TASEP hydrodynamics using microscopic characteristics

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The convergence of the totally asymmetric simple exclusion process to the solution of the Burgers equation is a classical result. In his seminal 1981 paper, Herman Rost proved the convergence of the density fields and local equilibrium when the limiting solution of the equation is a rarefaction fan. An important tool of his proof is the subadditive ergodic theorem. We prove his results by showing how second class particles transport the rarefaction-fan solution, as characteristics do for the Burgers equation, avoiding subadditivity. In the way we show laws of large numbers for tagged particles, fluxes and second class particles, and simplify existing proofs in the shock cases. The presentation is self contained.

Kewords and phrases Totally asymmetric simple exclusion process. Hydrodynamic limit. Burgers equation. Second class particles.

AMS Classification index Primary 60K35 82C

1 Introduction

In the totally asymmetric simple exclusion process there is at most a particle per site. Particles jump one unit to the right at rate 1, but jumps to occuppied sites are forbiden. Rescaling time and space in the same way, the density of particles converges to a deterministic function which satisfies the Burgers equation. This was first noticed by Rost [47], who considered an initial configuration with no particles at positive sites and with particles in each of the remaining sites. He then takes r in $[-1, 1]$ and proves that (a) the number of particles at time t to the right of rt, divided by t converges almost surely when $t \to \infty$ and (b) the limit coincides with the integral between r and ∞ of the solucion of the Burgers equation at time 1, with initial condition 1 to the left of the origin and 0 to its right. This is called convergence of the density fields. Rost also proved that the distribution of particles at time t around the position rt converges as t grows to a product measure whose parameter is the solution of the equation at the space-time point $(r, 1)$. This is called local equilibrium because the product measure is invariant for the tasep. These results were then proved for a large family of initial distributions and trigged an impressive set of work on the subject; see Section 10 later.

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The main novelty of this paper is a new proof of Rost theorem. Rost first uses the subadditive ergodic theorem to prove that the density field converges almost surely and then identifies the limit using couplings with systems of queues in tandem. Our proof shows convergence to the limit in one step, avoiding the use of subadditivity. For each $\rho \in [0,1]$ we couple the process starting with the 1-0 step Rost configuration with a process starting with a stationary product measure at density ρ and show that for each time t the Rost configuration dominates the stationary configuration to the left of R_t and the oposite domination holds to the right of R_t; see Lemma 9.1. Here R_t is a second class particle with respect to the stationary configuration. It is known that R_t/t converges to $(1-2\rho)$ and then the result follows naturally. A colorful and conceptual aspect of the proof is that $1-2\rho$ is the speed of the characteristic of the Burgers equation carrying the density ρ.

In order to keep the paper selfcontained we shortly introduce the Burgers equation and the role of characteristics and the graphical construction of the tasep which induces couplings and first and second class particles. We also include a simplified proof of the hydrodynamic limit in the increasing shock case, using second class particles. In the way we recall the law of large numbers for a tagged particle in equilibrium, which in turn implies law of large numbers for the flux of particles along moving positions and for tagged and isolated second class particles.

Section 2.1 introduce the Burgers equation and describe the role of characteristics. Section 3 gives the graphical construction of the tasep and describes its invariant measures. Section 4 contains some heuristics for the hydrodynamic limits and states the hydrodynamic limit results. Section 5 contains a proof a the law of large numbers for the tagged particle. Section 6 includes the graphical construction of the coupling and describes the two-class system associated to a coupling of two processes with ordered initial configurations. Section 7 contains the proof of the law of large numbers for the flux and the second class particles. In Section 8 we prove the hydrodynamic limit for the increasing shock and in Section 9 we prove Rost theorem, the hydrodynamics in the the rarefaction fan. Finally in Section 10 we make comments and give references to the previous results and other related works.

2 The Burgers equation

The one-dimensional Burgers equation is used as a model of transport. The function $u(r,t) \in [0,1]$ represents the density of particles at the space position $r \in \mathbb{R}$ at time $t \in \mathbb{R}^+$. The density must satisfy

$$\frac{\partial u}{\partial t} = -\frac{\partial [u(1-u)]}{\partial r} \tag{2.1}$$

The initial value problem for (2.1) is to find a solution under the initial condition $u(r,0) = u_0(r)$, $r \in \mathbb{R}$, where $u_0 : \mathbb{R} \to [0,1]$ is given. In this note we only consider the following family of initial conditions:

$$u_0(r) = u^{\lambda,\rho}(r) := \begin{cases} \lambda & \text{if } r \leq 0 \\ \rho & \text{if } r > 0 \end{cases} \tag{2.2}$$

where $\rho, \lambda \in [0,1]$. Lax [36] explains how to treat this case. Differentiating (2.1) we get

$$\frac{\partial u}{\partial t} = -(1-2u)\frac{\partial u}{\partial r} = 0 \tag{2.3}$$

so that u is constant along $w(t)$ with $w(0) = r$, the trajectory satisfying $\frac{d}{dt}w = (1 - 2u)$. That is, u propagates with speed $(1 - 2u)$: $u(w(t), t) = u_0(w(0))$. These trajectories are called *characteristics*. If different characteristics meet, carrying two different solutions to the same point, then the solution has a shock or discontinuity at that position. In our case the discontinuity is present in the initial condition. The cases $\lambda < \rho$ and $\lambda > \rho$ are qualitative different.

Shock case When $\lambda < \rho$ the characteristics starting at $r > 0$ and $-r$ have speed $(1-2\rho)$ and $(1-2\lambda)$ respectively and meet at time $t(r) = r/(\rho-\lambda)$ at position $(1-\lambda-\rho)r/(\rho-\lambda)$. Take $a < b$ large enough to guarantee that the shock is inside $[a, b]$ for times in $[0, t]$. By

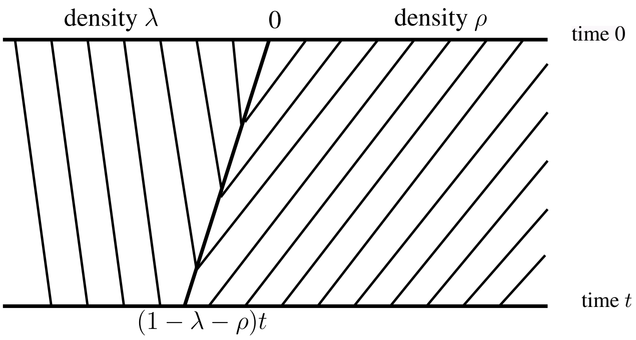

Figure 2.1: Shocks and characteristics in the Burgers equation. The characteristics starting at r and $-r$ that go at velocity $1 - 2\rho$ and $1 - 2\lambda$ respectively with $\rho > \lambda$. The center line is the shock that travels at velocity $1 - \rho - \lambda$.

conservation of mass:

$$\frac{d}{dt}\int_a^b u(r,t)\,dr \;=\; u(a,t)(1-u(a,t)) \;-\; u(b,t)(1-u(b,t)) \tag{2.4}$$

Since $\int_a^b u(r,t)\,dr = \lambda(y_t - a) + \rho(b - y_t)$, where y_t is the position of the shock at time t, we have

$$y_t'(\lambda - \rho) = \lambda(1 - \lambda) - \rho(1 - \rho)$$

and $y_t = (1-\lambda-\rho)t$. We conclude that for $\lambda < \rho$, the solution of the initial value problem $u(r, t)$ is ρ for $r > vt$ and λ for $r < vt$, that is,

$$u(r, t) = u^{\lambda,\rho}(r - vt).$$

The rarefaction fan When $\lambda > \rho$ the characteristics emanating at the left of the origin have speed $(1-2\lambda) < (1-2\rho)$, the speed to the right and there is a family of characteristics emanating from the origin with speeds $(1 - 2\alpha)$ for $\lambda \geq \alpha \geq \rho$. The solution is then

$$u(r,t) = \begin{cases} \lambda & \text{if } r < (1-2\lambda)t \\ \dfrac{t-r}{2t} & \text{if } (1-2\lambda)t \leq r \leq (1-2\rho)t \\ \rho & \text{if } r > (1-2\rho)t \end{cases} \tag{2.5}$$

