Relational type-checking of MELL proof-structures

Giulio Guerrieri
Luc Pellissier
Lorenzo Tortora de Falco

3 IV 16

Motivation

Given a syntactic structure R, defines its semantics $[\![\mathrm{R}]\!]$.
Elements of $[\![\mathrm{R}]\!]$ carry a lot of information on R:

- on its execution time (Carvalho, Pagani, and Tortora de Falco 2011),
- characterize it completely:
  - if R is of type **bool**;
  - in general.

How tractable is ¿ $x \in [\![\mathrm{R}]\!]$?

An old question: the coherent case

Girard 1987, 3.16 Remark (ii) :

> *"Assume, just for a minute, that we try to compute the semantics of a proof* π*, i.e., we try to decide, given* $z \in |A|$*, whether or not* $z \in \pi^\star$*. In all cases but [the cut] and [the existential quantification], the problem is immediately reduced to several simpler problems of the same type, with an effective bound on these problems. [...]*
> *This basic remark is the key to a semantic approach to computation, which will be undertaken somewhere else"*

Multiplicative Linear Logic

Multiplicative proof-structures

$$A, B, C ::= X \mid X^{\perp} \mid 1 \mid \perp \mid A \otimes B \mid A \parr B\,.$$

Figure: The formulæ

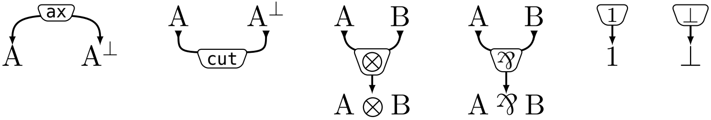
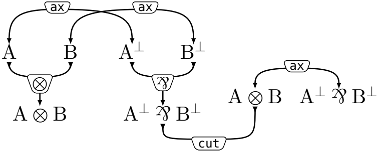
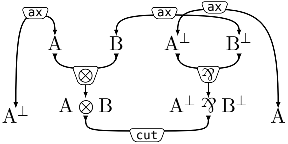


Figure: The cells

Proof-structures





Associate to every formula a set, to every proof-structure a relation between sets.

Definition (web of a formula)

$$
\begin{aligned}
|X^{\perp}| = |X| &= \mathcal{A}t, \text{ for any propositional variable } X;\\
|1| = |\perp| &= \{()\};\\
|A \otimes B| = |A \parr B| &= |A| \times |B|
\end{aligned}
$$

Let $a \in |A|, b \in |B|$.

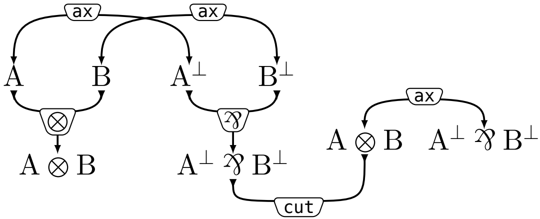


Figure: A proof-structure R

Let $a \in |\mathrm{A}|, b \in |\mathrm{B}|$.

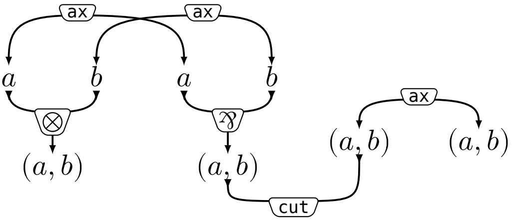


Figure: An experiment of R

$$\begin{aligned} [\![\mathrm{R}]\!] &= \{((a,b),(a,b)), a \in |\mathrm{A}|, b \in |\mathrm{B}|\} \\ &\subseteq |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^\perp \parr \mathrm{B}^\perp| \end{aligned}$$

Intersection types

$\vdash \mathrm{M} : \mathrm{A} \cap \mathrm{B}$ means M can be used both as an A and as a B.

Relational semantics are a non-idempotent type-system.

Our initial question ¿$x \in [\![\mathrm{R}]\!]$? becomes ¿is R typable at type x?

Recognition

Recognition of the relational interpretation

Suppose I have $((a,b),(a,b)) \in |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^{\perp} \parr \mathrm{B}^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![\mathrm{R}]\!]$?

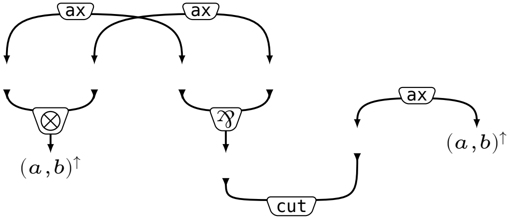


Figure: Successful recognition

Recognition of the relational interpretation

Suppose I have $((a,b),(a,b)) \in |A \otimes B| \times |A^{\perp} \parr B^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![R]\!]$?

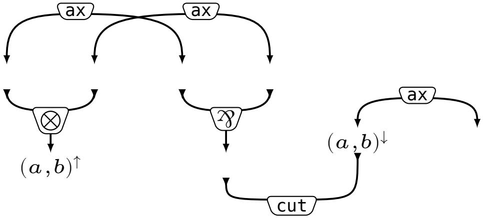


Figure: Successful recognition

Suppose I have $((a,b),(a,b)) \in |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^{\perp} \parr \mathrm{B}^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![\mathrm{R}]\!]$?

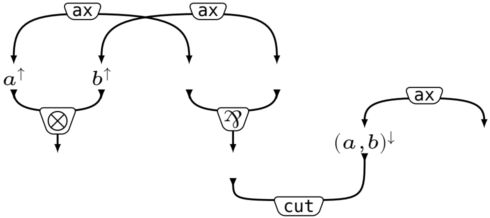


Figure: Successful recognition

Suppose I have $((a,b),(a,b)) \in |A \otimes B| \times |A^{\perp} \parr B^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![R]\!]$?

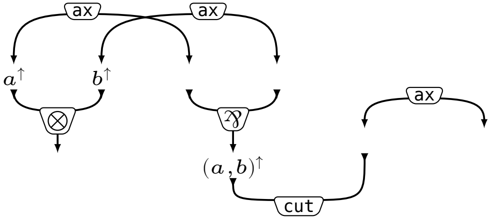


Figure: Successful recognition

Recognition of the relational interpretation

Suppose I have $((a, b), (a, b)) \in |A \otimes B| \times |A^\perp \parr B^\perp|$.
¿Is $((a, b), (a, b)) \in [\![R]\!]$?

