Multi-level adaptive vertex-centered finite volume methods for diffusion problems

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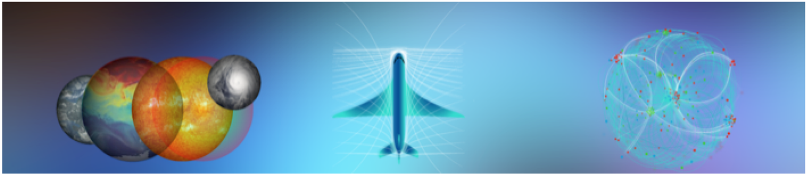
- Adaptive FE-FV are now widely used in the numerical solution of (PDEs) to achieve better accuracy with minimum degrees of freedom.
- We first solve the PDE to get the solution on the current mesh.
- The error is estimated using the solution, and used to mark a set of triangles that are to be refined.
- Triangles are refined in such a way to keep mesh regularity and conformity.



Modelization   Simulation   High Performance Calculation

- A typical loop of (AFE-FVM) through local refinement involves:

- ✓ Conformity of the mesh
- ✓ Prevent the propagation of refinement levels
- ✓ Efficiency of estimator
- ✓ Convergence of error
- ✓ Performance of CPU time

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a given nonlinear function.
Typically, $a(x) = x^{p-2}$ for some real number $p \in (1, +\infty)$. Let σ such that

$$\sigma(\xi) = a(|\xi|)\xi \quad \forall \xi \in \mathbb{R}^d \tag{1}$$

where $|.|$ is the Euclidean norm in $\mathbb{R}^d$. Then, for a given source function $f : \Omega \to \mathbb{R}$, the nonlinear Laplace problem consists in looking for $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\mathrm{div}(\sigma(\nabla u)) &= f \text{in } \Omega \\ u &= g \text{on } \partial\Omega \end{aligned} \tag{2}$$

Problem: find $p \in H_0^1(\Omega)$, $(\mathcal{S})$
$$\begin{cases} -\mathrm{div}(\mathbb{K}\nabla p) = f & \text{in } \Omega \quad \subset \mathbb{R}^{d=2,3} \\ p = g & \text{on } \partial\Omega \end{cases} \tag{3}$$

Unicity and Existence

Assumptions:

- $(H1)$ $\mathbb{K} \in \mathbb{L}^{\infty}(\Omega)$.
- $(H2)$ $f \in \mathbb{L}^2(\Omega)$.
- $(\mathcal{S})$ has a unique solution.

Remark: The problem (2) represents,for instance, the extension of the problem (3) which takes into account the nonlinear dependence of the Darcy velocity on the pressure head gradient ∇p. Note that (2) and (3) coincide, for $a(x) = x^{p-2}$, when $p = 2$.

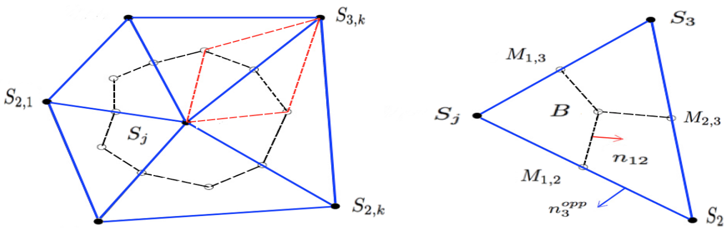


Figure: Dual cell

- S_1, S_2, and S_3 are the vertices of a triangle T,
- B its barycentre, Σ_1^{opp}, Σ_2^{opp} and Σ_3^{opp} The edges$[S_2S_3]$, $[S_1S_3]$ et $[S_1S_2]$;
- $\overrightarrow{n}_1^{opp}$, $\overrightarrow{n}_2^{opp}$ and $\overrightarrow{n}_3^{opp}$ outgoing unit normals such that $\overrightarrow{n_{pq}} \perp \overrightarrow{M_{pq}B}$ and $\overrightarrow{n_{pq}}.\overrightarrow{S_pS_q} > 0$

