Partitionning smartgrids into self-sufficient sets

J. David R. Grappe M. Lacroix E. Traversi

Laboratoire d'Informatique de Paris Nord, Université de Paris 13, Sorbonne Paris Cité, 99, Avenue J.-B. Clement 93430 Villetaneuse, France.

Abstract

We consider the problem of partitionning a network into microgrids. We mainly focus on the complexity results related to the different variants of this problem.

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The electric landscape in France is in deep mutation. The electric production is changing, moving from a small number of production plants with high electric power to a huge number of production units, each delivering a small electric power. From a legal point of view, it is now possible since 2017 to gather consumers and producers in a private local network called *microgrid*. In such microgrids, the consumers use the electricity generated by the producers belonging to this microgrid. The only electric exchange between a microgrid and the outside is the one necessary to obtain the equilibrium between electric consumption and production of the whole microgrid.

The merging of consumers and producers geographically close into microgrids presents several advantages. From an energetic aspect, the transportation of the electricity from producers which are geographically close to the

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consumers is more efficient. Also, there may exist local rules regulating the production of electricity such as 100% of renewable energy. Moreover, the partition of the whole electric network into microgrids tends to reduce the effects of electric problems such as blackouts. Indeed, when a problem locally appears in a microgrid, this latter can be more easily disconnected from the rest of the network stopping the propagation of the electric problem.

A microgrid is interesting only if the exchange between this microgrid and the outside is very small. This implies that ideally the local production should correspond to the local consumption. Moreover, the electric network of the microgrid must be sufficient to ensure the transportation of the electricity from the producers to the consumers. We consider in this work the problem of partitionning a network into microgrids under the previous requirements. We mainly focus on the complexity results related to the different variants of this problem.

We model the electrical network (smartgrid) with a graph whose vertices represent the producers/consumers/relays and whose edges are the electrical connexions. Accordingly, we associate to each vertex a weight which corresponds to its consumption or production. We think of microgrids as connected subgraphs of the smartgrid, and call a microgrid *self-sufficient* if its own production matches its consumption.

The main problem we are interested in consists in partitionning the smartgrid into a number of self-sufficient microgrids. Since this problem turns out to be NP-complete even in very restricted classes of graphs, we will also consider a variant called combinatorial in which the production and consumption are all unitary. In this variant, we may forget the weights on the vertices and, instead, have red vertices representing the producers and blue vertices for the consumers.

The second problem we will study asks one to find a self-sufficient microgrid containing a given subset of the vertex set. The weighted version of this problem is already NP-complete when the graph is a star. Consequently, we focus on the combinatorial version of the self-sufficient augmentation problem. We prove that it is close to the so-called graph motif problem. Consequently, it is NP-hard in general and polynomial for graphs of fixed treewidth.

1 Complexity results for self-sufficient partitions

In this section, we prove that the problem of partitioning a series-parallel graph into a fixed number of self-sufficient sets is NP-complete but becomes polynomial-time solvable in outerplanar graphs. Outerplanar graphs being a large subclass of series-parallel graphs, these two results, Theorems 1.1 and 1.2, establish the complexity behaviour of the SELF-SUFFICIENT PARTITION problem.

We now formally describe the SELF-SUFFICIENT PARTITION problem. Given a graph $G = (V, E)$ and $w : V \to \mathbb{Z}$, a subset of vertices X of V is called *self-suficient* if the graph $G[X]$ which X induces is connected and its own consumption matches its production, that is, $w(X) = 0$.

SELF-SUFFICIENT PARTITION:

Instance: A graph $G = (V, E)$, $w : V \to \mathbb{Z}$ and $k \in \mathbb{Z}_+$.

Solution: A partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of V such that P_i is self-sufficient for all $i = 1, \ldots, k$.

1.1 NP-completeness

In this section, we prove the NP-completeness of SELF-SUFFICIENT PARTITION using the well-known NP-complete problem PARTITION [2].

PARTITION:

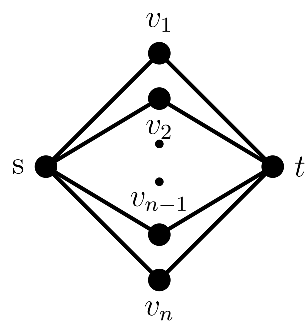
Instance: A multiset of positive integers $p_1, \ldots, p_n$.