The characteristic starting at the origin with speed $(1 - 2\alpha)$ carries the solution α:

$$u\big((1-2\alpha)t, t\big) = \alpha, \qquad \lambda \geq \alpha \geq \rho. \tag{2.6}$$

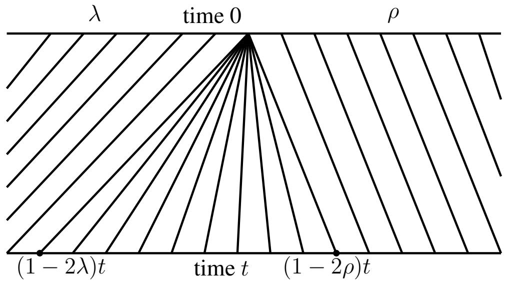

Figure 2.2: The rarefaction fan. Here $\lambda > \rho$.

The above solution is a *weak solution*, that is, for all $\Phi \in C_0^\infty$ with compact support,

$$\int\int\left(\frac{\partial \Phi}{\partial t}u + \frac{\partial \Phi}{\partial r}u(1-u)\right)drdt = 0. \tag{2.7}$$

The solution may be not unique, but (2.5) comes as a limit when $\beta \to 0$ of the unique solution of the (viscid) Burgers equation

$$\frac{\partial u}{\partial t} = -\frac{\partial[u(1-u)]}{\partial r} + \beta\frac{\partial^2 u}{\partial r^2}. \tag{2.8}$$

This solution, called *entropic*, is selected by the hydrodynamic limit of the tasep, as we will see.

3 The totally asymmetric simple exclusion process

We construct now the totally asymmetric simple exclusion process (tasep). Call *sites* the elements of $\mathbb{Z}$ and *configurations* the elements of the space $\{0,1\}^{\mathbb{Z}}$, endowed with the product topology. When $\eta(x) = 1$ we say that η has a *particle* at site x, otherwise there is a *hole*.

Harris graphical construction We define directly the graphical construction of the process, a method due to Harris [30]. The process in $\{0,1\}^{\mathbb{Z}}$ is given as a function of an initial configuration η and a Poisson process ω on $\mathbb{Z} \times \mathbb{R}^+$ with rate 1; ω is a random discrete subset of $\mathbb{Z} \times \mathbb{R}$. When $(x,t) \in \omega$ we say that there is an arrow $x \to x+1$ at time t. Fix a time $T > 0$. For almost all ω there is a double infinite sequence of sites

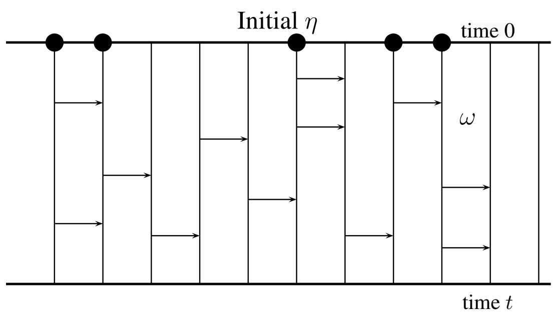

Figure 3.1: A typical ω, represented by arrows and the initial configuration η, where particles are represented by dots.

$x_i = x_i(\omega)$, $i \in \mathbb{Z}$ with no arrows $x_i \to x_i+1$ in $(0,T)$. The space $\mathbb{Z}$ is then partitioned into finite boxes $[x_i+1, x_{i+1}] \cap \mathbb{Z}$ with no arrows connecting boxes in the time interval $[0,T]$. Take ω satisfying this property and an arbitrary initial configuration η and construct η_t, $0 \le t \le T$, as a function of η and ω, as follows.

Since the boxes are finite, we can label the arrows inside each box by order of appearance. Take a box. If the first arrow in the box is (x,t) and at time $t-$ there is a particle at x and no particle at $x+1$, then the particle follows the arrow $x \to x+1$ so that at time t there is a particle at $x+1$ and no particle at x. If before the arrow from x to $x+1$ there is a different event (two particles, two holes or a particle at $x+1$ and no particle at x), then nothing happens: the configuration after the arrow is exactly the same as before. Repeat the procedure for the following arrows until the last arrow in the box. Proceed to next box and obtain a particle configuration depending on the initial η and the Poisson realization ω, denoted $\eta_t[\eta,\omega]$, $0 \le t \le T$. For times greater than T, use η_T as initial

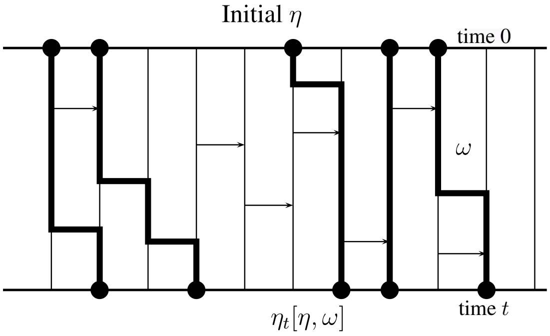

Figure 3.2: A typical construction. Particles follow arrows when destination site is empty.

configuration and repeat the procedure to construct the process between T and $2T$, using the arrows of ω with times in $[T,2T]$ and so on. In this way we have constructed the process

$$(\eta_t[\eta,\omega] : t \ge 0).$$

The process satisfies the almost sure Markov property

$$\eta_{t+s}[\eta,\omega] = \eta_s\big[\eta_t[\eta,\omega], \tau_t\omega\big], \tag{3.1}$$

where $\tau_t\omega := \{(x,s) : (x,t+s) \in \omega\}$ has the same distribution as ω and it is independent of $\omega \cap (\mathbb{Z} \times [0,t])$, by the properties of the Poisson process ω. This implies that the process η_t is Markov. Usually we omit the dependence on ω in the notation.

Product measures Let

$$U = (U(x) : x \in \mathbb{Z}) := \text{ iid random variables uniformly distributed in } [0,1]. \tag{3.2}$$

For $\rho \in [0,1]$ define $\eta^\rho = \eta^\rho[U]$ by

$$\eta^\rho(x) := \mathbf{1}\{U(x) < \rho\}. \tag{3.3}$$

The distribution of η^ρ is a Bernoulli product measure. Define

$$f_A(\eta) := \prod_{x\in A} \eta(x). \tag{3.4}$$

If ζ is a random configuration in $\{0,1\}^{\mathbb{Z}}$, then $(Ef_A(\zeta) : A \subset \mathbb{Z}, \text{finite})$ characterizes the distribution of ζ. In particular, the distribution of η^ρ is characterized by $Ef_A(\eta^\rho) = \rho^{|A|}$, where $|A|$ is the cardinality of A.

Take U independent of ω and call

$$\eta_t^\rho := \eta_t[\eta^\rho, \omega] \tag{3.5}$$

We denote P and E the probability and expectation associated to the probability space induced by the independent random elements U and ω.

Lemma 3.1. *For each $\rho \in [0,1]$, the distribution of η^ρ is invariant for the tasep. That is, for any finite $A \subset \mathbb{Z}$ we have $E(f_A(\eta_t^\rho)) = \rho^{|A|}$, for all $t \geq 0$.*

This lemma is proved in Liggett [39]. The configurations $\zeta^{(n)}(x) := \mathbf{1}\{x \geq n\}$ are frozen because all particles are blocked. In the same paper Liggett shows that all the invariant measures are combination of the Bernoulli product measures and the blocking measures, those concentrating mass on the frozen configurations $\eta^{(n)}$.