The approximation of the diffusive flux is based on an implicit scheme:

$$-\int_{\partial D_h} \mathbb{K}\nabla p.\vec{n}\, d\sigma = \int_{D_h} f(x)dx \qquad (4)$$

$$-\sum_{T\cap D_h\neq\emptyset} \int_{\partial D_h\cap T_h} \mathbb{K}_T\nabla p.\vec{n}\, d\sigma = \int_{D_h} f(x)dx \qquad (5)$$

We note the elementary diffusion terms by:

$$k_{12}^{flow}(T) = |T|\mathbb{K}_T \frac{|\Sigma_1^{opp}|}{2|T|}\frac{|\Sigma_2^{opp}|}{2|T|}\vec{n}_1^{opp}\vec{n}_2^{opp}$$

$$k_{13}^{flow}(T) = |T|\mathbb{K}_T \frac{|\Sigma_1^{opp}|}{2|T|}\frac{|\Sigma_3^{opp}|}{2|T|}\vec{n}_1^{opp}\vec{n}_3^{opp}$$

Finally, the finite volume scheme for the flow equation is written:

$$\sum_{T\in D_h} k_{12}^{flow}(T)(p_2-p_1) + k_{13}^{flow}(T)(p_3-p_1) = \int_{D_h} f(x)dx \qquad (6)$$

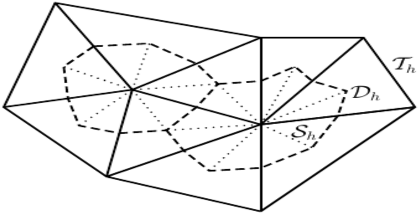


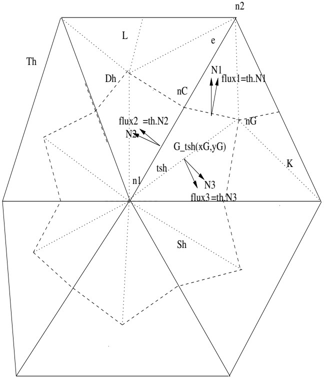
Figure: Primal mesh $\mathcal{T}_h$, Dual mesh $\mathcal{D}_h$ and the fine simplicial mesh $\mathcal{S}_h$

Remark: the flux $-\mathbb{K}\nabla p \in \mathrm{H}(\mathrm{div},\Omega)$ but $-\mathbb{K}\nabla p_h \notin \mathrm{H}(\mathrm{div},\Omega)$
Flux reconstruction (exploits the local conservativity):

$$\mathbf{t}_h \in \mathrm{RTN}_0(\mathcal{S}_h) \subset \mathrm{H}(\mathrm{div},\Omega)$$
$$(\mathrm{div}\ \mathbf{t}_h, 1)_D = (f, 1)_D, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}$$

Construction of $\mathbf{t}_h$ by Direct Prescription : We solved the following system $(\mathcal{S}')$:

$$(\mathcal{S}') \begin{cases} \mathbf{t}_h \cdot \overrightarrow{N1} = -\mathbb{K}\nabla \mathbf{p}_h \cdot \overrightarrow{N1} \\ \mathbf{t}_h \cdot \overrightarrow{N2} = -w_{K,s}\left(\mathbb{K}_{|K}\nabla \mathbf{p}_h \cdot \overrightarrow{N2}\right) - w_{L,s}\left(\mathbb{K}_{|L}\nabla \mathbf{p}_h \cdot \overrightarrow{N2}\right) \\ \mathbf{t}_h \cdot \overrightarrow{N3} = -\mathbb{K}\nabla \mathbf{p}_h \cdot \overrightarrow{N3} \end{cases}$$