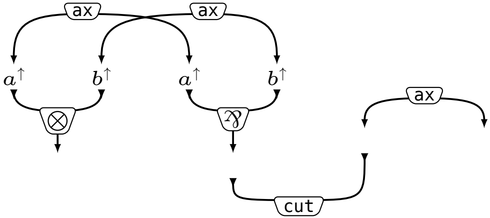


Figure: Successful recognition

Suppose I have $((a, b), (a, b)) \in |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^{\perp} \parr \mathrm{B}^{\perp}|$.
¿Is $((a, b), (a, b)) \in [\![\mathrm{R}]\!]$?

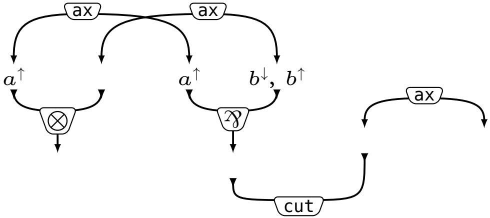


Figure: Successful recognition

Suppose I have $((a,b),(a,b)) \in |A \otimes B| \times |A^{\perp} ⅋ B^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![R]\!]$?

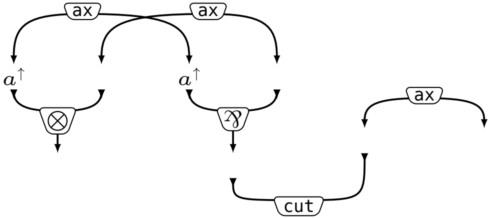


Figure: Successful recognition

Suppose I have $((a, b), (a, b)) \in |A \otimes B| \times |A^{\perp} \parr B^{\perp}|$.
¿Is $((a, b), (a, b)) \in [\![R]\!]$?

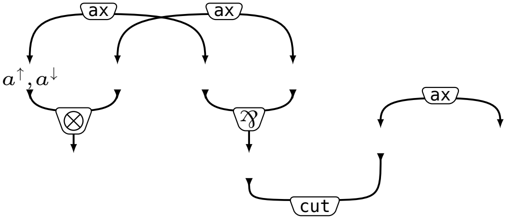


Figure: Successful recognition

Suppose I have $((a,b),(a,b)) \in |A \otimes B| \times |A^{\perp} \parr B^{\perp}|$.
¿Is $((a,b),(a,b)) \in [\![R]\!]$?

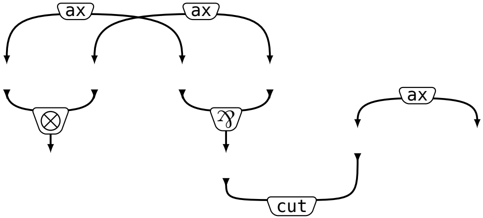


Figure: Successful recognition

Suppose I have $((a, b), (a, c)) \in |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^\perp \parr \mathrm{B}^\perp|$.
¿Is $((a, b), (a, c)) \in [\![\mathrm{R}]\!]$?

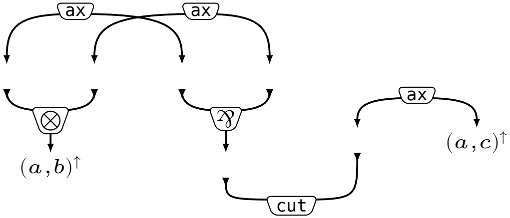


Figure: Failing recognition

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \parr B^\perp|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

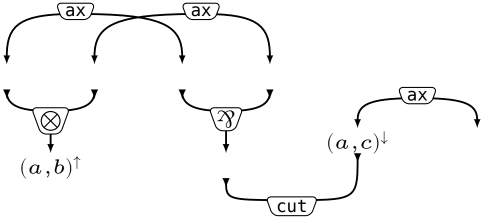


Figure: Failing recognition

Recognition of the relational interpretation

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \parr B^\perp|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

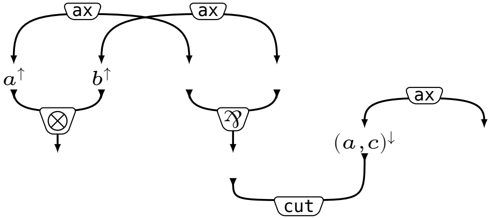


Figure: Failing recognition

Suppose I have $((a,b),(a,c)) \in |\mathrm{A} \otimes \mathrm{B}| \times |\mathrm{A}^{\perp} \parr \mathrm{B}^{\perp}|$.
¿Is $((a,b),(a,c)) \in [\![\mathrm{R}]\!]$?

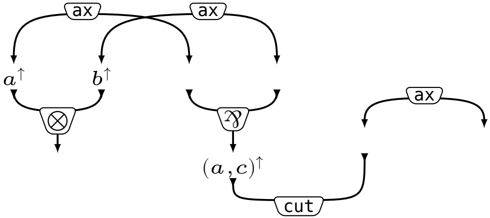


Figure: Failing recognition

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \parr B^\perp|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

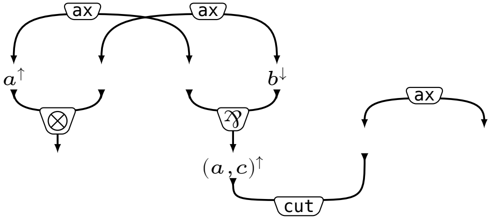


Figure: Failing recognition

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^\perp \parr B^\perp|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

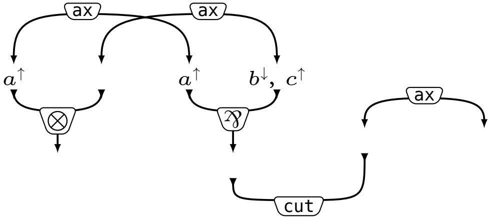


Figure: Failing recognition

Recognition of the relational interpretation

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^{\perp} \parr B^{\perp}|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

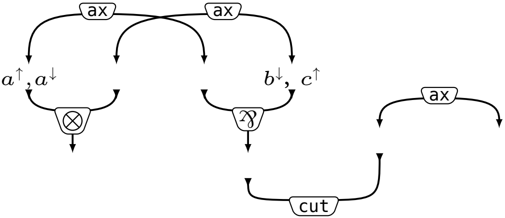


Figure: Failing recognition

Suppose I have $((a, b), (a, c)) \in |A \otimes B| \times |A^\perp ⅋ B^\perp|$.
¿Is $((a, b), (a, c)) \in [\![R]\!]$?

