Study of the NP-completeness of the Compact Table problem
NP-completeness comes to wargaming

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Journées Automates Cellulaires 2008
Outline

1. **Compact table problem**
   - Random-choices tables
   - Formal description
   - Other applications

2. **NP-Completeness**
   - General case
   - Fixed amplitude case
   - Bounded number of results case

3. **Conclusion and perspectives**
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3. Conclusion and perspectives
Random tables

- Set of initial conditions
- Finite number of results
- → 2-D table:

<table>
<thead>
<tr>
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<td>δ</td>
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<td>δ</td>
</tr>
</tbody>
</table>

- Dimension reduction: A: +0, B: +10, C: +20, D: +30

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| α | α | α | β | β | β | β | γ | γ | γ | α | β | β | β | γ | γ | δ | δ | δ | δ | δ |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| α | γ | γ | γ | γ | γ | γ | δ | δ | δ | α | α | β | β | γ | γ | γ | δ | δ | δ |

- Tiny font, because very long!
Some lines may overlap partially...

Known problem! This is the superword problem. NP-complete. But here, zero overlap!

However, we can also shuffle around the lines...

Example

\[ \alpha \beta \beta \beta \alpha \alpha \gamma \gamma \gamma \]

- A: \(+6 \rightarrow \alpha \alpha \alpha \beta \beta \beta \beta \gamma \gamma \gamma \)
- B: \(+0 \rightarrow \alpha \beta \beta \beta \gamma \gamma \delta \delta \delta \delta \)
- C: \(+12 \rightarrow \alpha \gamma \gamma \gamma \gamma \gamma \gamma \delta \delta \delta \delta \)
- D: \(+9 \rightarrow \alpha \alpha \beta \beta \gamma \gamma \gamma \gamma \delta \delta \)

J.-C. Dubacq, J.-Y. Moyen
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Compact tables
Shortening the information

- Some lines may overlap partially...
- Known problem! This is the superword problem. NP-complete. But here, zero overlap!
- However, we can also shuffle around the lines...

Example

\[
\delta\delta\delta\delta\gamma\gamma\alpha\beta\beta\beta\alpha\alpha\gamma\gamma\gamma\delta\delta\gamma\gamma
\]

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- B: +0 → αββγγδδδδ
- C: +12 → αγγγγγγδδδ
- D: +9 → ααββγγγγδδ
Initial motivation
Real wargames, no computers involved

- Dimension reduction is crucial (easily-readable tables)
- Size of reduced table important (easily-learnable tables)
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Initial motivation
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- Dimension reduction is crucial (easily-readable tables)
- Size of reduced table important (easily-learnable tables)
If the set of initial conditions matches the set of outcomes, we get a probabilistic automaton. Efficient representation of probabilistic automata.
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2. NP-Completeness
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3. Conclusion and perspectives
### Compact Table problem

**Instance**  
Alphabet \( \Sigma \), integer \( \ell \), set of words \( S \subseteq \Sigma^\ell \), integer \( k \)

**Answer**  
YES if there exists a word \( \tau \in \Sigma^k \) such that for any word \( u \in S \), there exists a permutation \( \sigma \) and words \( v \) and \( w \) such that \( \tau = v \cdot \sigma(u) \cdot w \), NO in all other cases.

### Compact table of order \( \ell \)

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## Formal Descriptions of Decision Problems

### Compact Table problem

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3 Conclusion and perspectives
DNA single strand analysis

- Weights of A, C, G and T molecules are different
- Replicate an unknown DNA strand, cut it in small pieces
- Centrifugate and weight each small piece
- Infer the ACGT percentages
- Reconstruct the shortest possible single-strand DNA sequence possible with CT.
- Will not work, since it is NP-complete.
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References:
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Compact Table problem
Theorem

The Hamiltonian Path problem can be reduced to the Compact Table problem. Thus, the Compact Table problem is NP-complete.

Proof.
We define Σ to be the set $E \cup V$. Each vertex $v$ is associated to a word $\tau_v$ of $\Sigma^\ell$ which is the set of edges adjacent to $v$ (in no particular order) and padded (since $G$ is not forced to be regular) by as many occurrences of $v$ as deemed necessary. $k$ is determined to be $n(\ell - 1) + 1$. Being in NP is straightforward.
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Example of construction

Hamiltonian path $ABCDEFG$ corresponding to the word (of length $n(\ell - 1) + 1 = 22$)

$$\tau = adbBBcdheDaEgEEhijGGf$$

$\tau_A = abdf$
$\tau_B = BBbc$
$\tau_C = cdeh$
$\tau_D = Daeg$
$\tau_E = EEgi$
$\tau_F = Fhij$
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NP-Completeness proof

**HP exists → CT exists**

Along the Hamiltonian path, edges can be collapsed, yields a word of length \( n(\ell - 1) + 1 \). *An edge is never used twice!*

**CT exists → HP exists**

Overlap only on edges, so \( \tau \) describes a path in \( G \). Because of length constraints, the path goes only once through each vertex.