- $\mathbb{K}_{|K}$ ($\mathbb{K}_{|L}$) is an approximation of the tensor of permeability on the triangle K (L)
- $\overrightarrow{N1}$, $\overrightarrow{N2}$ and $\overrightarrow{N3}$: unit normal vectors.
- Harmonic averaging : $w_{K,s} = \frac{\mathbb{K}_K}{\mathbb{K}_K+\mathbb{K}_L}$, $w_{L,s} = \frac{\mathbb{K}_L}{\mathbb{K}_K+\mathbb{K}_L}$

Error estimator:

$$|||p - p_h|||_\Omega^2 = \left\|\mathbb{K}^{\frac{1}{2}}\nabla(p - p_h)\right\|_\Omega^2 = \int_\Omega (\mathbb{K}^{\frac{1}{2}}\nabla p + \mathbb{K}^{-\frac{1}{2}} t_h)^2 \qquad (7)$$

$$|||p - p_h|||^2 \leq \sum_{D \in \mathcal{D}_h} \left(\underbrace{m_D \|f - \mathrm{div}\ \mathbf{t}_h\|_D}_{\text{residual error}} + \underbrace{\left\|\mathbb{K}^{\frac{1}{2}}\nabla p_h + \mathbb{K}^{-\frac{1}{2}}\mathbf{t}_h\right\|_D}_{\text{flux error}} \right)^2$$

- $m_{D,a} = \dfrac{C_{P,D} h_D^2}{c_{a,D}}$ if $D \in D_h^{int}$, $\quad m_{D,a} = \dfrac{C_{F,D} h_D^2}{c_{a,D}}$ if $D \in D_h^{ext}$
- $C_{P,D}$ is equal $\dfrac{1}{\pi^2}$ if D is convexe, $C_{F,D}$ is equal to 1 on general.
- Effectivity index: $\dfrac{\left(\sum_{D \in D_h} (\eta_{R,D} + \eta_{DF,D})^2\right)^{\frac{1}{2}}}{|||p - p_h|||_\Omega}$

T: $=$ Triangulation of Ω, for all $\tau \in T$ we define $v(\tau)$ the "newest vertex".

$E(\tau)$: $=$ Is the longest edge of τ, $v(\tau)$ is the vertex opposite to $E(\tau)$.

- (R1): The first step consists in dividing the elements by joining $v(\tau)$ to the middle I of $E(\tau)$.
- (R2): I becomes the "newest vertex" of each of the two created triangles.
- (R3): Neighbor refinement by $R1$ and conformity.