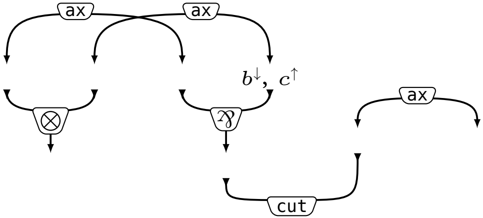
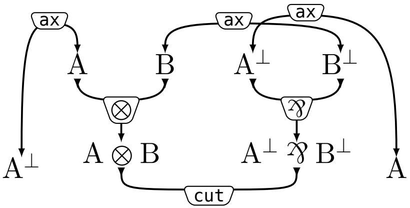


Figure: Failing recognition

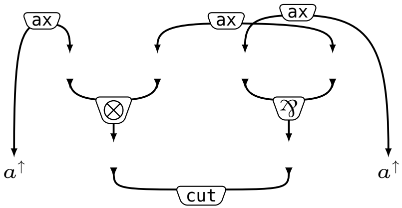
Semantically hidden cycles

Let $(a, a) \in |A^\perp| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?



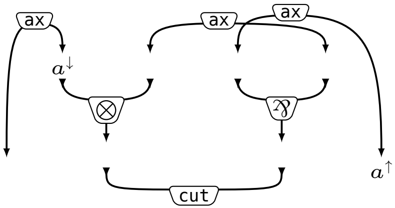
Semantically hidden cycles

Let $(a, a) \in |\mathrm{A}^{\perp}| \times |\mathrm{A}|$.
¿Is $(a, a) \in [\![\mathrm{R}]\!]$?



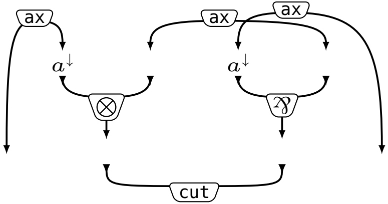
Semantically hidden cycles

Let $(a, a) \in |A^{\perp}| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?

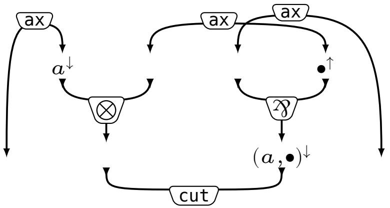


Semantically hidden cycles

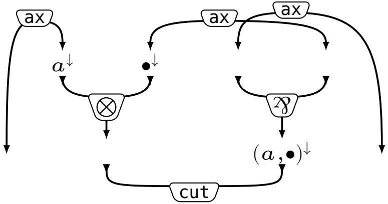
Let $(a, a) \in |\mathrm{A}^{\perp}| \times |\mathrm{A}|$.
¿Is $(a, a) \in [\![\mathrm{R}]\!]$?



Let $(a, a) \in |\mathrm{A}^{\perp}| \times |\mathrm{A}|$.
¿Is $(a, a) \in [\![\mathrm{R}]\!]$?

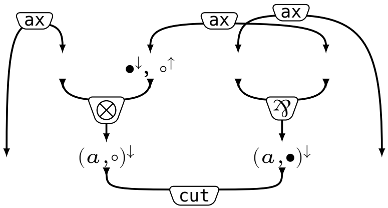


Let $(a, a) \in |A^{\perp}| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?

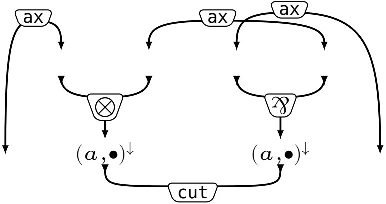


Semantically hidden cycles

Let $(a, a) \in |\mathrm{A}^{\perp}| \times |\mathrm{A}|$.
¿Is $(a, a) \in [\![\mathrm{R}]\!]$?

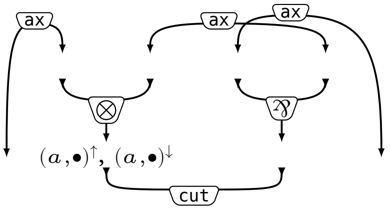


Let $(a, a) \in |A^{\perp}| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?



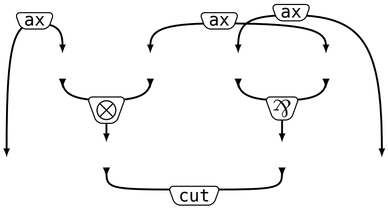
Semantically hidden cycles

Let $(a, a) \in |A^{\perp}| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?



Semantically hidden cycles

Let $(a, a) \in |A^{\perp}| \times |A|$.
¿Is $(a, a) \in [\![R]\!]$?



Relational Interaction Abstract Machine for MLL

Variant of a (coloured) Vector Addition System:

- one counter per port;
- each counter's value is a formal sum of relational elements with coefficients ± 1;
- displacement transitions, that moves tokens from counters to counters;
- unification transitions.

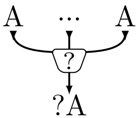
Reduce the problem to the accessibility of 0 by a machine.

Theorem

$x \in [\![\Phi]\!]$ if and only if the machine associated with Φ, initialized on x, stops with zero on all counters.
It can do so in $\mathrm{O}(|\text{cells of } \Phi|)$.

