Expected Reachability-Price Games

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August 29, 2019

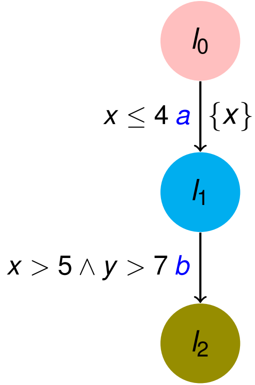
Overview and Contribution

Expected reachability-price games on priced probabilistic timed automata (PPTA)

Conditions to reduce to a finite stochastic game arena.

Decidability for PPTA with single clock and price-rates restricted to $\{0, 1\}$.

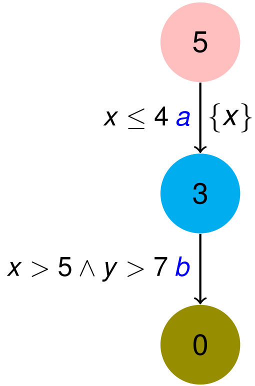
Timed Automaton (TA)



$$\bowtie \in \{<, \le, =, >, \ge\}$$

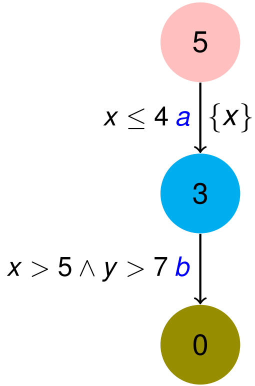
$$a, b \in Act$$

Priced Timed Automaton



location cost (per time unit)

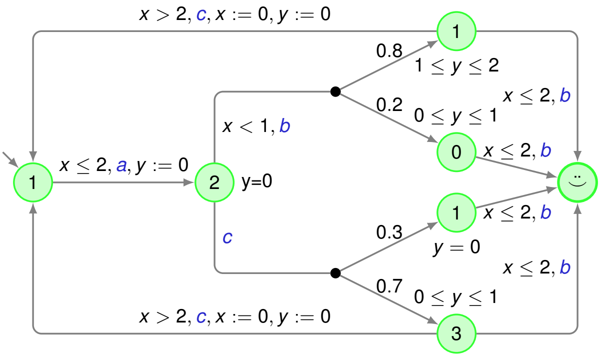
Priced Timed Automaton



location cost (per time unit)

Cost bounded reachability: given threshold k and a goal location, whether exists a timed path ρ such that $cost(\rho) \leq k$?

Priced Probabilistic Timed automata



The semantics is given by an uncountable MDP with a set of timed actions from $Act \times \mathbb{R}_{\geq 0}$.

Reachability on (Priced) Probabilistic Timed automata

Reachability with cost $\leq k$ and probability $\geq p$?

Undecidable for three clocks and clock rates in $\{0, 1\}$

Undecidable even for two clocks and integer clock rates

Undecidability of cost-bounded reachability in priced probabilistic timed automata
J Berendsen, T Chen, DN Jansen - ... on Theory and Applications of Models ..., 2009 - Springer

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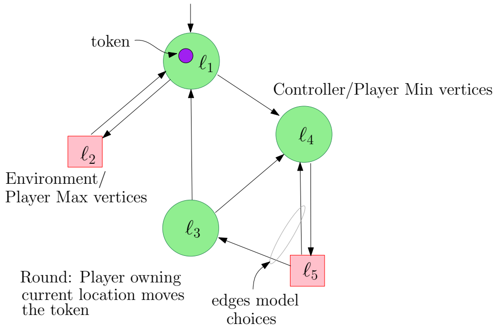
Optimal expected cost problem is decidable for probabilistic timed automata.

Concavely-Priced Probabilistic Timed Automata
M Jurdziński, M Kwiatkowska, G Norman, A Trivedi - CONCUR 2009

Two-player reachability games

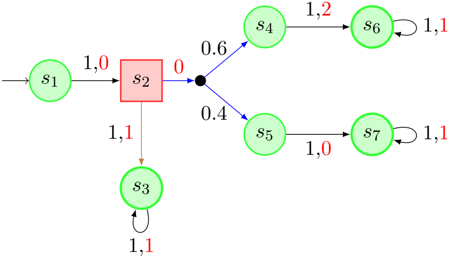
Mathematical model for supervisory controller synthesis.