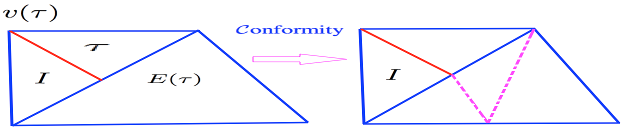


Figure: Bisect a triangle and Completion by Newest-Vertex-Bisection strategy

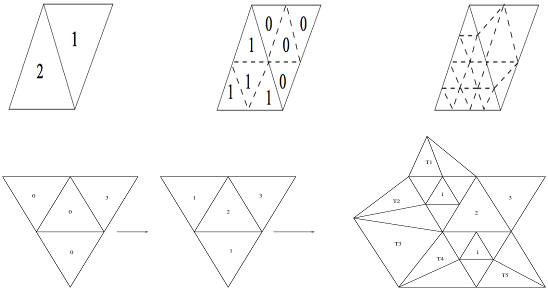


Figure: Mesh refinement with ADAPT and conformity with propagation levels

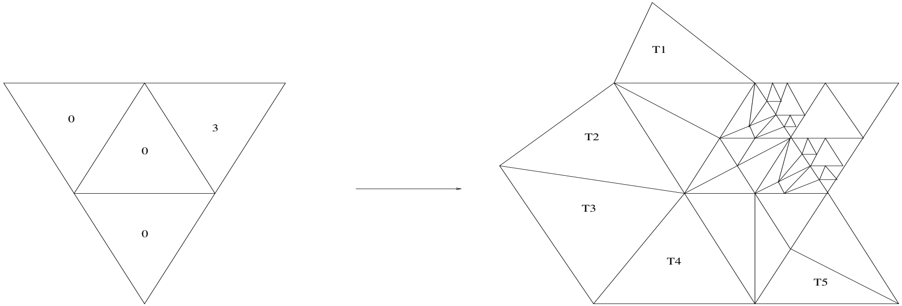
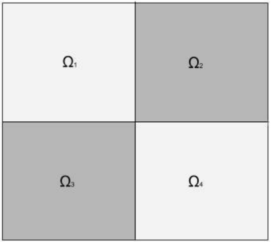


Figure: Mesh refinement with ADAPT-NEWEST

- First we proceed in a first time by refining our mesh by the ADAPT strategy, then for the conformity one uses the method Newest vertex bisection.
- There is no more propagation of the refinement on triangles T_1, T_2 et T_4.

Test with an analytical solution, $\alpha = 0,127$



- Problem:

$$\begin{aligned} -\mathrm{div}(\mathbb{K}\nabla p) &= f \quad \text{in } \Omega = (-1,1)^2 \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

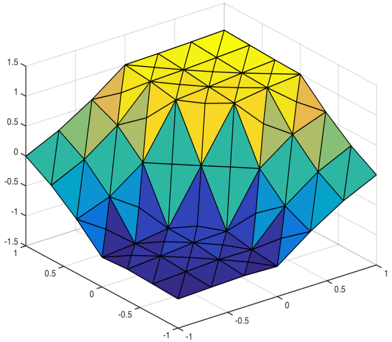
- heterogeneous permeability

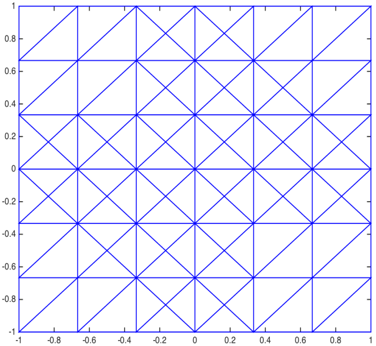
$$\mathbb{K} = \begin{cases} 1.\mathbb{I}_2 & \text{if } x \in \Omega_{1,4} \\ 100.\mathbb{I}_2 & \text{else.} \end{cases}$$

- Solution
  $p \in H^{1+\alpha}(\Omega)$, $a_i, b_i =$ const.

$$p(r,\theta) = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

Regular mesh: $\alpha = 0.127$





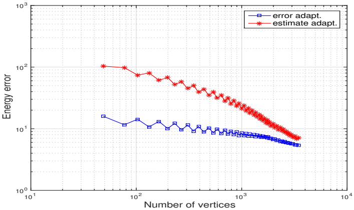
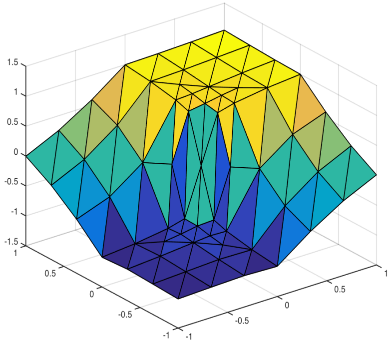


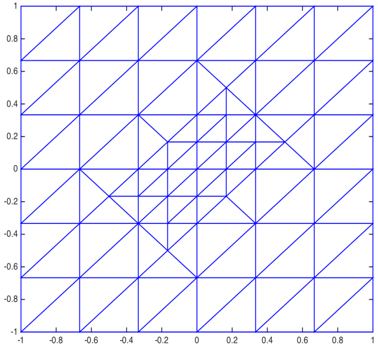
Figure: Energy Error, Estimator, Efficiency