The exponentials

We add the following cells:



and the ability to box entire proof-structures:

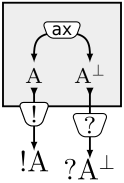


Figure: A box

Example

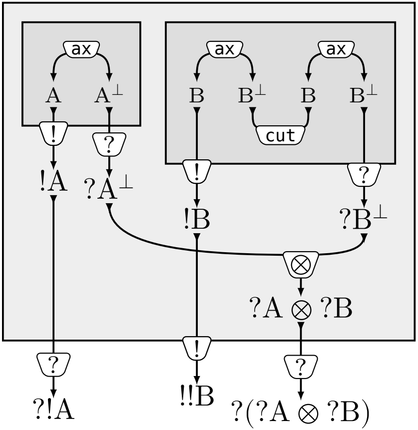


Figure: A MELL-ps R

$$|!A| = |?A| = \mathcal{M}_{\text{fin}}(|A|)$$

An experiments assigns the muultiset of its assignments on the auxiliary ports to the principal port of a ?-cell.
An experiment of a box is any number of experiments of its contents.

Example

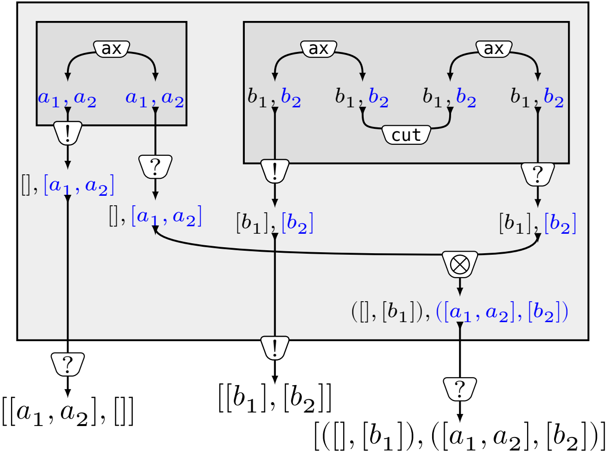
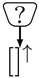


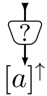
Figure: A MELL-ps R

Weakenings



Weakenings

Derelictions



Derelictions

Boxes

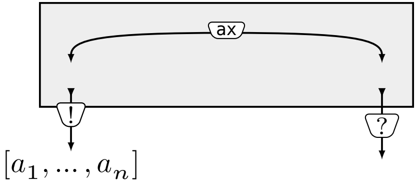


Figure: A box

Boxes

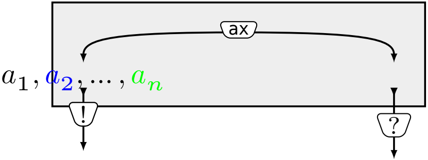


Figure: A box

Boxes

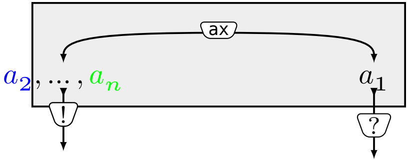


Figure: A box

Boxes

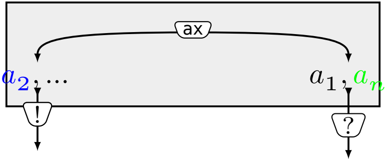
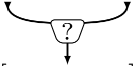


Figure: A box

Contractions?

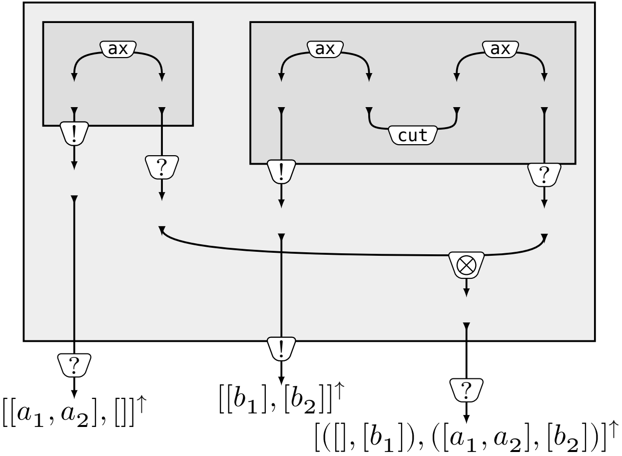


$[a_1, \ldots, a_n]$

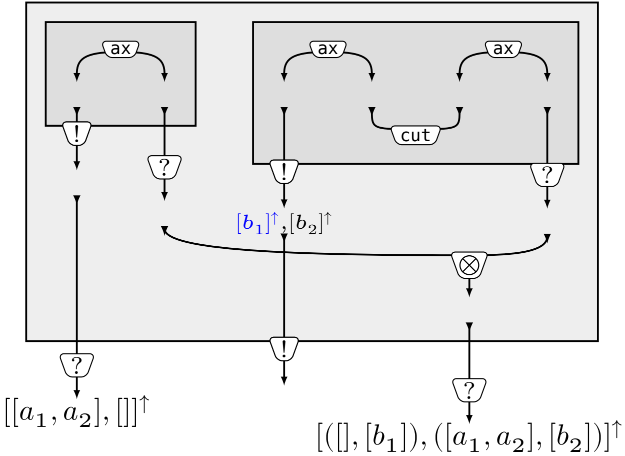
Either:

- try everything;
- restrict structures geometrically.

