Asymptotics of the Stirling numbers of the second kind revisited: A saddle point approach

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Introduction

Let \( \binom{n}{m} \) be the Stirling number of the second kind. Their generating function is given by

\[
\sum_n \frac{m!}{n!} \binom{n}{m} z^n = f(z)^m, \\

f(z) := e^z - 1.
\]

In the sequel all asymptotics are meant for \( n \to \infty \).

Let us first summarize the related literature. The asymptotic Gaussian approximation in the central region is proved in Harper [7]. See also Bender [1], Sachkov [13] and Hwang [10].

In the non-central region, most of the previous papers use the solution of

\[
\frac{\rho e^\rho}{e^\rho - 1} = \frac{n}{m}. \\
\text{(1)}
\]

As shown in the next section, this actually corresponds to a Saddle point.
Let us mention

- Hsu [8]:

For $t = o(n^{1/2})$

$$\begin{align*}
\begin{pmatrix} n + t \\ n \end{pmatrix} &= \frac{n^{2t}}{2^t t!} \left[ 1 + \frac{f_1(t)}{n} + \frac{f_2(t)}{n^2} + \ldots \right], \\
f_1(t) &= \frac{1}{3} t(2t + 1).
\end{align*}$$
Moser and Wyman [12]:

For $t = o(\sqrt{n})$, 

$$
\begin{align*}
\binom{n}{n-t} &= \binom{n}{t} q^{-t} \left[ 1 + \frac{(t)^2}{12} q + \frac{(t)^2}{288} q^2 + \ldots \right], \\
q &= \frac{2}{n-t}.
\end{align*}
$$

For $n - m \to \infty$, $n \to \infty$, 

$$
\begin{align*}
\binom{n}{m} &= \frac{n!(e^\rho - 1)^m}{2\rho^n m!(\pi m \rho H)^{1/2}} \left[ 1 - \frac{1}{m \rho} \left( \frac{15 C_3^2}{16 \rho^2 H} - \frac{3 C_4}{4 \rho H^2} \right) + \ldots \right], \\
H &= \frac{e^\rho (e^\rho - 1 - \rho)}{2(e^\rho - 1)^2},
\end{align*}
$$

$C_3, C_4$ are functions of $\rho$. 

Good [6]:

\[
\binom{n+t}{t} = \frac{(t+n)! (e^\rho - 1)^t}{t! \rho^{t+n} [2\pi t (1 + \kappa - (1 + \kappa)^2 e^{-\rho})]^{1/2}} \times
\]

\[
\times \left[ 1 + \frac{g_1(\kappa)}{t} + \frac{g_2(\kappa)}{t^2} + \ldots \right],
\]

\[
\kappa := \frac{n}{t},
\]

\[
g_1(\kappa) = \frac{3\lambda_4 - 5\lambda_3^2}{24},
\]

\[
\lambda_i = \frac{\kappa_i(\rho)}{\sigma^i},
\]

\[
\sigma = \kappa_2(\rho)^{1/2},
\]

\[
\kappa_1 = \kappa, \kappa_2 = (\kappa_1 + 1)(\rho - \kappa_1).
\]
Bender [1]:

\[
\binom{n}{m} \sim \frac{n!e^{-\alpha m}}{m!\rho^{n-1}(1 + e^\alpha)\sigma \sqrt{2\pi n}},
\]

\[
\frac{n}{m} = (1 + e^\alpha) \ln(1 + e^{-\alpha}),
\]

\[
\rho = \ln(1 + e^{-\alpha}),
\]

\[
\sigma^2 = \left(\frac{m}{n}\right)^2 \left[1 - e^\alpha \ln(1 + e^{-\alpha})\right].
\]