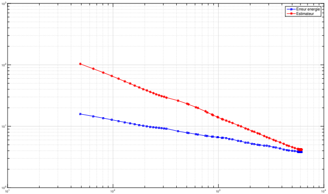
NewestVB approach

iter	DoFs	η	ϵ_1	ϵ_2	f_η	CPU
1	128	103.3915	15.836	0.30586	6.5289	0.679237
6	436	67.4077	10.0475	0.13689	6.7089	0.668070
12	942	44.08	8.8284	0.074364	4.993	0.074364
24	2170	18.4356	7.4593	0.02977	2.4715	1.426176
59	7162	7.0814	5.3987	0.021182	1.3117	4.063058
Total						**75.13**

Regular mesh : $\alpha = 0.127$







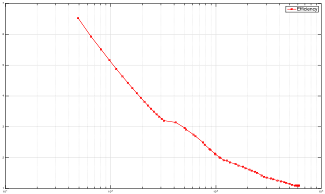


Figure: Estimateur, erreur energie (gauche), efficacité (droite)

AdaptNVB approach

iter	DoFs	η	ϵ_1	ϵ_2	f_η	CPU
1	240	103.3915	15.836	0.30586	6.5289	0.671513
6	1040	42.7369	10.677	0.081534	4.0027	0.918080
13	2160	20.8044	7.846	0.035467	2.4715	1.462513
24	4920	9.3623	5.8007	0.022808	1.614	2.870125
29	7296	6.9263	5.104	0.022277	1.357	3.995373
Total						**35.13**

Irregular mesh : $\alpha = 0.127$

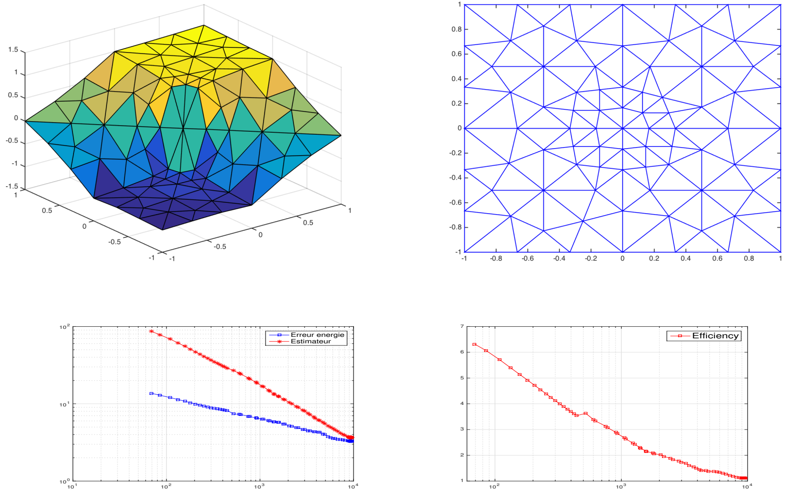


Figure: Estimateur, erreur energie (gauche), efficacité (droite)

Nested adaptive vertex-centered finite volume

The diffusion in a two-dimensional closed medium $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ is described by the following equation

$$\begin{aligned} -\nabla \cdot (\mathbb{K}\nabla u(\mathbf{x})) &= f(\mathbf{x}), \qquad \forall\, \mathbf{x} \in \boldsymbol{\Omega}, \\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \forall\, \mathbf{x} \in \partial\boldsymbol{\Omega}, \end{aligned} \tag{8}$$

where f is the external force, g the boundary source, and $\mathbb{K}$ is a piecewise constant diffusion coefficient.