Solution: A partition of $\{1, \ldots, n\}$ into two subsets S_1 and S_2 such that $\sum_{i \in S_1} p_i = \sum_{i \in S_2} p_i$.

A graph is *series-parallel* if it does not contain K_4 as a minor. A *cut vertex* is a vertex whose removal yiealds at least two connected components. When no removal of a single vertex disconnects a graph, the latter is said 2-connected. Loops and bridges are called trivial 2-connected graphs. The non trivial 2-connected components of a graph are the maximal 2-connected subgraphs of the graph, i.e., the components obtained after removing the loops and bridges. Seriesparallel graph admit the following constructive characterization: a graph is series-parallel if all its non trivial 2-connected components can be built, starting from the circuit of length two, by repeatedly applying the following operations: add a parallel edge to an existing edge; or subdivide an existing edge, that is replace the edge by a path of length two.

Theorem 1.1 SELF-SUFFICIENT PARTITION *is NP-complete even if* $k = 2$ *and* G *is a* 2*-connected series-parallel graph.*

Proof. We reduce the problem to PARTITION. Let $p_1, \ldots, p_n$ be an instance of Partition and define $q = \frac{1}{2}\sum_{i=1}^{n} p_i$. Let $G = (V, E)$ be the graph with $n+2$ vertices $s, t, v_1, \ldots, v_n$ and the $2n$ edges sv_i and v_it for $i = 1, \ldots, n$. Note that G is series-parallel. Let $w(s) = w(t) = -q$ and $w(v_i) = p_i$.



Let $\{P_1, P_2\}$ be a partition of V. By construction, if s and t belong to the same P_i, then $G[V \setminus P_i]$ is not connected. Hence, without loss of generality, if $G[P_1]$ and $G[P_2]$ are both connected, then $s \in P_1$ and $t \in P_2$. Therefore, there exists $I \subseteq \{1, \ldots, n\}$ such that $P_1 = \{s, v_i : i \in I\}$ and $P_2 = \{t, v_j : j \notin I\}$. Suppose that both P_i's are self-sufficient. Then, since $w(P_1) = w(P_2) = 0$ and $w(s) = w(t) = -q$, we have $w(\{v_i : i \in I\}) = \sum_{i \in I} p_i = q$ and $w(\{v_j : j \notin I\}) = \sum_{i \notin I} p_i = q$. Therefore, $\{I, V \setminus I\}$ is a solution of Partition. Conversely, note that if S_1, S_2 is a solution of Partition, then $\{s, v_i = i \in S_1\}$ and $\{t, v_j = j \in S_2\}$ are both self-sufficient and form a partition of V, hence are a solution of Self-sufficient partition. □

1.2 Polynomial cases

In this section, we prove that if the desired number of self-sufficient sets of the partition is fixed, and if the graph is outerplanar, then the problem of self-sufficient partition can be solved in polynomial time. A graph is *outerplanar* if it can be drawn on the plane so that all its vertices belong to the external face. Equivalently, a graph is outerplanar if it contains neither K_4 nor $K_{2,3}$ as a minor. Recall that series-parallel graphs are the graphs with no K_4-minor. Therefore, the following result and Theorem 1.1 establish the complexity boundary of the self-sufficient partition problem.

Theorem 1.2 *If G is 2-connected outerplanar and k is fixed, then* Self-sufficient partition *is polynomial-time solvable.*

Proof. Let G be outerplanar and 2-connected and k be fixed. The goal is to find a partition of G into k self-sufficient sets of vertices. Let C be the cycle which forms the external face of G. By the following claim, enumeration gives an algorithm that runs in $O(n^{2k})$.

Claim 1.3 *If $\mathcal{P}$ is a self-sufficient partition of G, then there exists $P \in \mathcal{P}$ such that the vertices of P form a subpath of C.*

Proof. Suppose not, then there exists distinct P and Q in $\mathcal{P}$ such that C traverses, in this order, a set of vertices X_P of P, a set of vertices X_Q of Q, a set of vertices Y_P of P, and then a set of vertices Y_Q of Q. Since both $G[P]$ and $G[Q]$ are connected, we may assume that $G[X_P \cup Y_P]$ and $G[X_Q \cup Y_Q]$ contain an edge e_P and e_Q, respectively. But then e_P and e_Q are crossing, a contradiction to the fact that G is outerplanar with external face C. □