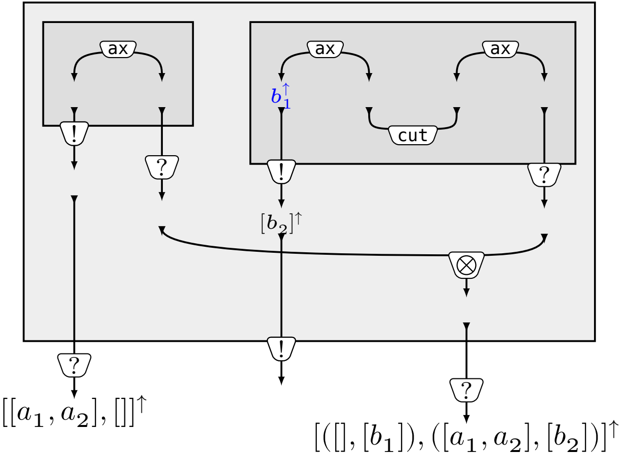
Example



Example

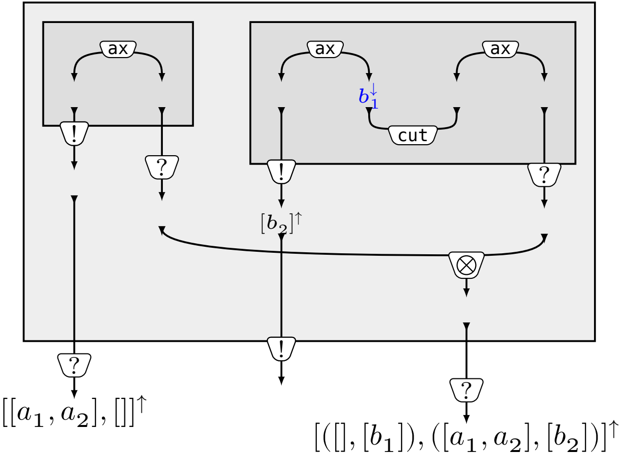

ax
ax
ax
cut
!
?
!
?
$[b_1]^{\uparrow}, [b_2]^{\uparrow}$
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example

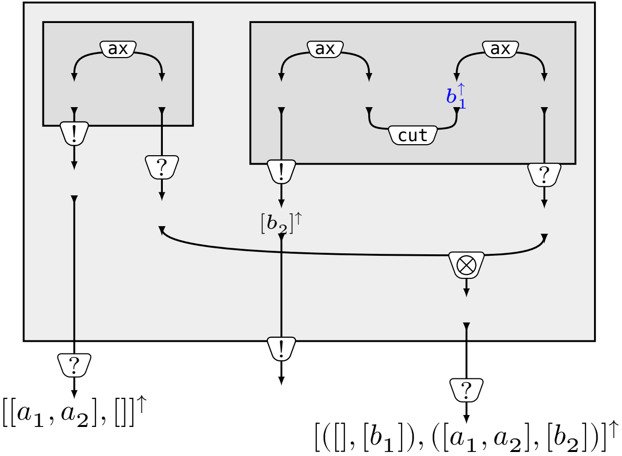


ax
ax
ax
$b_1^{\uparrow}$
cut
!
?
!
?
$[b_2]^{\uparrow}$
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

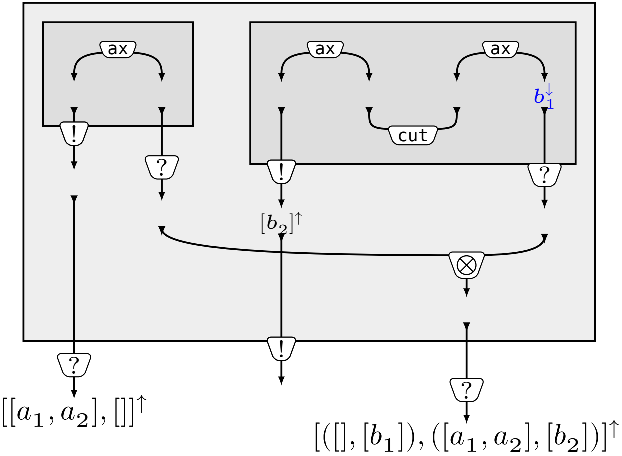
Example


ax
ax
ax
$b_1^{\downarrow}$
cut
!
?
!
?
$[b_2]^{\uparrow}$
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

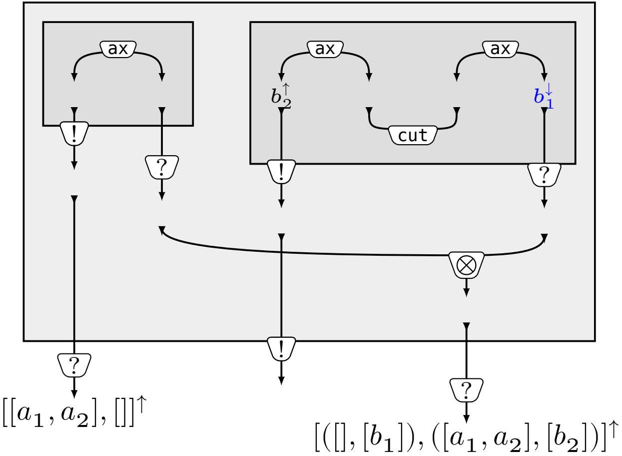
Example


ax
ax
ax
cut
$b_1^{\uparrow}$
!
?
!
?
$[b_2]^{\uparrow}$
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

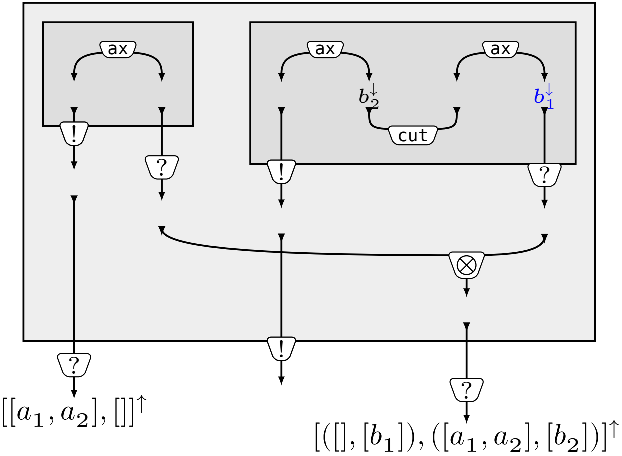
Example


ax
ax
ax
$b_1^{\downarrow}$
cut
!
?
!
?
$[b_2]^{\uparrow}$
$\otimes$
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

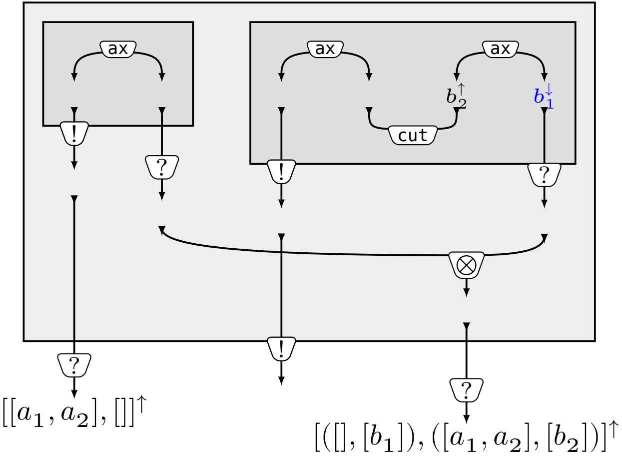
Example


ax
ax
ax
$b_2^{\uparrow}$
$b_1^{\downarrow}$
cut
!
?
!
?
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example