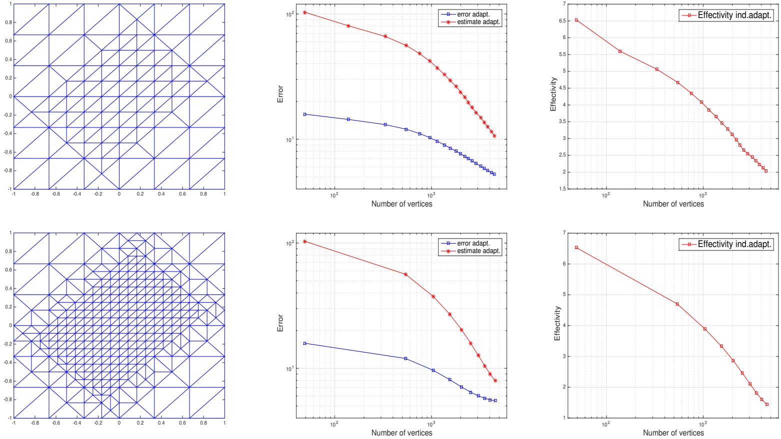


Figure: Results using conventional approach (left) and using nested approach (right).

Table: Comparison between the conventional and nested approaches. CPU times are in seconds.

Conventional approach

iter	DoFs	η	ϵ_1	ϵ_2	f_η	CPU
1	240	103.3915	15.836	0.30586	6.5289	0.71
6	2256	42.0556	10.298	0.079115	4.0838	1.45
20	9903	10.617	5.2289	0.020684	2.0305	5.5
Total						**60.691**

Nested approach

iter	DoFs	η	ϵ_1	ϵ_2	f_η	CPU
1	1046	103.3915	15.836	0.30586	6.5289	0.39
3	3054	37.5234	9.6343	0.065962	3.8948	1.5
10	10082	7.9307	5.5007	0.024378	1.4418	6.8
Total						**35.13**

Algorithm

Parameter	Signification
$\mathcal{I}_{adiv}$	Table have the values 0 or 1, 1: the triangle that contains this vertice must be refined, and 0 otherwise
$\mathcal{M}$arker	Table that indicates whether an edge should be marked or not.
$\mathcal{N}_{Lev}$	maximum number of multi-level refinement.
$\mathcal{N}_{adiv}$	contains 0 or 1, 0: the edge is not yet divided, 1: the node is already created in the middle of the edge.
$\mathcal{N}_{Ref}$	Maximum level of refinement.

- First step construction of dual mesh $\mathcal{D}_h$, simplical mesh $\mathcal{S}_h$.
- We compute the estimators on the simplicial mesh edges which will allow us to calculate the estimator in the node that surrounds these edges.
- With the aid of this estimator, a threshold is defined in order to obtain the iadiv in the node.
- After calculating iadiv in the cell (this information will allow us to know if the cell must be adapted or not and also know its level of refinement.

- If for example the level of refinement is 3 one proceeds as follows:
  We will iterate 3 times and in each iteration we will make an adaptation (Adapt strategy) and the conformity (new-vertex).

Parameter	Signification
$\mathcal{I}_{adiv}$	Table have the values 0 or 1, 1: the triangle that contains this vertice must be refined, and 0 otherwise
$\mathcal{M}$arker	Table that indicates whether an edge should be marked or not.
$\mathcal{N}_{Lev}$	maximum number of multi-level refinement.
$\mathcal{N}_{adiv}$	contains 0 or 1, 0: the edge is not yet divided, 1: the node is already created in the middle of the edge.
$\mathcal{N}_{Ref}$	Maximum level of refinement.

for t:=1 **to** $\mathcal{N}_{Ref}$ **do**
    Creation of all data structures
    Calcul of numerical solution
    calcul of estimator
    Calcul of exact error
    Compute of criteria
    for k:=1 **to** $\mathcal{N}_{Lev}$ **do**
        Mesh refinement with Adapt strategy
        Conformity with NewestVB startegy
        Marking of new triangles created to be refined
    end
end

Conclusion

- Coupled method Adpat-Newest to refine the mesh.
- Convergence of error to .
- Performance of CPU time using multi-level adaptation.

THANK YOU FOR YOUR ATTENTION !!