4 The Hydrodynamic Limit

Heuristic derivation of Burgers equation from tasep Using the forwards Kolmogorov equation for the function $f(\eta) = \eta(x)$ we get

$$\frac{d}{dt}E(\eta_t(x)) = E\big[-\eta_t(x)(1-\eta_t(x+1)) + \eta_t(x-1)(1-\eta_t(x))\big], \tag{4.1}$$

Fix an $\varepsilon > 0$ which will go later to zero and define

$$u^\varepsilon(r,t) := E[\eta_{\varepsilon^{-1}t}(\varepsilon^{-1}r)],$$

where $\varepsilon^{-1}r$ is an abuse of notation for integer part of $\varepsilon^{-1}r$. Putting the ε's in (4.1) we get

$$\begin{aligned}\frac{d}{dt}u^\varepsilon(r,t)) = \varepsilon^{-1}E\big[&-\eta_{t\varepsilon^{-1}}(r\varepsilon^{-1})(1-\eta_{t\varepsilon^{-1}}(r\varepsilon^{-1}+1)) \\ &+ \eta_{t\varepsilon^{-1}}(r\varepsilon^{-1}-1)\,(1-\eta_{t\varepsilon^{-1}}(r\varepsilon^{-1}))\big].\end{aligned} \tag{4.2}$$

Assume that there exist a limit

$$u(r,t) := \lim_{\varepsilon\to 0} u^\varepsilon(r,t)$$

and that the distribution of $\eta_{\varepsilon^{-1}t}$ around $\varepsilon^{-1}r$ is approximately product, that is,

$$\lim_{\varepsilon\to 0} E\big[\eta_{t\varepsilon^{-1}}(r\varepsilon^{-1})\,\eta_{t\varepsilon^{-1}}(r\varepsilon^{-1}+1)\big] \;=\; (u(r,t))^2.$$

Assume further that $u(r,t)$ is differentiable in r. In this case, the right hand side of (4.2) must converge to minus the derivative of $u(r,t)(1-u(r,t))$, that is, the limiting $u(r,t)$ must satisfy the Burgers equation. This heuristic argument may also be a script of a proof of the convergence of the tasep density to a solution of the Burgers equation. Instead, we show directly the convergence in the terms described below by (4.5) and (4.6) below.

Hydrodynamics limit statements Consider the Burgers equation with initial data u_0 such that there exists a unique entropic weak solution $u(r,t)$ for the initial value problem (2.1)-(2.2). Take the uniform random variables U defined in (3.2) and define

$$\zeta^\varepsilon(x) := \mathbf{1}\{U(x) \le u_0(\varepsilon x)\}. \tag{4.3}$$

That is, for each $\varepsilon > 0$, the random configuration ζ^ε is a sequence of independent Bernoulli random variables with varying parameter induced by u_0 for the mesh ε. Let ζ^ε_t be the tasep with random initial configuration ζ^ε:

$$\zeta^\varepsilon_t := \eta_t[\zeta^\varepsilon, \omega]. \tag{4.4}$$

Theorem 4.1 (Hydrodynamic limits)**.** *Let $u(r,t)$ be the solution of the Burgers equation with initial condition u_0. Let ζ^ε be given by* (4.3) *and ζ^ε_t be the tasep with initial condition ζ^ε defined in* (4.4)*. Then,*

Convergence of the density fields *For all real numbers $a < b$,*

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x: a\le \varepsilon x \le b} \zeta^\varepsilon_{\varepsilon^{-1}t}(x) = \int_a^b u(r,t)dr, \qquad a.s. \tag{4.5}$$

Local-equilibrium *At the continuity points of $u(r,t)$,*

$$\lim_{\varepsilon\to 0} E[f_A(\tau_{\varepsilon^{-1}r}\zeta^\varepsilon_{\varepsilon^{-1}t})] \;= u(r,t)^{|A|}, \tag{4.6}$$

where the translation operator τ_z is defined by $(\tau_z\eta)(x) = \eta(\lfloor x+z \rfloor)$, here $\lfloor z \rfloor$ is the integer part of z.

The limit (4.6) gives weak convergence of the particle distribution at the points of continuity of $u(r,t)$ to the invariant product measure $\nu^{u(r,t)}$. When $A = \{0\}$, the limit (4.6) is the so called *density profile*:

$$\lim_{\varepsilon\to 0} E[\zeta^\varepsilon_{\varepsilon^{-1}t}(\varepsilon^{-1}r)] \;=\; u(r,t), \tag{4.7}$$

ignoring the integer parts, as abuse of notation. We provide proofs of (4.5) and (4.6) for the initial condition $u_0 = u^{\lambda,\rho}$ defined in (2.2). In Section 10 we give references to the proof of the general case.

Sketch of proof of hydrodynamic limits When $u_0 = u^{\lambda,\rho}$, the configurations ζ^ε do not depend on ε and are given by $\zeta^\varepsilon \equiv \eta^{\lambda,\rho}$, defined by

$$\eta^{\lambda,\rho}(x) := \begin{cases} \mathbf{1}\{U(x) \le \lambda, \} & \text{if } x \le 0 \\ \mathbf{1}\{U(x) \le \rho, \} & \text{if } x > 0. \end{cases} \tag{4.8}$$

where $U(x)$ are defined in (3.2).

The proofs are based on coupling of the tasep with different initial conditions. A crucial property of the coupling is attractivity, meaning that initial coordinatewise ordered configurations keep their order under the coupled evolution. The coupling naturally introduce first and second class particles. During the proof we will prove laws of large numbers for (a) a tagged particle for the stationary process η^λ_t, (b) the flux of η^λ_t particles along a traveller with constant speed, (c) a second class particle for the process with initial shock configuration $\eta^{\lambda,\rho}$ with $\lambda < \rho$ and (d) a second class particle for the stationary process η^λ_t. We will see the microscopic counterpart of Figures 2.1 and 2.2.

5 The tagged particle

Given a configuration η tag the particles of η as follows:

$$X(i)[\eta] := \begin{cases} \max\{x \le 0 : \eta(x) = 1\} & \text{if } i = 0 \\ \min\{x > X(i-1) : \eta(x) = 1\} & \text{if } i > 0 \\ \max\{x < X(i+1) : \eta(x) = 1\} & \text{if } i < 0. \end{cases} \tag{5.1}$$

We are interested in configurations with a particle at the origin. So, define

$$\tilde{\eta}(x) := \begin{cases} 1 & \text{if } x = 0 \\ \eta(x) & \text{otherwise} \end{cases}; \qquad \tilde{\eta}_t := \eta_t[\tilde{\eta}, \omega]. \tag{5.2}$$

The positions of the particles at time t can be recovered from the graphical construction of Figure 3.2 by following the thick trajectories. Call $X_t(i)[\tilde{\eta}, \omega]$ the position of the i-th particle at time t; when η and ω are understood we just denote $X_t(i)$. Call $X_t := X_t(0)$ the position of the tagged particle initially at the origin and define the process as seen from that tagged particle by

$$\tau_{X_t}\eta_t[\tilde{\eta}, \omega] \tag{5.3}$$

where the shifted configuration $\tau_y\eta$ is defined by $(\tau_y\eta)(x) = \eta(y + x)$. Take the random

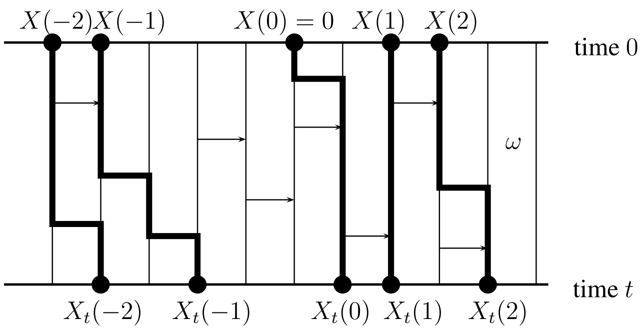

Figure 5.1: Trajectories of the tagged particles.

configuration η^ρ and add a particle at the origin to get $\tilde{\eta}^\rho$. The law of $\tilde{\eta}^\rho$ is the Bernoulli product measure conditioned to have a particle at the origin. The distribution of $\tilde{\eta}^\rho$ is invariant for the process as seen from the tagged particle (see [15], for instance):

Lemma 5.1. $\tau_{X_t}\tilde{\eta}^\rho_t$ *has the same distribution as* $\tilde{\eta}^\rho$.