It is easy to see that \(\rho\) here coincides with the solution of (1). Bender’s expression is similar to Moser and Wyman’ result.
Bleich and Wang [2]:
Let $\rho_1$ be the solution of

$$\frac{\rho_1 e^{\rho_1}}{e^{\rho_1} - 1} = \frac{n + 1}{m}.$$ 

Then

$$\binom{n}{m} = \frac{n!(e^{\rho_1} - 1)^m}{(2\pi(n + 1))^{1/2}m!\rho_1^n(1 - G)^{1/2}} \times$$

$$\times \left[ 1 - \frac{A}{24(n + 1)(1 - G)^3} + O(1/n^2) \right],$$

where

$$A := 2 + 18G - 20G^2(e^{\rho_1} + 1)$$

$$+ 3G^3(e^{2\rho_1} + 4e^{\rho_1} + 1) + 2G^4(e^{2\rho_1} - e^{\rho_1} + 1),$$

and

$$G = \frac{\rho_1}{e^{\rho_1} - 1}.$$ 

The series is convergent for for $m = o(n^{2/3})$. 
Temme [15]:

\[
\begin{align*}
\left\{ \frac{n}{m} \right\} &= e^A m^{n-m} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k f_k(t_0) m^{-k}, \\
f_0(t_0) &= \left( \frac{t_0}{(1 + t_0)(\rho - t_0)} \binom{n}{m} \right)^{1/2}, \\
t_0 &= \frac{n}{m} - 1,
\end{align*}
\]

where \( A \) is a function of \( \rho, n, m \).
Tsylova [16]:
Let $m = tn + o(n^{2/3})$.

$$\binom{n}{m} = \frac{(\gamma n)^n}{\sqrt{2\pi \delta n}(\gamma n)^m} \exp \left[ -(m - tn)^2 / (2\delta n) \right] (1 + o(1)),$$

$$\gamma(1 - e^{-1/\gamma}) = \gamma,$$

$$\delta = e^{-1/\gamma}(t - e^{-1/\gamma}).$$

After some algebra, this coincides with Moser and Wyman’ result.
Chelluri, Richmond and Temme [3]:
They prove, with other techniques, that Moser and Wyman expression is valid if $n - m = \Omega(n^{1/3})$ and that Hsu formula is valid for $y - x = o(n^{1/3})$

Erdos and Szekeres: see Sachkov [13], p.164:
Let $m < n/\ln n$,

$$\binom{n}{m} = \frac{m^n}{m!} \exp \left[ \left( \frac{n}{m} - m \right) e^{-n/m} \right] (1 + o(1)).$$
All these papers simply use $\rho$ as the solution of (1). They don’t compute the detailed dependence of $\rho$ on $\alpha$ for our range, neither the precise behaviour of functions of $\rho$ they use. Moreover, most results are related to the case $\alpha < 1/2$.

We will use multiseries expansions: multiseries are in effect power series (in which the powers may be non-integral but must tend to infinity) and the variables are elements of a scale: details can be found in Salvy and Shackell [14]. The scale is a set of variables of increasing order. The series is computed in terms of the variable of maximum order, the coefficients of which are given in terms of the next-to-maximum order, etc. Actually we implicitly used multiseries in our analysis of Stirling numbers of the first kind in [11].

Let us finally mention that Hsu [9] consider some generalized Stirling numbers.

In Sec.2, we revisit the asymptotic expansion in the central region and in Sec.3, we analyse the non-central region $j = n - n^\alpha$, $\alpha > 1/2$. We use Cauchy’s integral formula and the saddle point method.
Consider the random variable $J_n$, with probability distribution

$$
\mathbb{P}[J_n = m] = Z_n(m),
$$

$$
Z_n(m) := \binom{n}{m} B_n,
$$

where $B_n$ is the $n$th Bell number. The mean and variance of $J_n$ are given by

$$
M := \mathbb{E}(J_n) = \frac{B_{n+1}}{B_n} - 1,
$$

$$
\sigma^2 := \mathbb{V}(J_n) = \frac{B_{n+2}}{B_n} - \frac{B_{n+1}}{B_n} - 1.
$$
Let \( \zeta \) be the solution of

\[
\zeta e^\zeta = n.
\]

This immediately leads to

\[
\zeta = W(n),
\]

where \( W \) is the Lambert function (we use the principal branch, which is analytic at 0). We have the well-known asymptotic

\[
\zeta = \ln(n) - \ln \ln(n) + \frac{\ln \ln(n)}{\ln(n)} + O\left(\frac{1}{\ln(n)^2}\right). \tag{2}
\]

To simplify our expressions in the sequel, let

\[
F := e^\zeta, \quad G := e^{\zeta/2}.
\]

The multiseries’ scale is here \( \{\zeta, G\} \).
Our result can be summarized in the following local limit theorem