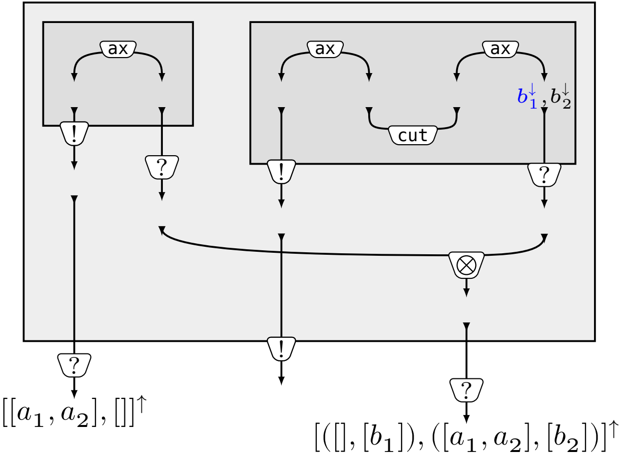

ax
ax
ax
$b_2^{\downarrow}$
$b_1^{\downarrow}$
cut
!
?
!
?
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example

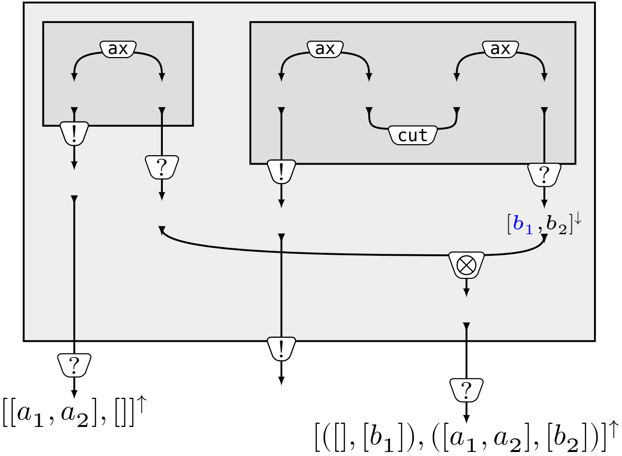


ax
ax
ax
$b_2^{\uparrow}$
$b_1^{\downarrow}$
cut
!
?
!
?
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

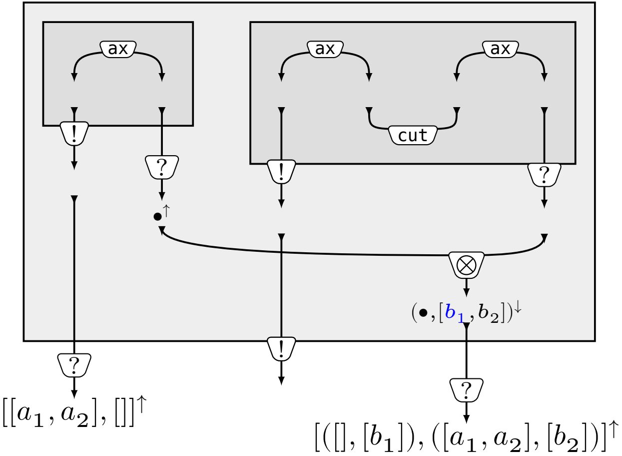
Example


ax
ax
ax
$b_1^{\downarrow}, b_2^{\downarrow}$
cut
!
?
!
?
$\otimes$
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

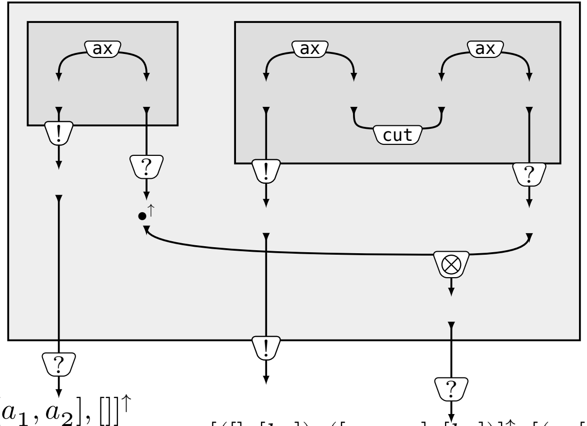
Example



Example



ax
ax
ax
!
?
cut
!
?
$\bullet^{\uparrow}$
$\otimes$
$(\bullet, [b_1, b_2])^{\downarrow}$
!
?
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

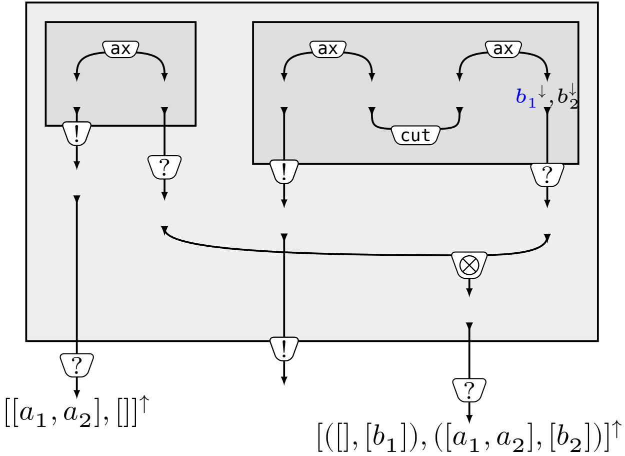


$[[a_1, a_2], []]^{\uparrow}$

$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}, [(\bullet, [b_1, b_2])]^{\downarrow}$

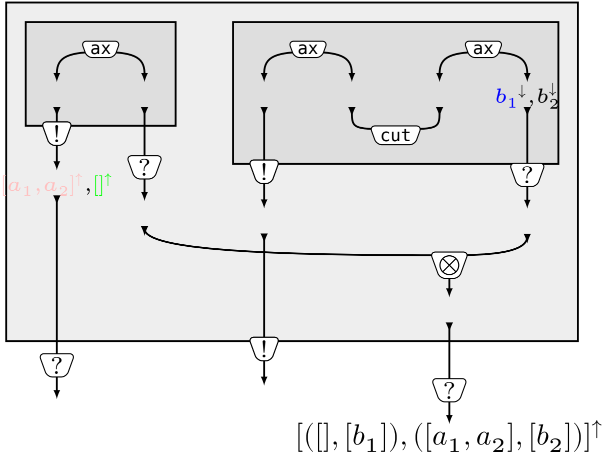
Failure...