Proposition 5.2 (Law of large numbers for the tagged particle)**.** *Let* X_t *be the position of the tagged particle initially at the origin for the process with random initial configuration* $\tilde{\eta}^\rho$*. Then,*

$$\lim_{t\to\infty} \frac{X_t}{t} = (1 - \rho), \qquad a.s. \tag{5.4}$$

Sketch proof. A proof based in Burke theorem [9] goes as follows. Think particles as servers and holes as customers in a infinite system of queues in series. Labeling the servers, the queue of server-i is the block of successive holes to the right of $X_t(i)$. Each time server-i jumps to the right, a customer is served and goes to the queue of server-$(i-1)$. Burke theorem says that if the initial random configuration is $\tilde{\eta}^\rho$, then the marginal distribution of the process $(X_t, t \geq 0)$ is a Poisson process of rate $(1-\rho)$. See [21] for a proof in this context. As a corollary we get the law of large numbers (5.4).

An alternative proof without using such strong result is the argument of Saada [48]. She proves that the process as seen from the tagged particle $\tau_{X_t}\tilde{\eta}^\rho$ is ergodic, which in turn implies the law of large numbers. □

6 Coupling and two-class tasep

The graphical construction provides a natural coupling of the tasep starting with two or more different configurations. Let η, η' be initial configurations and define the coupling

$$\big((\eta_t, \eta'_t) : t \geq 0\big) := \big((\eta_t[\eta, \omega], \eta_t[\eta', \omega]) : t \geq 0\big).$$

This amounts to use the same arrows for both marginals. By construction, each marginal

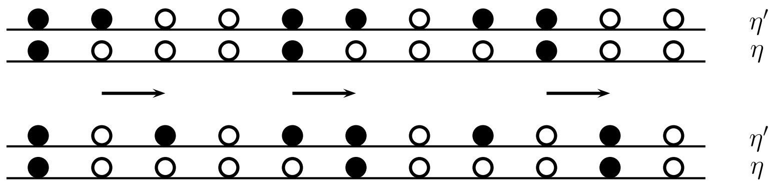

Figure 6.1: Coupling. Configurations η' and η before and after 3 possible arrows.

of the coupling has the distribution of the tasep. Particles at site x of each marginal try to jump at the same time, but the jump occurs only if the destination site $x+1$ is empty in the corresponding marginal.

Denote $\eta \leq \eta'$ if $\eta(x) \leq \eta'(x)$ for all $x \in \mathbb{Z}$.

Lemma 6.1. *Attractivity. For all $t \geq 0$ we have*

$$\eta \leq \eta' \quad \textit{implies} \quad \eta_t \leq \eta'_t \qquad \textit{a.s.} \tag{6.1}$$

Discrepancy conservation. If $\eta \leq \eta'$, then the number of discrepancies is conserved:

$$\sum_x (\eta'(x) - \eta(x)) = \sum_x (\eta'_t(x) - \eta_t(x)). \tag{6.2}$$

Proof. To show (6.1) it is sufficient to check that if $\eta_{t-} \leq \eta'_{t-}$ and $(t,x) \in \omega$, that is, there is an arrow from x to $x+1$ at time t, then $\eta_t \leq \eta'_t$, that is, the domination still holds after the arrow. The same exploration shows that the number of discrepancies does not change after the arrow. □

First and second class particles Fix $\eta \leq \eta'$ and call

$$\sigma_t := \eta_t[\eta, \omega], \qquad \xi_t := \eta_t[\eta', \omega] - \eta_t[\eta, \omega]. \tag{6.3}$$

By definition $\sigma_t \in \{0,1\}^{\mathbb{Z}}$ and by attractivity, $\xi_t \in \{0,1\}^{\mathbb{Z}}$. We call *first class* the σ particles and *second class* the ξ particles. The process $((\sigma_t, \xi_t) : t \geq 0)$ is Markov; it can be constructed directly as function of ω and the initial configurations σ and ξ, as follows. At each site there is at most one particle, either ξ or σ. Arrows involving ξ-ξ, σ-σ, ξ-0,

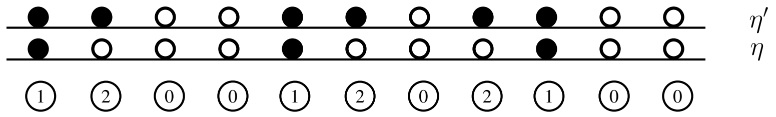

Figure 6.2: The (σ, ξ) configuration associated to (η, η') of figure 6.1. σ particles are labelled 1, ξ particles are labelled 2 and holes are labelled 0.

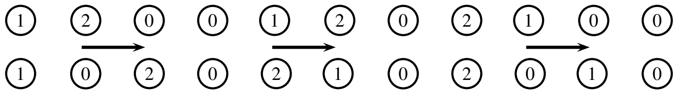

Figure 6.3: Another way of looking at the coupling. We see three possible jumps of first and second class particles associated to the configuration η' and η of figure 6.1.

σ-0 particles, use the same rules as the tasep, but arrows involving σ-ξ particles follow the rules (a) if $\sigma \to \xi$ then the particles interchange positions and (b) if $\xi \to \sigma$, then nothing happens. In other words, ξ particles behave as particles when interacting with holes and as holes when interacting with σ particles.

The vector (σ_t, ξ_t) depends on the initial configuration $(\sigma, \xi) = (\eta, \eta' - \eta)$ and on ω. When this needs to be stressed we denote

$$(\sigma_t, \xi_t) = (\sigma_t, \xi_t)[(\sigma, \xi), \omega)] = (\sigma_t[(\sigma, \xi), \omega)], \xi_t[(\sigma, \xi), \omega)]), \tag{6.4}$$

either way.

7 Law of large numbers

Flux Let $(y_t : t \geq 0)$ be an arbitrary trajectory in $\mathbb{R}$ with $y(0) = 0$. Define the *flux* of particles along y_t by

$$F_{y_t}(t)[\eta, \omega] := \sum_{i \leq 0} \mathbf{1}\{X_t(i)[\eta, \omega] > y_t\} - \sum_{i > 0} \mathbf{1}\{X_t(i)[\eta, \omega] \leq y_t\} \tag{7.1}$$

Consider the configuration $\tilde{\eta}$ defined from η in (5.2), having a particle at the origin. Recall

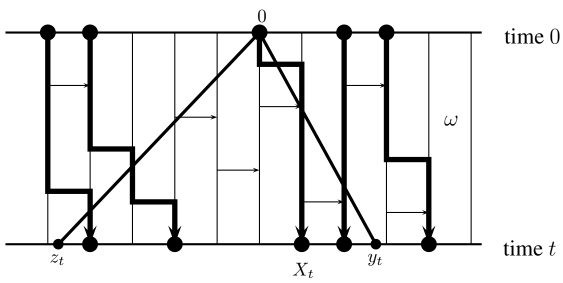

Figure 7.1: The flux along trajectory y_t is -1 and the flux along trajectory z_t is 3.