**Theorem 2.1**

Let \( x = (m - M)/\sigma \). Then

\[
Z_n(m) = \frac{\binom{n}{m}}{B_n} = e^{-x^2/2} \frac{(1 + \zeta)^{1/2}}{\sqrt{2\pi G}} \left[ 1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1 + \zeta)^{3/2}} + O(1/G^2) \right].
\]

**Proof.** By Salvy and Shackell [14], we have

\[
M = F + A_1 + O(1/F),
\]

\[
\sigma^2 = \frac{F}{1 + \zeta} + A_3 + O(1/F),
\]

\[
\frac{B_n}{n!} = \exp(T_1)H_0, \tag{3}
\]

\[
T_1 = -\ln(\zeta)\zeta F + F - \zeta/2 - \ln(\zeta) - 1 - \ln(2\pi)/2, \tag{4}
\]
\[ A_1 = -\frac{2 + 3/\zeta + 2/\zeta^2}{2(1 + 1/\zeta)^2}, \]
\[ A_3 = -\frac{2 + 8/\zeta + 11/\zeta^2 + 9/\zeta^3 + 2/\zeta^4}{2(1 + 1/\zeta)^4}, \]
\[ H_0 = \frac{1}{(1 + 1/\zeta)^{1/2}} \left[ 1 + A_5/F + \mathcal{O}(1/F^2) \right], \]
\[ A_5 = -\frac{2 + 9/\zeta + 16/\zeta^2 + 6/\zeta^3 + 2/\zeta^4}{24(1 + 1/\zeta)^3}. \]
This leads to (from now on, we only provide a few terms in our expansions, but of course we use more terms in our computations), using expansions in $G$,

$$
\sigma = \frac{G}{(1 + \zeta)^{1/2}} + \frac{A_3 (1 + \zeta)^{1/2}}{2G} + O\left(\frac{1}{G^3}\right),
$$

$$
\sigma \sim \frac{G}{\sqrt{\zeta}} \sim \frac{\sqrt{n}}{\ln(n)}.
$$
We now use the Saddle point technique (for a good introduction to this method, see Flajolet and Sedgewick [4], ch. VIII). Let \( \rho \) be the saddle point and \( \Omega \) the circle \( \rho e^{i\theta} \). By Cauchy's theorem,

\[
Z_n(m) = \frac{n!}{m!B_n 2\pi i} \int_{\Omega} \frac{f(z)^m}{z^{n+1}} \, dz
\]

\[
= \frac{n!}{m!B_n \rho^n 2\pi} \int_{-\pi}^{\pi} f(\rho e^{i\theta})^m e^{-ni\theta} \, d\theta
\]

\[
= \frac{n!}{m!B_n \rho^n 2\pi} \int_{-\pi}^{\pi} e^{m \ln(f(\rho e^{i\theta})) - ni\theta} \, d\theta
\]

\[
= \frac{n!}{m!B_n \rho^n 2\pi} \int_{-\pi}^{\pi} \exp \left[ m \left\{ -\frac{1}{2} \kappa_2 \theta^2 - \frac{i}{6} \kappa_3 \theta^3 + \ldots \right\} \right],
\]

(5)

\[
\kappa_i(\rho) = \left( \frac{\partial}{\partial u} \right)^i \ln(f(\rho e^u)) |_{u=0}.
\]

(6)

See Good [5] for a neat description of this technique.
Let us now turn to the Saddle point computation. $\rho$ is the root (of smallest module) of

$$m \rho f'(\rho) - nf(\rho) = 0, \text{ i.e.}$$

$$\frac{\rho e^\rho}{e^\rho - 1} = \frac{n}{m},$$

which is, of course identical to (1). After some algebra, this gives

$$\rho = \frac{n}{m} + W\left(-\frac{n}{m}e^{-n/m}\right).$$
In the central region, we choose

\[ m = M + \sigma x = F + \frac{x}{(1 + \zeta)^{1/2}} G + A_1 + \frac{x A_3 (1 + \zeta)^{1/2}}{2G} + \mathcal{O}(1/G^2). \]