Reachability objective: Does Player Min have a strategy to reach ℓ_3?

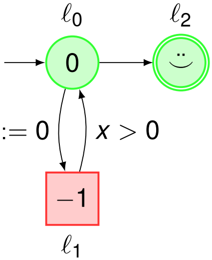
A strategy for a player from a vertex *v* that he owns is an edge/action chosen from *v* given a finite run ending in *v*.

Stochastic game



Round: choose a distribution

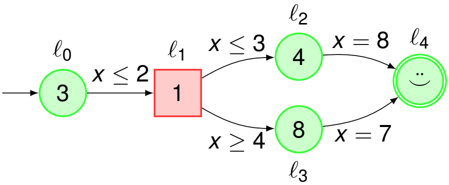
Two-player reachability-price timed games



Player Max does not have an optimal positional strategy

Adding negative prices to priced timed games
T Brihaye, G Geeraerts, SN Krishna, L Manasa... - ... on Concurrency Theory, 2014 - Springer

Two-player reachability-price timed games

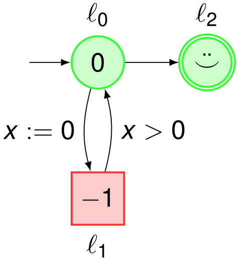


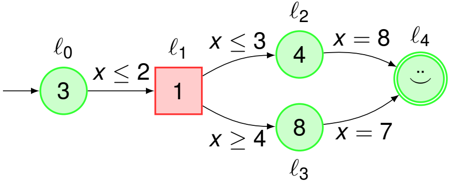
Optimal strategy for Min player: transition from ℓ_0 to ℓ_1 at time $4/3$

A similar example with two clocks appears in the following paper.

Optimal strategies in priced timed game automata
P Bouyer, F Cassez, E Fleury, KG Larsen - International Conference on …, 2004 - Springer

Two-player reachability-price timed games





- Negative prices: may not be positional strategies for Max player
- Arbitrary positive prices: optimal strategies may not be boundary

Reachability price games on Timed automata

Cost-bounded reachability

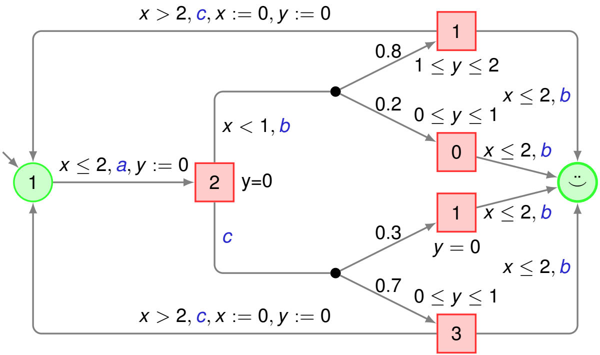
- Undecidable for three clocks with costs $0, 1$.

Improved undecidability results on weighted timed automata
P Bouyer, T Brihaye, N Markey - Information Processing Letters, 2006 - Elsevier

- Undecidable for two clocks with both positive and negative prices
- Decidability for one-clock bi-valued (a set of two integers from $\{-d, 0, d\}$) price timed automata

Adding negative prices to priced timed games
T Brihaye, G Geeraerts, SN Krishna, L Manasa... - ... on Concurrency Theory, 2014 - Springer

Priced Probabilistic Timed Game Arena (PTGA)



$\mathcal{T} = (\mathrm{T}, L_{\mathrm{Min}}, L_{\mathrm{Max}})$ The semantics is given by a stochastic game arena $[\![\mathcal{T}]\!] = ([\![\mathrm{T}]\!], S_{\mathrm{Min}}, S_{\mathrm{Max}})$.