X_t is the position of the tagged particle of $\tilde{\eta}$ initialmente at the origin and observe that the flux of $\tilde{\eta}$ particles along the tagged particle X_t is null:

$$F_{X_t}(t)[\tilde{\eta}, \omega] \equiv 0. \tag{7.2}$$

Hence we have the following alternative expression for the flux of $\tilde\eta$ particles.

$$F_{y_t}(t)[\tilde\eta,\omega] = \sum_x \tilde\eta_t(x)\,\big(\mathbf{1}\{y_t < x \le X_t\} - \mathbf{1}\{X_t < x \le y_t\}\big); \tag{7.3}$$

only one of the indicator functions is no null in each term of (7.4). And, since η and $\tilde\eta$ have at most one discrepancy which is conserved by (6.2),

$$F_{y_t}(t)[\eta,\omega] = \sum_x \eta_t(x)\,\big(\mathbf{1}\{y_t < x \le X_t\} - \mathbf{1}\{X_t < x \le y_t\}\big) \;+\; O(1). \tag{7.4}$$

Proposition 7.1. *Let $a \in \mathbb{R}$. Then,*

$$\lim_{t\to\infty} \frac{F_{at}(t)[\eta^\rho,\omega]}{t} = \rho[(1-\rho)-a], \qquad a.s. \tag{7.5}$$

Proof. Using (7.4) we can write

$$\begin{aligned} F_{at}(t)[\eta^\rho,\omega] = &\sum_x \eta_t^\rho(x)\,\big(\mathbf{1}\{at < x \le (1-\rho)t\} - \mathbf{1}\{(1-\rho)t < x \le at\}\big) \\ &+ \sum_x \eta_t^\rho(x)\,\big(\mathbf{1}\{(1-\rho)t < x \le X_t\} - \mathbf{1}\{X_t < x \le (1-\rho)t\}\big) + O(1) \end{aligned}$$

Dividing by t and taking $t\to\infty$, the first term converges a.s. to $\rho[(1-\rho)-a]$ because $\eta_t^\rho \sim \eta^\rho$ by Lemma 3.1. The absolute value of the second term is bounded by $|X_t-(1-\rho)t|/t$ which goes to zero a.s. by Proposition 5.2. □

Tagged second class particle Take $0\le\lambda<\rho\le 1$ and define the two-class process

$$(\sigma_t,\xi_t) := (\eta_t^\lambda, \eta_t^\rho - \eta_t^\lambda) \tag{7.6}$$

The marginal laws of σ_t and $\sigma_t+\xi_t$ are stationary but the process (σ_t,ξ_t) is not stationary. To put a second class particle at the origin define $\underset{\sim}{\eta}$ as the configuration

$$\underset{\sim}{\eta}(x) := \begin{cases} 0 & \text{if } x = 0 \\ \eta(x) & \text{otherwise.} \end{cases} \tag{7.7}$$

and recall $\tilde\eta$ defined in (5.2) as the configuration η with a particle at the origin. Now define

$$(\underset{\sim}{\sigma}_t, \tilde\xi_t) := (\underset{\sim}{\eta}_t^\lambda, \tilde\eta_t^\rho - \underset{\sim}{\eta}_t^\lambda). \tag{7.8}$$

The initial configuration for this process is identical to (σ,ξ) out of the origin while at the origin there is a second class particle: $\sigma(0)=0$ and $\xi(0)=1$.

Proposition 7.2. *Take $\lambda<\rho$ and let $Y_t^{\lambda,\rho}$ be the position of the tagged ξ particle for the process* (7.8)*, initially located at the origin, $Y_0^{\lambda,\rho}=0$. Then,*

$$\lim_{t\to\infty} \frac{Y_t^{\lambda,\rho}}{t} = 1-\lambda-\rho, \qquad a.s. \tag{7.9}$$

Proof. Denote $G_{y_t}(t)[(\utilde{\sigma},\tilde{\xi}),\omega]$ the flux of $\tilde{\xi}$ particles along a trajectory y_t for the process $(\utilde{\sigma}_t,\tilde{\xi}_t)$. This flux is the difference of the flux of $\tilde{\eta}^\lambda$ and $\utilde{\eta}^\rho$ particles:

$$G_{y_t}(t)[(\utilde{\sigma},\tilde{\xi}),\omega] = F_{y_t}(t)[\utilde{\eta}^\lambda,\omega] - F_{y_t}(t)[\tilde{\eta}^\rho,\omega] \tag{7.10}$$
$$= F_{y_t}(t)[\eta^\lambda,\omega] - F_{y_t}(t)[\eta^\rho,\omega] + O(1) \tag{7.11}$$

the error of order 1 comes from (7.4). Taking $y_t = at$ for some real number a, by the law of large numbers (7.5),

$$\lim_{t\to\infty} \frac{G_{at}(t)[(\utilde{\sigma},\tilde{\xi}),\omega]}{t} = [\rho(1-\rho)-\lambda(1-\lambda)] - a(\rho-\lambda), \quad \text{a.s.} \tag{7.12}$$

The limit is negative for $a > 1-\lambda-\rho$ and positive for $a < 1-\lambda-\rho$. On the other hand, $G_{at}(t)$ is non increasing in a and, by exclusion, the flux of $\tilde{\xi}$ particles along $Y_t^{\lambda,\rho}$ is null: $G_{Y_t^{\lambda,\rho}}(t) \equiv 0$. This implies (7.9). □

Isolated second class particle Take $\alpha \in (0,1)$. To create a second class particle for the configuration η^α we consider the coupling

$$(\utilde{\eta}_t^\alpha, \tilde{\eta}_t^\alpha - \utilde{\eta}_t^\alpha) \tag{7.13}$$

and call R_t^α the position at time t of the second class particle in the coupling (7.13).

Proposition 7.3. *We have*

$$\lim_{t\to\infty} \frac{R_t^\alpha}{t} = 1-2\alpha, \qquad a.s. \tag{7.14}$$

Proof. Take $\alpha < \rho$ and consider the coupling

$$(\utilde{\eta}_t^\alpha, \tilde{\eta}_t^\rho - \utilde{\eta}_t^\alpha) \tag{7.15}$$

and, as before, denote $Y_t^{\alpha,\rho}$ the position of the tagged second class particle initially at the origin for this process. Recalling that we are using the same U and ω in the couplings (7.13) and (7.15) we see that both R_t^α and $Y_t^{\alpha,\rho}$ see the same first class particles $\utilde{\eta}_t^\alpha$ but while R_t^α sees no other particle, $Y_t^{\alpha,\rho}$ is blocked by the second class particles $(\tilde{\eta}_t^\rho - \utilde{\eta}_t^\alpha)$ to its right. For this reason,

$$R_t^\alpha \ge Y_t^{\alpha,\rho}, \qquad \text{if } \alpha < \rho. \tag{7.16}$$

On the other hand, take $\lambda < \alpha$ and consider the coupling

$$(\utilde{\eta}_t^\lambda, \tilde{\eta}_t^\alpha - \utilde{\eta}_t^\lambda). \tag{7.17}$$

The first class particles for $Y_t^{\lambda,\alpha}$ are $\eta_t^{\lambda,\alpha} \le \eta_t^\alpha$, the first class particles for R_t^α. See (8.4) and (8.5) below for more details. Hence

$$R_t^\alpha \le Y_t^{\lambda,\alpha}, \qquad \text{if } \lambda < \alpha. \tag{7.18}$$

Use the law of large numbers (7.9) to conclude. □

8 Proof of hydrodynamics: increasing shock

In this section we prove Theorem 4.1 in the shock case: $u_0 = u^{\lambda,\rho}$ given by (2.2) with $\lambda < \rho$. Recall the solution $u(r,t) = u^{\lambda,\rho}(r-(1-\lambda-\rho)t)$ and the fact that the initial tasep configuration $\eta^\varepsilon = \eta^{\lambda,\rho}$ does not depend on ε.