This leads to

\[ \ln(m) = \zeta + \frac{x}{(1 + \zeta)^{1/2}} G + \mathcal{O}(1/G^2), \]

\[ \frac{n}{m} = \zeta - \frac{\zeta x}{(1 + \zeta)^{1/2} G} + \frac{-A_1 \zeta + \zeta x^2/(1 + \zeta)}{G^2} + \mathcal{O}(1/G^3), \]

\[ \rho = \zeta - \frac{\zeta x}{(1 + \zeta)^{1/2} G} + \frac{\zeta (-A_1 + x^2/(1 + \zeta) - 1)}{G^2} + \mathcal{O}(1/G^3), \]

\[ \ln(\rho) = \ln(\zeta) - \frac{x}{(1 + \zeta)^{1/2} G} + \mathcal{O}(1/G^2). \]
Now we note that

\[ e^\rho - 1 = \rho e^\rho \frac{m}{n}, \]

\[ \ln(e^\rho - 1) = \rho + \ln(\rho) + \ln(m) - \ln(n), \]  \hspace{1cm} (7)

\[ \ln(n) = \zeta + \ln(\zeta), \]

(8)
so, by Stirling’s formula, with (4), the first part of (5) leads to

\[ \frac{n!}{m! B_n \rho^n} f(\rho)^m = \exp [T_2] H_1 H_2, \]

\[ T_2 = m(\rho + \ln(\rho) - \zeta - \ln(\zeta)) \]

\[ - (m + \ln(2\pi)/2 + \ln(m)/2) - \zeta F \ln(\rho) - T_1, \]

\[ H_1 = \frac{1}{H_0} = (1 + 1/\zeta)^{1/2} - \frac{A_5 (1 + 1/\zeta)^{1/2}}{G^2} + O(1/G^4), \]

\[ H_2 = 1 \left[ 1 + \frac{1}{12m} + \frac{1}{288m^2} + O(1/m^3) \right] \]

\[ = 1 - \frac{1}{12G^2} + \frac{x}{12G^3 (1 + \zeta)^{1/2}} + O(1/G^4). \]
Note carefully that there is a cancellation of the term $m \ln(m)$ in $T_2$. Using all previous expansions, we obtain

$$\exp(T_2) = e^{-x^2/2+\ln(\zeta)}H_3,$$

(9)

$$H_3 = 1 + \frac{x(-15\zeta - 6\zeta^2 - 6A_1 + x^2 - 12A_1\zeta - 6A_1\zeta^2 + 2x^2\zeta - 9}{6(1 + \zeta)^{3/2}G}$$

$$+ \mathcal{O}(1/G^2).$$

We now turn to the integral in (5). We compute

$$\kappa_2 = -\frac{\rho e^\rho(-e^\rho + 1 + \rho)}{(e^\rho - 1)^2} = \zeta - \frac{\zeta x}{(1 + \zeta)^{1/2}G} + \mathcal{O}(1/G^2),$$

and similar expressions for the next $\kappa_i$ that we don’t detail here. Note that $\kappa_3, \kappa_5, \ldots$ are useless for the precision we attain here.
Now we use the classical trick of setting

\[ m \left[ -\kappa_2 \theta^2 / 2! + \sum_{l=3}^{\infty} \kappa_l (i\theta)^l / l! \right] = -u^2 / 2. \]

Computing \( \theta \) as a series in \( u \), this gives, by inversion,

\[ \theta = \frac{1}{G} \sum_{1}^{\infty} a_i u^i, \]

with, for instance

\[ a_1 = \frac{1}{\zeta^{1/2}} + \frac{\zeta^{1/2}}{2G^2} + O(1/G^3). \]
Setting $d\theta = \frac{d\theta}{du} du$, we integrate on $[u = -\infty..\infty]$: this extension of the range can be justified as in Flajolet and Segewick [4], Ch. VIII. Now, inserting the term $\zeta$ coming in (9) as $e^{\ln(\zeta)}$, this gives

$$H_4 = \frac{\zeta^{1/2}}{\sqrt{2\pi G}} \left(1 + \frac{\zeta}{2G^2} + O(1/G^3)\right).$$

Finally, combining all expansions,

$$Z_n(m) = \binom{n}{m} B_n = e^{-x^2/2} H_1 H_2 H_3 H_4 = R_1,$$

$$R_1 = e^{-x^2/2} \frac{(1 + \zeta)^{1/2}}{\sqrt{2\pi G}} \left[1 + \frac{x(-6\zeta + 2x^2\zeta + x^2 - 3)}{6G(1 + \zeta)^{3/2}} + O(1/G^2)\right].$$