Example

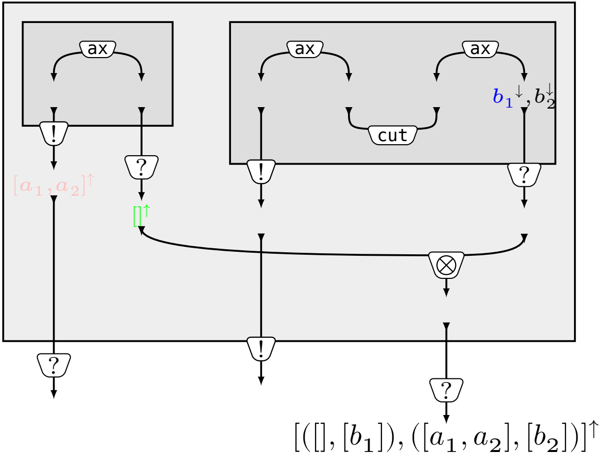


ax
ax
ax
$b_1^{\downarrow}, b_2^{\downarrow}$
cut
!
?
!
?
⊗
?
!
?
$[[a_1, a_2], []]^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

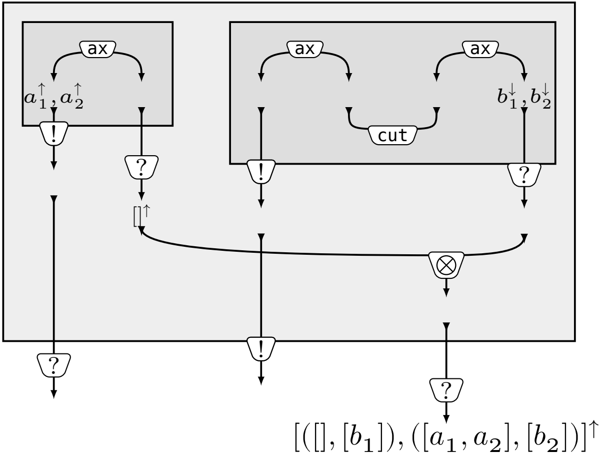
Example


ax
ax
ax
cut
$b_1^{\downarrow}, b_2^{\downarrow}$
$[a_1, a_2]^{\uparrow}, []^{\uparrow}$
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

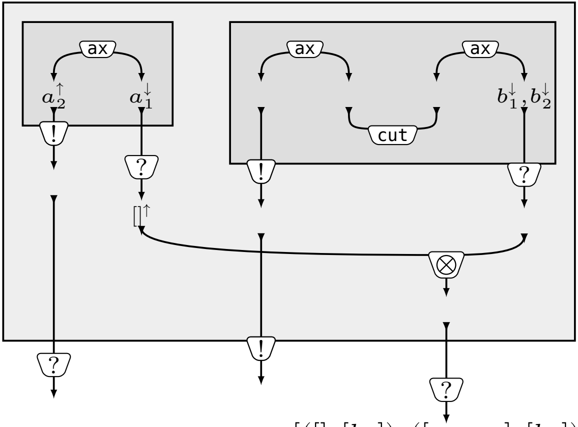
Example


ax
ax
ax
cut
$b_1{}^{\downarrow}, b_2^{\downarrow}$
!
?
!
?
$[a_1, a_2]^{\uparrow}$
$[]^{\uparrow}$
⊗
?
!
?
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example

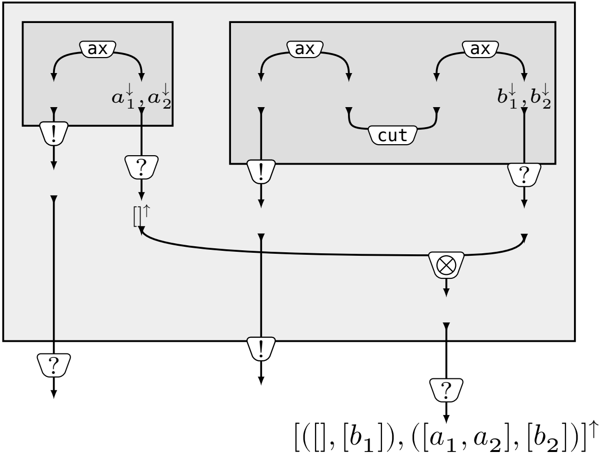

ax
ax
ax
$a_1^{\uparrow}, a_2^{\uparrow}$
$b_1^{\downarrow}, b_2^{\downarrow}$
cut
!
?
!
?
$[]^{\uparrow}$
$\otimes$
?
!
?
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example



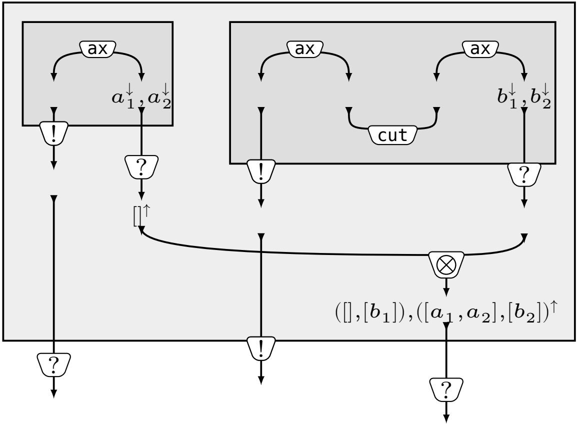
$[([], [b_1]), ([a_1, a_2], [b_2])]^{\uparrow}$