Let $\Gamma_z : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the *cut operator* defined by

$$\Gamma_z\eta(x) := \eta(x)\mathbf{1}\{x \geq z\}. \tag{8.1}$$

This operator, when applied to the configuration η cuts the η-particles to the left of z. The operator Γ_0, when applied to the second class particles ξ commutes with the dynamics in the following sense. If $\xi(0) = 1$ and Y_t is the position of the ξ particle initially at the origin, then

$$(\sigma_t[(\sigma,\xi),\omega], \Gamma_{Y_t}\xi_t[(\sigma,\xi),\omega]) = (\sigma_t[(\sigma,\Gamma_0\xi),\omega], \xi_t[(\sigma,\Gamma_0\xi),\omega]). \tag{8.2}$$

That is, to cut the initial ξ configuration to the left of the origin and evolve until time t is the same as to cut the ξ_t configuration to the left of Y_t.

Let (σ,ξ) be a two-class configuration and let

$$\eta := \sigma + \Gamma_0\xi. \tag{8.3}$$

Add a second class particle with respect to η_t at the origin at time zero; call R_t its position at time t. Add a ξ particle at the origin at time zero; call Y_t its position at time t. Then, using (8.2),

$$(\underset{\sim}{\eta}_t, R_t) = (\underset{\sim}{\sigma}_t + \Gamma_{Y_t}\tilde{\xi}_t, Y_t). \tag{8.4}$$

We use (8.4) with $(\underset{\sim}{\sigma},\tilde{\xi}) = (\underset{\sim}{\eta}^\lambda, \tilde{\eta}^\rho - \underset{\sim}{\eta}^\lambda)$ so that $\underset{\sim}{\eta}^{\lambda,\rho} = \underset{\sim}{\sigma} + \Gamma_0\tilde{\xi}$, $R_t^{\lambda,\rho}$ is a second class particle with respect to $\underset{\sim}{\eta}_t^{\lambda,\rho}$ and $Y_t^{\lambda,\rho}$ is a $\tilde{\xi}$ tagged particle for $(\underset{\sim}{\sigma}_t, \tilde{\xi}_t)$ to get:

$$(\underset{\sim}{\eta}_t^{\lambda,\rho}, R_t^{\lambda,\rho}) = (\underset{\sim}{\sigma}_t + \Gamma_{Y_t^{\lambda,\rho}}\tilde{\xi}_t, Y_t^{\lambda,\rho}). \tag{8.5}$$

Notice that

$$\eta_t^{\lambda,\rho}(x) = \begin{cases} \underset{\sim}{\eta}_t^{\lambda,\rho}(x) & \text{if } x \neq R_t^{\lambda,\rho} \\ \eta_t^{\lambda,\rho}(0) & \text{if } x = R_t^{\lambda,\rho} \end{cases} \tag{8.6}$$

Proof of local equilibrium (4.6) **for** $\lambda < \rho$ Let $A \subset \mathbb{Z}$ be a finite set and recall $f_A(\eta) := \prod_{x\in A}\eta(x)$. To simplify notation we use t as rescaling parameter and will show

$$\lim_{t\to\infty} Ef_A(\tau_{rt}\eta_t^{\lambda,\rho}) = \rho^{|A|}. \tag{8.7}$$

This corresponds to show (4.6) for macroscopic time $t = 1$. Take first $r > (1-\lambda-\rho)$ and denote $Y_t = Y_t^{\lambda,\rho}$ the position of the tagged ξ particle. By (8.5) and (8.6) we get

$$\begin{aligned} Ef_A(\tau_{rt}\underset{\sim}{\eta}_t^{\lambda,\rho}) &= Ef_A(\tau_{rt}(\underset{\sim}{\sigma}_t + \Gamma_{Y_t}\tilde{\xi}_t)) \\ &= E\big[f_A(\tau_{rt}(\sigma_t + \xi_t))\,\mathbf{1}\{Y_t < rt + \min A\}\big] \qquad (8.8) \\ &\quad + E\big[f_A(\tau_{rt}(\underset{\sim}{\sigma}_t + \Gamma_{Y_t}\tilde{\xi}_t))\,\mathbf{1}\{Y_t \geq rt + \min A\}\big] \\ &= Ef_A(\tau_{rt}(\sigma_t + \xi_t)) + O\big[P(Y_t \geq rt + \min A)\big] \qquad (8.9) \\ &= \rho^{|A|} + O\big[P(Y_t \geq rt + \min A)\big] \underset{t\to\infty}{\to} \rho^{|A|}, \qquad (8.10) \end{aligned}$$

where in (8.8) we used the definition (8.1) of Γ_z, in (8.9) the bound $|f_A| \le 1$ and in (8.10) the law of large numbers $Y_t/t \to 1-\lambda-\rho$. Finally, observe that $|f_A(\tau_{rt}\eta_t^{\lambda,\rho}) - f_A(\tau_{rt}\tilde{\eta}_t^{\lambda,\rho})| \le \mathbf{1}\{R_t - rt \in A\} \to 0$ a.s. if $r \ne (1-\rho-\lambda)$. This concludes the proof of (8.7) when $r > (1-\lambda-\rho)$.

When $r < (1-\lambda-\rho)$ the same argument shows

$$Ef_A(\tau_{rt}\eta_t^{\lambda,\rho}) = \lambda^{|A|} + O\big[P(Y_t \le rt + \max A)\big] \underset{t\to\infty}{\to} \lambda^{|A|}. \tag{8.11}$$

This finishes the proof of (8.7) for all $r \ne 1-\lambda-\rho$ and $\lambda < \rho$.

To show the result for the macroscopic variables r and t, that is (4.6), substitute t by $\varepsilon^{-1}t$ and rt by $\varepsilon^{-1}r$ in the above proof.

Proof of convergence of the density fields We use the same argument and notation as in the previous proof. Fix $1-\lambda-\rho < a < b$ and write

$$\frac{1}{t}\sum_{at\le x\le bt} \eta_t^{\lambda,\rho}(x) = \frac{1}{t}\sum_{at\le x\le bt} (\sigma_t(x) + \Gamma_{Y_t}\xi_t(x)) \underset{t\to\infty}{\to} \rho(b-a) \tag{8.12}$$

where we used the law of large numbers for the marginal distribution of $\sigma_t + \xi_t = \eta^\rho$ and the law of large numbers for the tagged second class particle $Y_t/t \to 1-\rho-\lambda$. The same argument applied to $a < b < 1-\lambda-\rho$ shows

$$\frac{1}{t}\sum_{at\le x\le bt} \eta_t^{\lambda,\rho}(x) = \frac{1}{t}\sum_{at\le x\le bt} (\sigma_t(x) + \Gamma_{X_t}\xi_t(x)) \underset{t\to\infty}{\to} \lambda(b-a) \tag{8.13}$$

using now the law of large numbers for $\sigma_t = \eta^\lambda$. Since for $a < 1-\lambda-\rho < b$ we have $\frac{1}{t}\sum_{at\le x\le bt} \eta_t^{\lambda,\rho}(x) \le b-a$, we can conclude.

9 Proof of hydrodynamics: rarefaction fan

Here we consider $\lambda > \rho$, when the solution is the rarefaction fan (2.5). An essential component of this proof is the law of large number for a second class particle Proposition 7.3. We first prove a crucial lemma. Recall that the processes η_t^ρ and $\eta_t^{\lambda,\rho}$ defined in (3.3) and (4.8) are all constructed with the same U and ω, so naturally coupled.