Note that the dominant term is equivalent to the dominant term of $1/\sqrt{2\pi \sigma}$, as expected. More terms in this expression can be obtained if we compute $M, \sigma^2, B_n/n!$ with more precision. Also, using (2), our result can be put into expansions depending on $n, \ln n, \ldots$
To check the quality of our asymptotic, we have chosen \( n = 3000 \). This leads to

\[
\zeta = 6.184346264 \ldots, \\
G = 22.02488900 \ldots, \\
M = 484.1556441 \ldots, \\
\sigma = 8.156422315 \ldots, \\
B_n = 0.2574879583 \ldots 10^{6965}, \\
B_{n as} = 0.2574880457 \ldots 10^{6965},
\]

where \( B_{n as} \) is given by (3). Figure 1 shows \( Z_n(m) \) and

\[
\frac{1}{\sqrt{2\pi\sigma}} \exp \left[- \left( \frac{m-M}{\sigma} \right)^2/2 \right].
\]
Figure 1: $Z_n(m)$ and $\frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\left(\frac{m-M}{\sigma}\right)^2/2\right]$
The fit seems quite good, but to have more precise information, we show in Figure 2 the quotient $Z_n(m) \sqrt{\frac{1}{2\pi\sigma}} \exp \left[ - \left( \frac{m-M}{\sigma} \right)^2 / 2 \right]$. The precision is between 0.05 and 0.10.
Figure 2: $Z_n(m) / \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ - \left( \frac{m-M}{\sigma} \right)^2 / 2 \right]$
Figure 3 shows the quotient $Z_n(m) / R_1$. The precision is now between 0.004 and 0.01.

\[ Z_n(m) / R_1 \]

\[ 460 \quad 470 \quad 480 \quad 490 \quad 500 \quad 510 \]

Figure 3: $Z_n(m) / R_1$
Large deviation, $m = n - n^\alpha$, \quad $\alpha > 1/2$

We set

$$\varepsilon := n^{\alpha - 1},$$

$$\frac{1}{\varepsilon} = n^{1-\alpha} \ll n^\alpha \ll n,$$

$$L := \ln(n).$$

The multiseries' scale is here $\{n^{1-\alpha}, n^\alpha, n\}$. 
Our result can be summarized in the following local limit theorem

**Theorem 3.1**

\[
\begin{align*}
\binom{n}{m} &= e^{T_1 R}, \\
T_1 &= n^\alpha (T_{11} L + T_{10}), \\
R &= \frac{1}{\sqrt{2\pi n^\alpha/2}} \left[ R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + O(1/n^3) \right], \\
R_0 &= R_{00} + \frac{R_{01}}{n^\alpha} + O(1/n^{2\alpha}), \\
R_1 &= R_{10} + \frac{R_{11}}{n^\alpha} + O(1/n^{2\alpha}), \\
R_2 &= R_{20} + \frac{R_{21}}{n^\alpha} + O(1/n^{2\alpha}),
\end{align*}
\]

where $T_{i,j}, R_{i,j}$ are power series in $\varepsilon$. 
Proof. Using again the Lambert function, we derive successively (again we only provide a few terms here, we use a dozen of terms in our expansions)

\[ m = n(1 - \varepsilon), \]
\[ \frac{n}{m} = \frac{1}{1 - \varepsilon}, \]
\[ \rho = 2\varepsilon + \frac{4}{3}\varepsilon^2 + \frac{10}{9}\varepsilon^3 + O(\varepsilon^4), \]
\[ \ln(m) = L - \varepsilon - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3), \]
\[ \ln(\rho) = -L(1 - \alpha) + \ln(2) + \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon^2 + O(\varepsilon^3). \]
For the first part of Cauchy’s integral, we have, noting that \(n \varepsilon = n^\alpha\), and using (7),

\[
\frac{n!}{m! \rho^n} f(\rho)^m = \exp(T) H_2,
\]

\[
T = m(\rho + \ln(\rho) - L) - (-m + \ln(m)/2) + (-n + nL + L/2) - n \ln(\rho)
\]

\[
= T_1 + T_0,
\]