**#P-completeness remark**

Transformation is not parcimonious (many possible permutations). But given an instance, number of solutions is either 0 or \((\ell - 1)^2 \prod_{1 \leq i \leq n} \frac{(\ell - 2)!}{(\ell - d(i))!}\).

Therefore, the problem is also #P-complete, even though the reduction is not (and probably cannot) be parcimonious.
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3 Conclusion and perspectives
Amplitude $\ell > 2$
Work is already done

**Theorem**

*The Compact Table problem of order $\ell > 2$ is NP-complete.*

**Proof.**

Our reduction reduces HP of degree $\ell$ to CT of order $\ell$. Since HP is still NP-complete with degree 3, done.
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The Compact Table problem of order $\ell = 2$ is in P.

Proof.

Consider the results as vertices, initial conditions are edges. One can see easily that giving the smallest word containing all lines of the table is akin to describe a graph containing all edges of the graph. Details about unconnected components are in the paper.
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**Theorem**

*CTP is solvable in linear time in case there are only two possible results.*

**Proof.**

Use a sequence of $m_1$ times the first result “0” followed by $m_2$ times the second result “1”, where $m_1$ is the largest number of “0” for any initial condition and $m_2$ is the largest number of “1”.

- Limited amplitude+limited outcomes, trivial (finite number of words).
- 2 results: CT method not efficient (prob. success enough)
2-results case
Everything is so easy now

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3-results case
Unfinished fun on triangles

Number of possible words of amplitude \( \ell \): \( \left( \frac{\ell + k - 1}{\ell} \right) \).

Proof.
Number of occurrences of each result, including 0, in bijection with words on \( \{x, y\} \) of length \( \ell + k - 1 \) with \( k - 1 \) \( y \) letters separating runs of \( x \) (run \( i \) is the number of occurrence of result \( i \)).

Superword of size \( \left( \frac{\ell + k - 1}{\ell} \right) + \ell - 1 \) containing all permutations? \( \rightarrow \) Open problem.

- 3 outputs, amplitude 1: \( abc \).
- Amplitude 2: \( caabbcc \).
- \( \ell = 3 \): \( abcccaaabbcc \).
- \( \ell = 4 \): \( abacbcbccccaaabbbbc \).
- Recurrence? 4-results? Beyond?
Number of possible words of amplitude $\ell$: $\binom{\ell + k - 1}{\ell}$.

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Summary and Open problems
Stuff we couldn’t do in time

### Answered Questions
- Works even if words of different lengths;
- Permutations do not help;
- They may even make things harder;
- Some things remain simple.

### Open Questions
- The superword problem is known to be NP-hard but approximable;
- For Compact table: not clear. Heuristics may apply, but ratio is not a constant.
- Restriction to 3 or more results: still open. 3 results may be possible (winding out from the inside to the outside).
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This slide intentionally left blank
Some details on proof in case $\ell = 2$
You probably asked for it!

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<tr>
<th></th>
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<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
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<td>$F$</td>
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<tr>
<td>$G$</td>
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\[
\begin{tikzpicture}
  \node (A) at (0,0) {$\alpha$};
  \node (B) at (2,0) {$B$};
  \node (C) at (2,-2) {$C$};
  \node (D) at (0,-2) {$D$};
  \node (E) at (1,-1) {$E$};
  \node (F) at (-1,-1) {$F$};
  \node (G) at (0,-3) {$G$};

  \draw (A) -- (B);
  \draw (B) -- (C);
  \draw (C) -- (D);
  \draw (D) -- (A);
  \draw (A) -- (E);
  \draw (B) -- (E);
  \draw (C) -- (E);
  \draw (D) -- (E);
  \draw (E) -- (F);
  \draw (E) -- (G);
\end{tikzpicture}
\]
We separate in $A$ (connected components with vertices of odd degree) and the other ones ($B$). We want to reach $(a, b, n/2) = (1, 0, 1)$.

$\alpha$ Adding one edge going from one component to itself: either $[0, 0, 1]$ between two even vertices, $[0, 0, 0]$ between an even vertex and an odd vertex, $[0, 0, -1]$ between two odd vertices. There is a special case for the last one: the move could also be $[-1, 1, -1]$.

$\beta$ Adding one edge between two components of $A$: $[-1, 0, 1]$ between two even vertices, $[-1, 0, 0]$ between an even vertex and an odd vertex, $[-1, 0, -1]$ between two odd vertices.
Some details on proof in case $\ell = 2$ (cont.)

- Adding one edge between one component of $A$ and one of $B$: $[0, -1, 1]$ if the vertex in the component in $A$ was of even degree, $[0, -1, 0]$ otherwise. There is always an even number of odd-degree vertices in a component, so $a$ never decreases this way.

- Adding one edge between two components of $B$: $[1, -2, 1]$ (always).

  - If $a = 0$, then $n = 0$ and $b > 1$. The transformation $\delta \gamma^{b-2}$ leads us to the final state and is of length $b + n - 1 = b - 1$.
  
  - If $a > 0$, then transformation $\beta^{a-1} \gamma^b \alpha^{n-a}$ leads us to the final state and is of length $b + n - 1$.

  In each case, there is only one subcase that decreases $b + n$; there may be some choice for the exact edge to be added.