S_{Min}: controlled by player Min

S_{Max}: controlled by player Max

Expected reachability-price game (ERPG)

Player Min attempts to reach a final state with expected cost as low as possible.

$$\mathrm{EReach}(s, \mu, \chi) \stackrel{\text{def}}{=} \mathbb{E}_{s}^{\mu,\chi} \left\{ \sum_{i=1}^{\min\{i \mid X_i \in F\}} \pi(X_{i-1}, Y_i) \right\} .$$

Amount that Player Min loses to Player Max.

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$$\text{Upper value} : \overline{\mathrm{Val}}(s) \stackrel{\text{def}}{=} \inf_{\mu \in \Sigma_{\mathrm{Min}}} \sup_{\chi \in \Sigma_{\mathrm{Max}}} \mathrm{EReach}(s, \mu, \chi).$$

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Lower value : $\underline{\mathsf{Val}}(s) \stackrel{\text{def}}{=} \sup_{\chi \in \Sigma_{\mathrm{Max}}} \inf_{\mu \in \Sigma_{\mathrm{Min}}} \mathrm{EReach}(s, \mu, \chi)$.

A game is determined if $\underline{\mathsf{Val}}(s) = \overline{\mathsf{Val}}(s)$ for all $s \in S$.

Proposition

Every expected reachability-price game is determined.

Expected reachability-price problem

Given an expected reachability-price game $\mathcal{T} = (\mathsf{T}, L_{\mathsf{Min}}, L_{\mathsf{Max}})$,

- initial state $s \in S$,
- a bound $B \in \mathbb{R}$

decide whether $\mathsf{Val}(s) \leq B$.

Optimality equations

- characterises the value in an expected reachability-price game.
- $P : S \to \mathbb{R}_{\geq 0}$ is a solution of optimality equations $\mathrm{Opt}(\mathcal{T})$,

  if, for all $s \in S$:

$$P(s) = \begin{cases} 0 & \text{if } s \in F \\ \inf_{\tau \in A(s)} \{\pi(s,\tau) + \sum_{s' \in S} p(s'|s,\tau) \cdot P(s')\} & \text{if } s \in S_{\mathrm{Min}} \setminus F \\ \sup_{\tau \in A(s)} \{\pi(s,\tau) + \sum_{s' \in S} p(s'|s,\tau) \cdot P(s')\} & \text{if } s \in S_{\mathrm{Max}} \setminus F. \end{cases}$$

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Proposition

If $P \models Opt(\mathcal{T})$, then $Val(s) = P(s)$ for all $s \in S$ and, for every $\varepsilon > 0$, both players have pure ε-optimal strategies.

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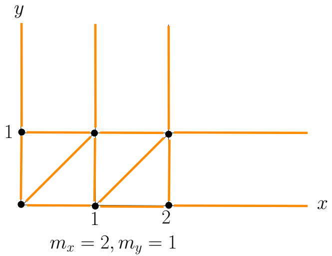
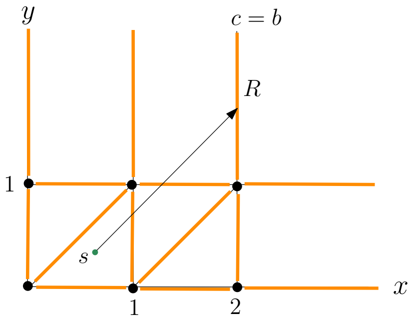
The problem of solving an expected reachability-price game on $\mathcal{T}$ can be reduced to solving the optimality equations $\mathrm{Opt}(\mathcal{T})$.

