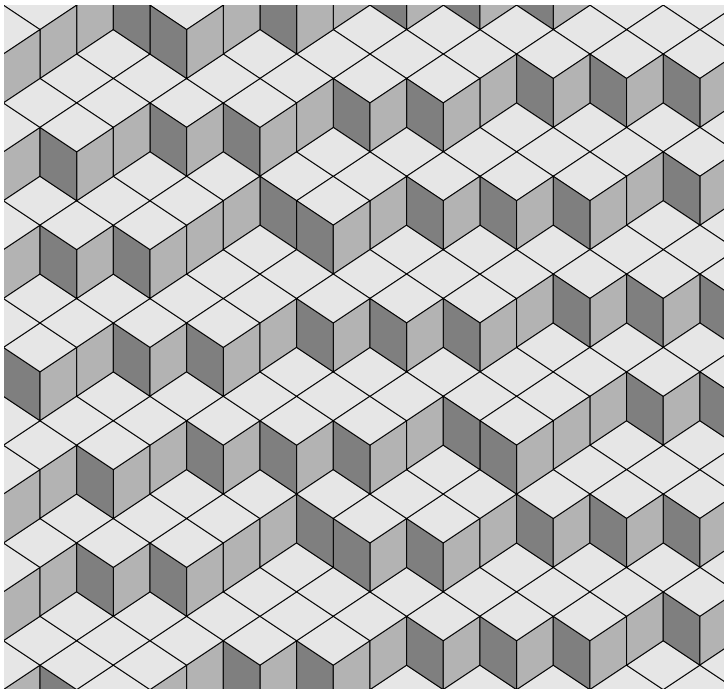
Generalized substitution: map on unit faces of $\mathbb{R}^3$ (Arnoux-Ito).

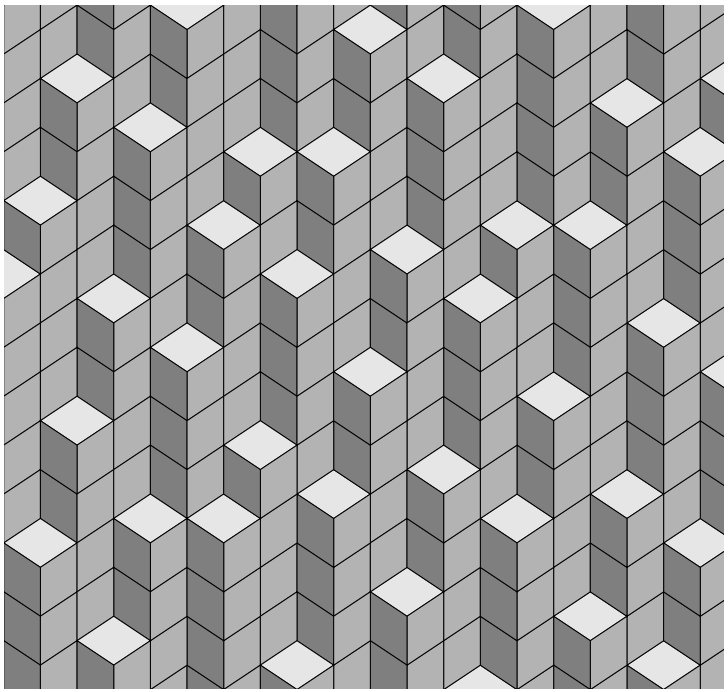
**Theorem**

*Generalized substitions map stepped surfaces to stepped surfaces and stepped planes to stepped planes.*









# Conclusion

| | | |
|---|---|---|
| 2-letter words | $\longrightarrow$ | stepped surfaces |
| Sturmian words | $\longrightarrow$ | aperiodic stepped planes |
| Sturmian substitutions | $\longrightarrow$ | generalized substitutions |
| complexity $n+1$ | $\longrightarrow$ | ? |
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Penrose stepped surfaces?

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