Lemma 9.1. *Take $\lambda > \rho$ and for each $\alpha \in [0,1]$ let R_t^α be a second class particle initially at the origin for the process η_t^α as defined in* (7.13)*. Then*

$$\eta_t^{\lambda,\rho}(x) = \begin{cases} \eta_t^\rho(x) & \text{if } x > R_t^\rho \\ \eta_t^\lambda(x) & \text{if } x < R_t^\lambda. \end{cases} \tag{9.1}$$

Furthermore, for $\lambda \ge \alpha \ge \rho$ we have

$$\eta_t^{\lambda,\rho}(x) \le \eta_t^\alpha(x), \quad \text{for } x > R_t^\alpha, \tag{9.2}$$

$$\eta_t^\alpha(x) \le \eta_t^{\lambda,\rho}(x), \quad \text{for } x < R_t^\alpha. \tag{9.3}$$

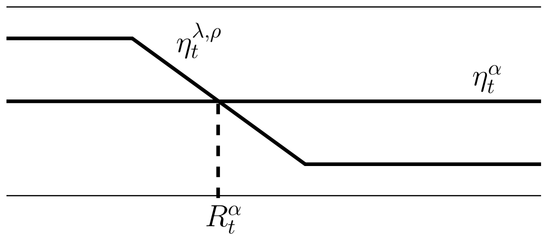

Figure 9.1: Macroscopic schema of (9.2) and (9.3). The configuration $\eta_t^{\lambda,\rho}$ dominates η_t^α to the left of R_t^α and the opposite happens to its right.

Proof. The first identity in (9.1) holds at time 0 because $R_0^\rho = 0$ and the initial configurations $\eta^{\lambda,\rho}(x) = \eta^\rho(x)$ for all $x > 0$, by definition. Hence we can define

$$(\sigma_t, \xi_t) := (\eta_t^\rho, \eta_t^{\lambda,\rho} - \eta_t^\rho),$$

where σ_t are first class particles and ξ_t are second class particles and clearly $\xi_0(x) = 0$ for $x > 0$. Force $\xi_0(0) = 1$. Then R_t^ρ, the second class particle for η^ρ, coincides with the position of the rightmost ξ_t particle:

$$R_t^\rho = \max\{y : \xi_t(y) = 1\}.$$

This is because ξ particles interact by exclusion and the ξ particle initially at the origin

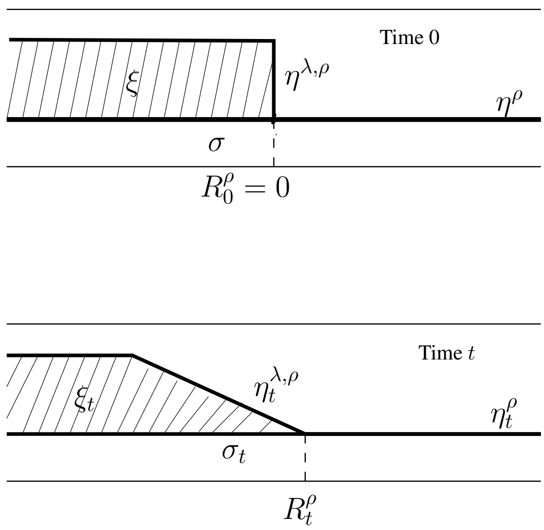

Figure 9.2: Macroscopic schema of the coupling to show (9.1). There are no ξ_t particles to the right of R_t^ρ at time t.

does not feel the ξ particles to its left. Furthermore the ξ particle initially at the origin is a second class particle with respect to η^ρ particles. This proves the first identity in (9.1).

The second identity in (9.1) is obtained in the same way by defining

$$(\sigma_t, \xi_t) := (\eta_t^{\lambda,\rho}, \eta_t^\lambda - \eta_t^{\lambda,\rho})$$

and observing that

$$R_t^\lambda = \min\{y : \xi_t(y) = 1\},$$

that is, $\xi_t(x) = 0$ for $x < R_t^\lambda$.

To show (9.2) and (9.3) recall $\lambda \geq \alpha \geq \rho$ and observe that

$$\eta_t^{\lambda,\rho}(x) \leq \eta_t^{\lambda,\alpha}(x) = \eta_t^{\alpha}(x), \qquad \text{for } x > R_t^\alpha \tag{9.4}$$

$$\eta_t^{\alpha}(x) = \eta_t^{\alpha,\rho}(x) \leq \eta_t^{\lambda,\rho}(x), \qquad \text{for } x < R_t^\alpha, \tag{9.5}$$

where the inequalities hold by attractivity and the identities are (9.1). □

Corollary 9.2. *Let $\lambda \geq \alpha > \beta \geq \rho$. Then,*

$$P\Big(\liminf_{t\to\infty} \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbf{1}\{x \in ((1-2\alpha)t,(1-2\beta)t)\} \geq 2(\alpha-\beta)\beta\Big) = 1 \tag{9.6}$$

$$P\Big(\limsup_{t\to\infty} \frac{1}{t}\sum_x \eta_t^{\lambda,\rho}(x)\mathbf{1}\{x \in ((1-2\alpha)t,(1-2\beta)t)\} \leq 2(\alpha-\beta)\alpha\Big) = 1 \tag{9.7}$$

Proof. From (9.2)-(9.3),

$$\sum_x \eta_t^{\alpha}(x)\mathbf{1}\{x \in (R_t^\beta, R_t^\alpha)\} \leq \sum_x \eta_t^{\lambda,\rho}(x)\mathbf{1}\{x \in (R_t^\beta, R_t^\alpha)\}, \tag{9.8}$$

from where,

$$\begin{aligned}\sum_x &\eta_t^{\alpha}(x)\mathbf{1}\{x \in ((1-2\beta)t,(1-2\alpha)t)\}\\ &\leq \sum_x \eta_t^{\lambda,\rho}(x)\mathbf{1}\{x \in ((1-2\beta)t,(1-2\alpha)t)\}\\ &\qquad + 2|R_t^\beta-(1-2\beta)t| + 2|R_t^\alpha-(1-2\alpha)t)|\end{aligned} \tag{9.9}$$

Divide by t, take $t \to \infty$ and use the law of large numbers for $\eta_t^\alpha \sim \eta^\alpha$ and for R_t^α, R_t^β to get (9.6). The same argument shows (9.7). □

Proof of convergence of the density fields Fix $r \in (1-2\lambda, 1-2\rho)$ and use the bound (9.7) with $\beta = k/n$ and $\alpha = (k-1)/n$ to obtain

$$\begin{aligned}\limsup_t \frac{1}{t} &\sum_{x\in(rt,(1-2\rho)t)} \eta_t^{\lambda,\rho}(x)\\ &= \limsup_t \frac{1}{t}\sum_{k=1}^n\sum_x \eta_t^{\lambda,\rho}(x)\,\mathbf{1}\big\{x \in [t(1-2\tfrac{k}{n}), t(1-2\tfrac{k-1}{n})] \cap [rt,(1-2\rho)t)\big\}\\ &\leq \sum_{k=1}^n \frac{k}{n}\frac{2}{n}\,\mathbf{1}\big\{\rho \leq \tfrac{k}{n} \leq \tfrac{1-r}{2}\big\}\\ &\underset{n\to\infty}{\longrightarrow} \int_\rho^{\frac{1-r}{2}} 2r'dr' \;=\; \Big(\frac{1-r}{2}\Big)^2 - \rho^2 \;=\; \int_r^{1-2\rho} u(r',1)dr'.\end{aligned}$$

The same argument using (9.6) shows that

$$\liminf_t \frac{1}{t}\sum_{x\in(rt,(1-2\rho)t)} \eta_t^{\lambda,\rho}(x) \geq \int_r^{1-2\rho} u(r',1)dr'.$$