Region graph



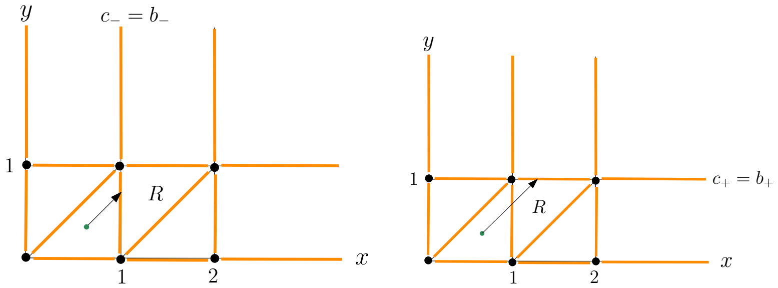
1. for each $x \in C$, either both $v(x) > m_x$ and $v'(x) > m_x$ or $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor$
2. for each $x \in C$ such that $v(x) \leq m_x$
$$frac(v(x)) = 0 \text{ iff } frac(v'(x)) = 0$$
3. for all $x, y \in C$ such that $v(x) \leq m_x$ and $v(y) \leq m_y$,
$$frac(v(x)) \leq frac(v(y)) \text{ iff } frac(v'(x)) \leq frac(v'(y)).$$

Boundary region abstraction (PTGA)



s to thin region R: $((b, c, a), R)$

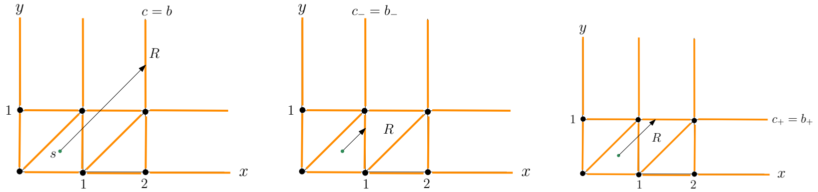
Boundary region abstraction (PTGA)



s to thick region R in the future:

- infimum delay action: $((b_-, c_-, a), R)$
- supremum delay action: $((b_+, c_+, a), R)$

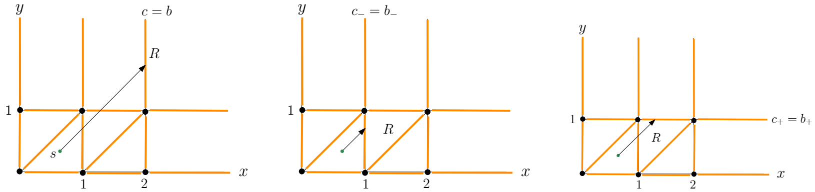
Boundary region abstraction (PTGA)



Summarise the boundary timed actions:

finitely many actions from each state

Boundary region abstraction (PTGA)



Summarise the boundary timed actions:

finitely many actions from each state

$$\widehat{S} = \{((\ell,\nu),(\ell,\zeta)) \mid (\ell,\zeta) \in \mathcal{R} \wedge \nu \in \overline{\zeta}\}$$

Boundary region abstraction (PTGA)

Lemma

For every state of a boundary region abstraction, its reachable sub-graph is finite.

The reachable sub-graph from the initial valuation corresponds to the standard corner-point abstraction.

On the optimal reachability problem of weighted timed automata
P Bouyer, T Brihaye, V Bruyère, JF Raskin - Formal Methods in System ..., 2007 - Springer

ERPG on Boundary region abstraction

Non-expansive and monotonically decreasing functions

A function $F : X \to \mathbb{R}$ is non-expansive if $|F(\nu) - F(\nu')| \leq \|\nu - \nu'\|$ for all $\nu, \nu' \in X$.

Nice functions

$F : \widehat{S} \to \mathbb{R}_{\geq 0}$ is regionally nice if for every region $(\ell, \zeta) \in \mathcal{R}$ the function $F((\ell, \cdot), (\ell, \zeta))$ is nice.

Properties of nice functions

Non-expansive and monotonically decreasing functions

1. **Minimum and Maximum.** $F, F' : \widehat{S} \to \mathbb{R}$ are regionally nice functions.

   Then $\min(F, F')$ and $\max(F, F')$ are also regionally nice.

2. **Convex Combination.** $\langle f_i \rangle_{i=1}^{n}$ are nice functions then for $\langle p_i \in [0, 1] \rangle_{i=1}^{n}$ with $\sum_{i=1}^{n} p_i = 1$. Then $\sum_{i=1}^{n} p_i \cdot f_i$ is nice.

3. **Limit.** The limit of a sequence of nice functions is nice.