\[
T_1 = n^\alpha (T_{11} L + T_{10}),
\]

\[
T_{11} = 2 - \alpha,
\]

\[
T_{10} = 1 - \ln(2) - \frac{4}{3} \varepsilon - \frac{5}{9} \varepsilon^2 + \mathcal{O}(\varepsilon^3),
\]

\[
T_0 = \frac{1}{2} \varepsilon + \frac{1}{4} \varepsilon^2 + \mathcal{O}(\varepsilon^3),
\]
\[ H_1 = \exp(T_0) = 1 + \frac{1}{2} \varepsilon + \frac{3}{8} \varepsilon^2 + O(\varepsilon^3), \]

\[ H_2 = \left[ 1 + \frac{1}{12n} + \frac{1}{288n^2} + O\left(\frac{1}{n^3}\right) \right] \bigg/ \left[ 1 + \frac{1}{12m} + \frac{1}{288m^2} + O\left(\frac{1}{m^3}\right) \right] \]

\[ = 1 + \frac{\varepsilon}{12(\varepsilon - 1)n} + \frac{\varepsilon^2}{288(\varepsilon - 1)^2n^2} + O\left(\frac{\varepsilon^3}{n^3}\right). \]
Note again that there are cancellations, in $T_1$ of the terms $m \ln(m)$ and $\ln(2\pi)/2$.

Now we turn to the integral part. We obtain, for instance, using (6),

$$\kappa_2 = \varepsilon + \frac{4}{3} \varepsilon^2 + \frac{13}{9} \varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

$$\theta = \frac{1}{\sqrt{n}} \sum_{1}^{\infty} a_i u^i,$$

$$a_1 = \frac{1}{\sqrt{\varepsilon}} \left[ 1 - \frac{1}{6} \varepsilon^2 - \frac{1}{72} \varepsilon^4 + \mathcal{O}(\varepsilon^6) \right].$$
Integrating, this gives

\[ H_3 = \frac{1}{\sqrt{2\pi n^{\alpha/2}}} \left[ H_{31} + \frac{H_{32}}{n^{\alpha}} + O(1/n^{2\alpha}) \right], \]

\[ H_{31} = 1 - \frac{1}{6} \varepsilon - \frac{1}{72} \varepsilon^2 + O(\varepsilon^3), \]

\[ H_{32} = -\frac{1}{12} + \frac{1}{72} \varepsilon - \frac{71}{864} \varepsilon^2 + O(\varepsilon^3). \]
Now we compute

\[ \begin{pmatrix} n \\ m \end{pmatrix} = e^{T_1 H_1 H_2 H_3} = e^{T_1 R}, \]

with

\[
R = \frac{1}{\sqrt{2\pi n^{\alpha/2}}} \left[ R_0 + \frac{R_1}{n} + \frac{R_2}{n^2} + \mathcal{O}(1/n^3) \right],
\]

\[
R_0 = R_{00} + \frac{R_{01}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}),
\]

\[
R_1 = R_{10} + \frac{R_{11}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}),
\]

\[
R_2 = R_{20} + \frac{R_{21}}{n^\alpha} + \mathcal{O}(1/n^{2\alpha}),
\]
Given some desired precision, how many terms must we use in our expansions? It depends on $\alpha$. For instance, in $T_1$, $n^\alpha \varepsilon^k \gg 1$ if $k < \alpha/(1 - \alpha)$. Also $\varepsilon^k$ in $R_{00}$ is less than $\varepsilon^\ell/n$ in $R_{10}/n$ if $k - \ell > 1/(1 - \alpha)$. Any number of terms can be computed by almost automatic computer algebra. We use Maple in this paper.
To check the quality of our asymptotic, we have chosen $n = 100$ and a range $\alpha \in [1/2, 9/10]$, i.e. a range $m \in [37, 90]$. We use 5 or 6 terms in our final expansions. Figure 4 shows the quotient $\binom{n}{m}/(e^{T_1 R})$. The precision is at least 0.0066. Note that the range $[M - 3\sigma, M + 3\sigma]$, where the Gaussian approximation is useful, is here $m \in [21, 36]$. 
Figure 4: \( \left\{ \frac{n}{m} \right\} / (e^{T_1} R) \)
Central region

Large deviation, \( m = n - n^\alpha \),

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