Now, note that there are $\binom{n}{2}$ subpaths of C. Let P be the vertex set of such a path and let $G' = G \setminus P$. The addition of P to any partition of G' into $k-1$ self-sufficient sets yields a partition of G into k self-sufficient sets. Since reapeating this process decreases k by one, and since there are at most $\binom{n}{2} \leq n^2$ subpaths at each step, all the solutions are enumerated in less than $(n^2)^k$ operations. In particular, if k is fixed, then this is polynomial in n. □

2 Self-sufficient augmentation

2.1 Weighted version

We now prove that the following problem of SELF-SUFFICIENT AUGMENTATION is NP-complete by reducing SUBSET SUM to it. SUBSET SUM is a well-known NP-complete problem, see [2].

SELF-SUFFICIENT AUGMENTATION:
Instance: A graph $G = (V, E)$, $w : V \to \mathbb{Z}$ and $X \subseteq V$.

Solution: A self-sufficient subset Y of V which contains X.

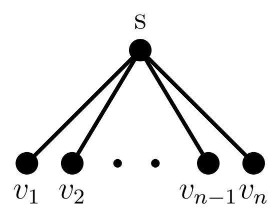
SUBSET SUM:
Instance: A multiset of integers $p_1, \dots, p_n$ and $q \in \mathbb{Z}$.

Solution: A subset I of $\{1, \dots, n\}$ such that $\sum_{i \in I} p_i = q$.

Theorem 2.1 SELF-SUFFICIENT AUGMENTATION *is NP-complete even if G is a star.*

Proof. We reduce the problem to SUBSET SUM. Let $p_1, \dots, p_n$ and q be an instance of SUBSET SUM. Wlog, we assume that p_i is nonzero for $i = 1, \dots, n$.

Let $G = (V, E)$ be the graph with $n+1$ vertices $s, v_1, \ldots, v_n$ and the n edges sv_i for $i = 1, \ldots, n$. Note that G is a star. Define w by $w(s) = -q$ and $w(v_i) = p_i$, and let $X = \{s\}$.



First, note that if $\sum i \in I p_i = q$ for some subset I of $\{1, \ldots, n\}$, then $X = \{s, v_i : i \in I\}$ is self-sufficient. Conversely, let us show that any self-sufficient subset of V which contains s induces a solution of SUBSET SUM. By construction, if $G[X]$ is connected for a subset X of V with $s \in X$, then X is a set of the form $X = \{s, v_i : i \in I\}$ for some subset I of $\{1, \ldots, n\}$. If moreover $w(X) = 0$, then $w(X \setminus \{s\}) = \sum_{i \in I} p_i = -w(s) = q$. Therefore, if X is a solution of SELF-SUFFICIENT AUGMENTATION, then the index set of the vertices of $X \setminus \{s\}$ is a solution of SUBSET SUM. □

2.2 Combinatorial version

Given the difficulty of the SELF-SUFFICIENT AUGMENTATION problem, shown in Theorem 2.1, we now consider the combinatorial variant of this problem. In this version, the vertices of the graph are either producers or consumers, and their production/consumption is not taken into acount: the goal of a microgrid is to have as many producers as consumers.

A *bicolored graph* is a pair (G, π) where $G = (V, E)$ is an undirected graph and $\pi = \{V_1, V_2\}$ is a bipartition of V representing the color of each vertex. The vertices of V_1 are colored in red and those of V_2 in blue. A subgraph $G' = (V', E')$ of a bicolored graph (G, π) is *self-sufficient* if it is connected and V' contains the same number of vertices of each class of π. This is the special case of the weighted version of self-sufficiency when w takes only $+1$ and -1 values.

MINIMUM SELF-SUFFICIENT PROBLEM:

Instance: Given a bicolored graph (G, π), a subset W of vertex.

Solution: A subgraph $G' = (V', E')$ of G such that V' is of minimum size and contains W.

GRAPH MOTIF PROBLEM:

Instance: A colored graph $G = (V, E)$, a multiset of colors M.

Solution: A subset $X \subseteq V$ which induces a connected graph and whose multiset of colors equals M.

2.2.1 Positive results

The minimum self-sufficient problem reduces to the graph motif problem, as shown in the following proposition.