This proves (4.5) for intervals $(a,b) \subset (1-2\lambda, 1-2\rho)$. Take now $a < 1-2\lambda$ and use the second identity in (9.1) and the law of large numbers for $\mathbb{R}_t^\lambda$ to conclude that

$$\lim_t \frac{1}{t} \sum_{x \in (at,(1-2\lambda)t)} \eta_t^{\lambda,\rho}(x) = \lambda(1-2\lambda-a) = \int_a^{1-2\lambda} u(r',1)dr'. \tag{9.10}$$

Take $b > 1-2\rho$ and use the first identity in (9.1) and the law of large numbers for $\mathbb{R}_t^\rho$ to conclude

$$\lim_t \frac{1}{t} \sum_{x \in ((1-2\rho)t,bt)} \eta_t^{\lambda,\rho}(x) = \rho(b-(1-2\rho)) = \int_{1-2\rho}^{b} u(r',1)dr'. \tag{9.11}$$

Proof of density profile and local equilibrium Take a finite integer set A and recall $f_A(\eta) = \prod_{x\in A} \eta(x)$. Take $\lambda \geq \alpha > \beta \geq \rho$. From (9.2)-(9.3) we have

$$B_t := \big\{R_t^\alpha < rt + x < R_t^\beta, x \in A\big\} \subset \Big\{f_A(\tau_{rt}\eta_t^\alpha) \geq f_A(\tau_{rt}\eta_t^{\lambda,\rho}) \geq f_A(\tau_{rt}\eta_t^\beta)\Big\}. \tag{9.12}$$

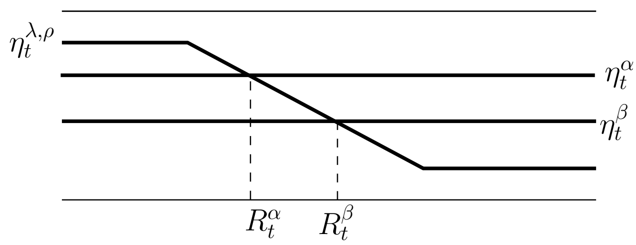

Figure 9.3: Macroscopic schema of (9.12).

Hence, denoting $\mathbf{1}B$ the indicator function of the set B, we have

$$E(f_A(\tau_{rt}\eta_t^\beta)\,\mathbf{1}B_t) \leq E(f_A(\tau_{rt}\eta_t^{\lambda,\rho})\,\mathbf{1}B_t) \leq E(f_A(\tau_{rt}\eta_t^\alpha)\,\mathbf{1}B_t).$$

By the law of large numbers for R_t^α and R_t^β, for $r \in ((1-2\alpha),(1-2\beta))$ we have $\lim_t \mathbf{1}B_t = 1$ a.s.. Hence, since $|f_A| \leq 1$, for $r \in ((1-2\alpha),(1-2\beta))$,

$$\beta^{|A|} \leq \liminf_t E(f_A(\tau_{rt}\eta_t^{\lambda,\rho})) \leq \limsup_t E(f_A(\tau_{rt}\eta_t^{\lambda,\rho})) \leq \alpha^{|A|}.$$

Take $\alpha \nearrow \frac{1-r}{2}$ and $\beta \searrow \frac{1-r}{2}$ to get

$$\lim_t E(f_A(\tau_{rt}\eta_t^{\lambda,\rho})) = \Big(\frac{1-r}{2}\Big)^{|A|} = u(r,1)^{|A|}.$$

This proves local equilibrium for r in the rarefaction fan $((1-2\lambda),(1-2\rho))$. For $r \geq 1-2\rho$ we know that $\eta_t^{\lambda,\rho}(x) = \eta_t^\rho(x)$ when $x > R_t^\rho$. This together with the law of large numbers for R_t^ρ allows to conclude. The same argument holds for $r < 1-2\lambda$.

10 Notes and references

There are many papers about hydrodynamics of interacting particles systems. We just quote some reviews and books. De Masi and Presutti [12, 10], Kipnis and Landim [32] and Lebowitz, Presutti and Spohn [37].

Lax (1972) shows the role of characteristics to solve the initial value problem of the Burgers equation. See also Evans [14]. Rezakhanlou [45] shows there that if the initial condition presents no decreasing discontinuity at a, then there is only one characteristic emanating from a. Rezakhanlou [46] shows that a local perturbation of the initial condition of the Burgers equation behaves like the characteristics or a shock.

The convergence of the hydrodynamic limit of the tasep to the Burgers equation has also different approaches and results. The local-equilibrium convergence (4.6) was proven by Liggett [38, 40] for the case $r = 0$, before the conection between the process and the Burgers equation appeared. The first paper realizing this connection was Rost [47] who studied the rarefaction fan case. Rost uses the sub-additive ergodic theorem to show almost sure convergence of the density fields and then a comparison with stationary systems of queues to identify the limit and to show local equilibrium; see also Liggett's book [41]. The result is generalized by Seppalainen [51, 50, 52, 49], who uses it to prove almost sure convergence of density fields for a large class of initial conditions. Proofs for more general initial profiles were provided by Benassi and Fouque [7], Benassi, Fouque, Saada, and Vares [8]. Andjel and Vares [3] prove convergence of the expectation of the density fields for general initial profiles for a class of processes including the tasep, without using subadditivity.

In dimension $d \geq 1$, Rezakhanlou [44] proves convergence in probability of the density fields while Landim [35] shows that this limit is enough to have local equilibrium. See also Landim [33, 34].

The use of law of large numbers for tagged and second class particles to show hydrodynamics in the tasep was used by Ferrari, Kipnis and Saada [25] and the author [16, 17] for the shock case. Sections 7 and 8 of this paper are based on [17], with some simplifications.

Further results not discussed in this paper Local equilibrium does not hold at the discontinuity points of the solution u. Wick [53], Andjel, Bramson and Liggett [2], De Masi, Kipnis, Presutti, and Saada [11] have proven partial results. The author [17] proved that the limit is a convex combination of the product measures with densities λ and ρ, depending if the second class particle for $\eta_t^{\lambda,\rho}$ is to the right or left of $(1 - \lambda - \rho)t$.

Microscopic interfaces. A second class particle with respect to a product initial configuration with densities $\lambda < \rho$ to the left and right of the origin, respectively, sees at any time t a measure that is absolutely continuous with respect to the product measure with a bounded Radom-Nikodim derivative. In fact, there exists an invariant measure for the process as seen from the second class particle which is absolutely continuous with respect to the product measure. This started with [25, 16, 17], then Derrida, Janowsky, Lebowitz and Speer [13] computed the measure, from where subsequent progress done by Ferrari Fontes Kohayakawa [22] and Angel [4] permitted Ferrari and Martin [26] to give a complete description of that measure in terms of the output of a discrete-time stationary MM1 queue.

Diffusive fluctuations. The flux or current of particles along lines different from the characteristic have variance of order t explicitely computed by Ferrari and Fontes [19], see also Ben Arous and Corwin [6]. For the second class particle in the shock also has variance of order t, computed in [20, 18].

The flux of particles along a characteristic has non-diffusive fluctuations, while a second class particle in a translation invariant Bernoulli measure has super diffussive behavior [19]. Ferrari and Spohn [29] compute the equilibrium current fluctuations along the characteristic of order $t^{1/3}$ and limit in distribution GUE Tracy-Widom distribution. For the growth process associated to the tasep Johansson [31] computes limiting fluctuations of order $t^{1/3}$, and find the limit distribution, see Prahoffer and Spohn [43] and Ben Arous and Corwin [6]. Balasz, Cator and Seppalainen [5] compute the order $t^{2/3}$ for the variance of the mentioned growth model.

The second class particle in the rarefaction fan converges almost surely to a uniform random variable in $[-1, 1]$. See Ferrari and Kipnis [24] for convergence in distribution and Mountford and Guiol [42] Ferrari, Pimentel and Martin [28, 27] for a.s. convergence. Further results can be found in Ferrari, Gonçalves and Martin [23] and Amir, Angel and Valko [1].

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