Optimality equations for ERPG on BRA

$P : \widehat{S} \to \mathbb{R}_{\geq 0}$ is a solution of optimality equations $\mathrm{Opt}(\widehat{\mathcal{T}})$:
$P \models \mathrm{Opt}(\widehat{\mathcal{T}})$, if for every $s \in \widehat{S}$:

$$P(s) = \begin{cases} 0 & \text{if } s \in \widehat{F} \\ \min\limits_{\alpha \in \widehat{A}(s)} \{\pi(s,\alpha) + \sum\limits_{s' \in S} p(s'|s,\alpha) \cdot P(s')\} & \text{if } s \in \widehat{S}_{\mathrm{Min}} \setminus \widehat{F} \\ \max\limits_{\alpha \in \widehat{A}(s)} \{\pi(s,\alpha) + \sum\limits_{s' \in S} p(s'|s,\alpha) \cdot P(s')\} & \text{if } s \in \widehat{S}_{\mathrm{Max}} \setminus \widehat{F} \end{cases}$$

ERPG on Boundary region abstraction

Consider $f : \widehat{S} \to \mathbb{R}$ over boundary region abstraction.

$\widetilde{f} : S \to \mathbb{R}$ over PTGA: $\widetilde{f}(\ell, \nu) = f((\ell, \nu), (\ell, [\nu]))$.

Let $\mathcal{T}$ be a binary-priced probabilistic timed game.

Theorem
If $P \models Opt(\widehat{\mathcal{T}})$ and P is regionally nice then $\widetilde{P} \models Opt(\mathcal{T})$.

Conditions for reducing expected reachability-price games over the boundary region abstraction.

One-clock binary-priced PTGA

Clock values in $\{0, 1\}$

Proposition

Let $\mathcal{T}$ be a one-clock binary-priced PTGA. If $P \models Opt(\widehat{\mathcal{T}})$, then P is regionally nice.

One-clock binary-priced PTGA

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Let $\mathcal{T}$ be a one-clock binary-priced PTGA. If $P \models Opt(\widehat{\mathcal{T}})$, then P is regionally nice.

Proof sketch: $\Psi : [\widehat{S} \to \mathbb{R}_{\geq 0}] \to [\widehat{S} \to \mathbb{R}_{\geq 0}]$

$$\Psi(f)(s) = \begin{cases} 0 & \text{if } s \in \widehat{F} \\ \min\limits_{\alpha \in \widehat{A}(s)} \{\pi(s,\alpha) + \sum\limits_{s' \in S} p(s'|s,\alpha) \cdot f(s')\} & \text{if } s \in \widehat{S}_{\mathrm{Min}} \setminus \widehat{F} \\ \max\limits_{\alpha \in \widehat{A}(s)} \{\pi(s,\alpha) + \sum\limits_{s' \in S} p(s'|s,\alpha) \cdot f(s')\} & \text{if } s \in \widehat{S}_{\mathrm{Max}} \setminus \widehat{F}. \end{cases}$$

Ψ^N is a contraction: Using Banach's fixed point theorem: Ψ can be used in an iterative scheme to converge to the solution of optimality equations $\mathrm{Opt}(\widehat{\mathcal{T}})$.

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Proposition

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Proof sketch:

Now show that the fixpoint is regionally nice.

If f is regionally nice, then so is $\pi(s, \alpha) + \sum_{s' \in S} p(s'|s, \alpha) \cdot f(s')$ for one-clock binary-priced PTGA.

Fixpoint is regionally nice follows from properties of nice functions: Minimum and maximum, convex combination, Limit

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Recall

If $P \models \mathrm{Opt}(\widehat{\mathcal{T}})$ and P is regionally nice then $\widetilde{P} \models \mathrm{Opt}(\mathcal{T})$.

The expected reachability-price game problem is decidable for one-clock binary-priced PTGA.

Conclusion

- Two-player expected reachability-price games
- Decidability of one-clock binary-priced PTGA
  - Reduction of expected reachability-price problem to the game on boundary region abstraction
  - Nice function
- Future Work: Complexity

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Thank You