Example



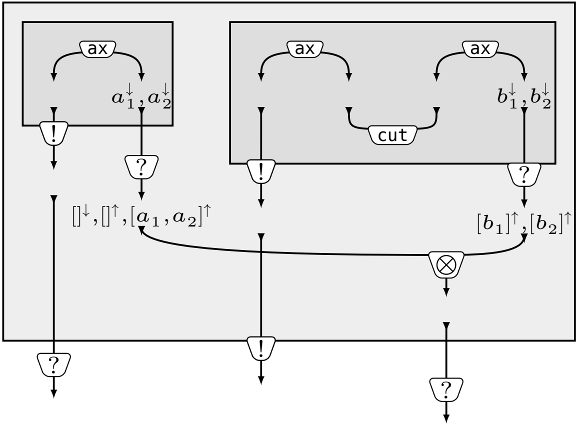
ax
ax
ax
$a_1^\downarrow, a_2^\downarrow$
$b_1^\downarrow, b_2^\downarrow$
cut
!
?
!
?
$[]^\uparrow$
⊗
?
!
?
$[([], [b_1]), ([a_1, a_2], [b_2])]^\uparrow$

Example



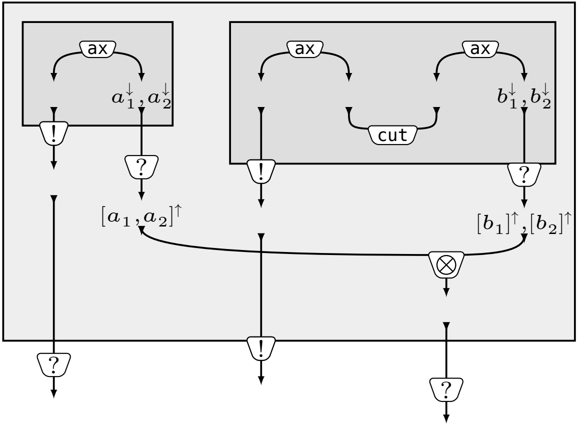
ax
$a_1^{\downarrow}, a_2^{\downarrow}$
!
?
$[]^{\uparrow}$
ax
ax
cut
$b_1^{\downarrow}, b_2^{\downarrow}$
!
?
⊗
$([],[b_1]),([a_1,a_2],[b_2])^{\uparrow}$
?
!
?

Example



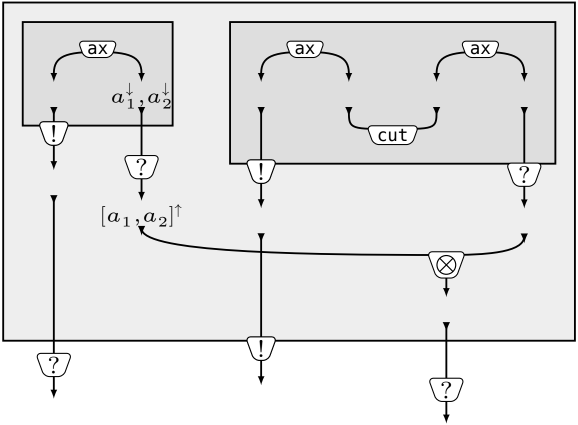
ax
ax
ax
$a_1^\downarrow, a_2^\downarrow$
$b_1^\downarrow, b_2^\downarrow$
cut
!
?
!
?
$[]^\downarrow, []^\uparrow, [a_1, a_2]^\uparrow$
$[b_1]^\uparrow, [b_2]^\uparrow$
$\otimes$
?
!
?

Example

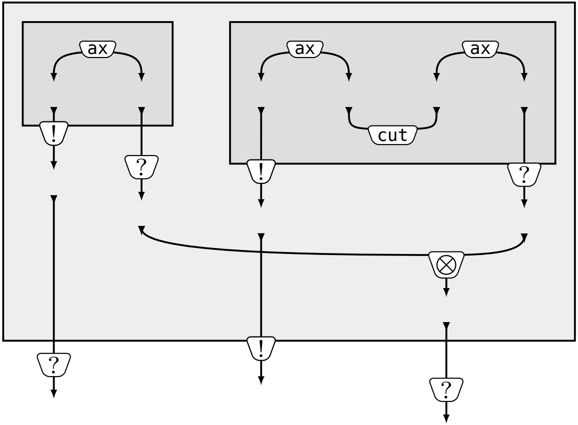


ax
$a_1^{\downarrow}, a_2^{\downarrow}$
!
?
ax
ax
cut
$b_1^{\downarrow}, b_2^{\downarrow}$
!
?
$[a_1, a_2]^{\uparrow}$
$[b_1]^{\uparrow}, [b_2]^{\uparrow}$
$\otimes$
?
!
?

Example



Example



Adapting the machine

Easy:

- add new rules corrisponding to new cells
- coefficients of relational elements of counters now in $\mathbf{Z}$.

But does not work on all MELL structures...

We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere:

Geometry

We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ???

We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ???

- Something to do with polarity;

We need to do non-choices coherently: connexity inside boxes

We need to be able to start somewhere: ???

- Something to do with polarity;
- contains the λ-calculus.

Result

Theorem

In a fragment of MELL *containing the λ-calculus, a machine M^{Φ} decides $x \in [\![\Phi]\!]$ in time $O(|x| \times |\text{cells of } \Phi|)$.*

Corollary (Terui, 2012)

Let

$$\mathsf{Str} := !(X \multimap X) \multimap !(X \multimap X) \multimap X \multimap X$$

$$\mathsf{Bool} := !X \multimap !X \multimap X$$

be the linear-logic translations of Church binary strings and booleans.
Let t *be a simply-typed λ-term of type* $\mathsf{Str}[A/X] \multimap \mathsf{Bool}$, *for arbitrary* A. *It decides a language $\mathcal{L}$.*
$\mathcal{L}$ is in **LinTIME** *(deterministic linear time).*