Proposition 2.2 *The minimum self-sufficient problem reduces to the graph motif problem.*

Proof. Let (G, π) be a bicolored graph with $G = (V, E)$ and $\pi = \{V_1, V_2\}$. (G, π) and a vertex subset $W \subseteq V$ define an instance of the minimum self-sufficient problem. The associated decision problem is to determine, given a positive integer k, whether there exists a self-sufficient subgraph $G' = (V', E')$ of G such that $W \subseteq V'$ and $|V'| \leq k$. We suppose, wlog that $|W \cap V_1| \leq |W \cap V_2|$ and set $d = |W \cap V_2| - |W \cap V_1|$. Any self-sufficient subgraph will contain at least $|W| + d$ vertices. Let $\ell = \lfloor \frac{k-|W|-d}{2} \rfloor$. Any self-sufficient subgraph will contain $|W| + d + 2j$ vertices for some $j \in \{0, \dots, \ell\}$.

We define an instance $(\tilde{G}, \tilde{M})$ of the graph motif problem as follows. Consider $\ell + 1$ copies of G to define $\tilde{G}$. Its vertices are colored in blue and red according to the color of the vertices in G. Moreover, in each copy, all the vertices of $W$are colored in a new color, say green. Let v_0 be a vertex of W. For the j^{th} copy of G ($j \in \{0, \dots, \ell\}$), add the path $P_j = v, u_1, w_1, \dots, u_{\ell-j}, w_{\ell-j}, v_1$ where all the vertices but v are new vertices, v being v_0 in this copy of G. the vertices $u_1, \dots, u_j$ are colored in red, $w_1, \dots, w_j$ in blue and v_1 in green.

Let $\tilde{M}$ be the multiset of colors defined as follows. It contains $|W| + 1$ times the green color, $\ell + d$ times the red color and ℓ times the blue color.

Now, let $G' = (V', E')$ be a solution of the self-sufficient problem and $|V'| = |W| + d + 2j$. A solution to the graph motif problem is obtained by taking all the vertices corresponding to those of V' in the j^{th} copy of G plus the vertices of P_j.

In a solution W to the graph motif problem, all the vertices of W belong to a same copy of G. Remove the vertices of added path in W provides a solution to the self-sufficient problem. □

Proposition 2.2 implies that the polynomial cases for the graph motif problem are also polynomial cases for the minimum self-sufficient problem. Since the graph motif problem is polynomial when there is a polynomial number of colors and G is of bounded treewidth, we obtain the following result.

Corollary 2.3 *The minimum self-sufficient problem is polynomial-time solvable if G has a fixed treewidth.*

In particular, in contrast to the weigthed case, the minimum self-sufficient problem is polynomial-time solvable for the graphs of treewidth 2, which are precisely the series-parallel graphs. Due to the transformation and the complexity of the graph motif problem, we obtain more precisely that the minimum self-sufficient problem can be solved in $O(n^{4w+2})$ where w denotes the treewidth of G. If G is a tree, we can obtain a much more efficient algorithm.

2.2.2 Negative results

We show a NP-hardness result for the minimum self-sufficient problem. It is similar to the one existing for the motif graph problem. The proof is based on the one of [1]. The proof uses the Exact Cover by 3-Sets problem.

	EXACT COVER BY 3-SETS (X3C):
Instance:	A a set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a collection $S = \{s_1, s_2, \ldots, s_n\}$ of 3-element subsets of X.
Solution:	A sub-collection $C \subseteq S$ such that every element of X is included in exactly one subset $s_i \in C$.

Proposition 2.4 *The minimum self-sufficient problem is NP-hard even if G bipartite with maximum degree less than or equal to four.*

Proof. We slightly modify the reduction given in Theorem 2 of [1] from X3C to the graph motif with two colors and G bipartite with maximum degree four. In their proof, the authors construct form an instance X, S of the X3C problem a bipartite graph G containing $2n+q$ white vertices and n black ones. The motif M to find is composed of $2n + 3q$ white vertices and q black ones.

We construct from G a bipartite G' by replacing edge $s'_n s''_n$ by a path $s'_n, v_1, v_2, \ldots, v_{2n+2q}, s''_n$, where $v_1, \ldots, v_{2n+2q}$ are new black vertices. We set W equal to the set of white vertices. There exists a motif M in G if and only if there exists a self-sufficient subgraph of G'. Note that in this case, we only seek for the existence of a self-sufficient subgraph, independantly from its size. However, it is always possible to add dummy nodes to always have a solution and focusing on self-sufficient subgraphs with size less than or equal to a given value. □

References

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[2] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.