Exact and asymptotic enumeration results for combinatorial objects

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Outline of the talk

1. Discrete parking problems
2. Pólya-Eggenberger urn models
3. Network models

Discrete parking problems

(partially together with Georg Seitz, TU Wien)

Discrete parking problems: Parking scheme

The parking scheme:

- Consider one-way street
- m parking spaces are in a row
- n drivers wish to park in these spaces
- Each driver has preferred parking space to which he drives
- If parking space is empty $\Rightarrow$ he parks there
- If not, he drives on and parks in the next free parking space if there is one
- If all remaining parking spaces are occupied $\Rightarrow$ leaves without parking

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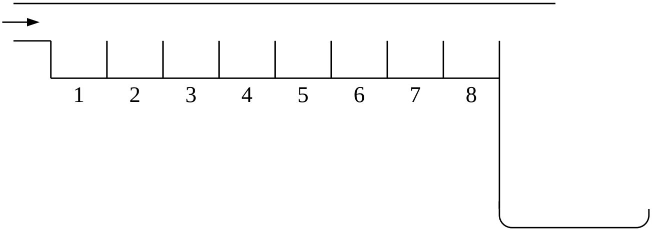
Example: 8 parking spaces, 8 cars

Parking sequence: 3, 6, 3, 8, 6, 7, 4, 5



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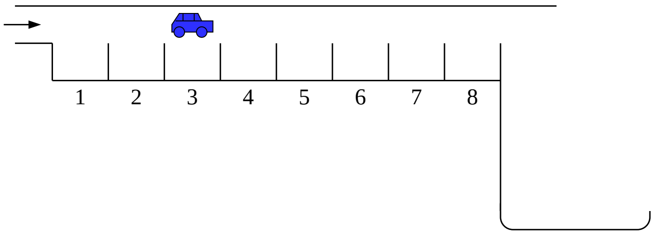

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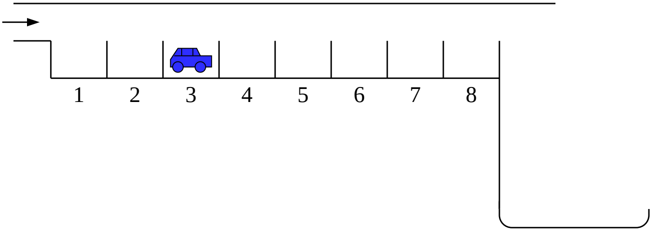

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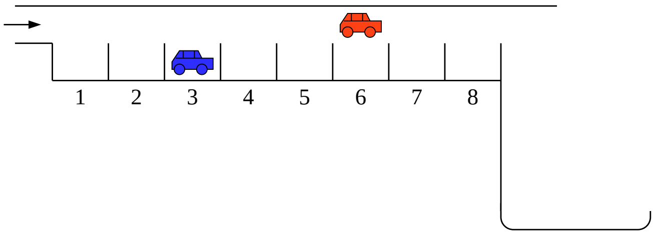

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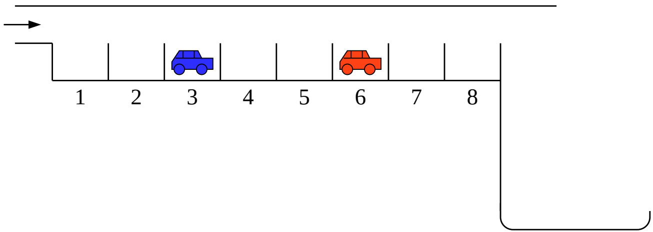

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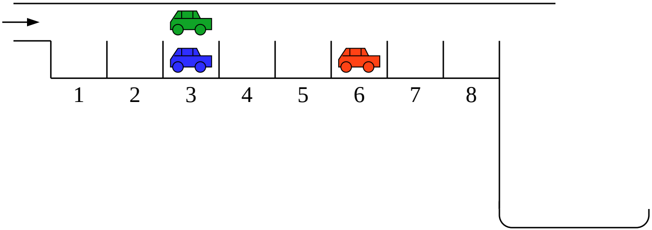

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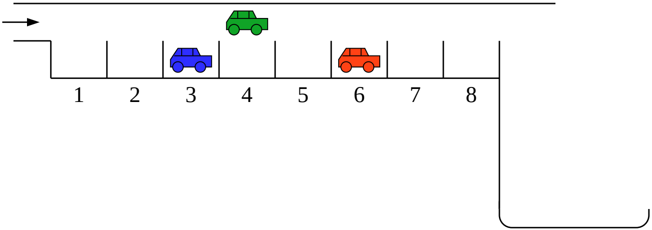

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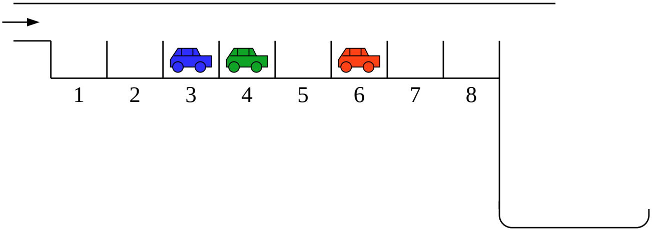

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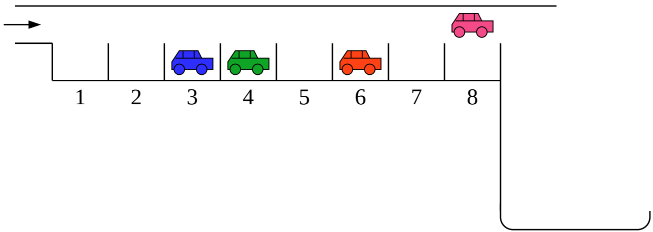

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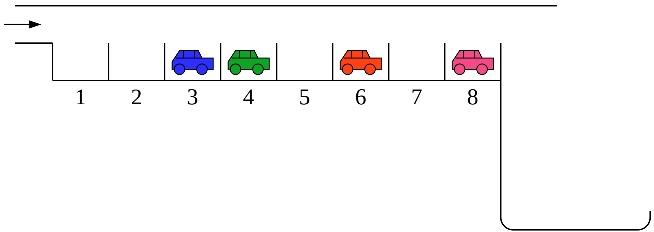

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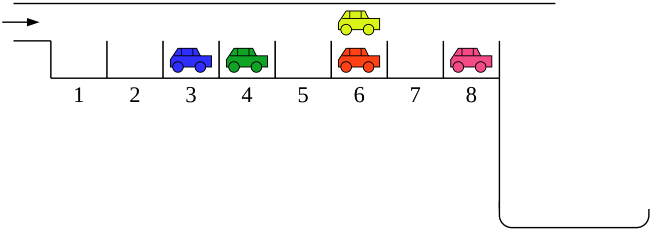

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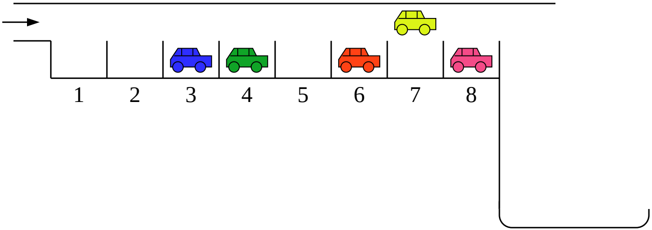

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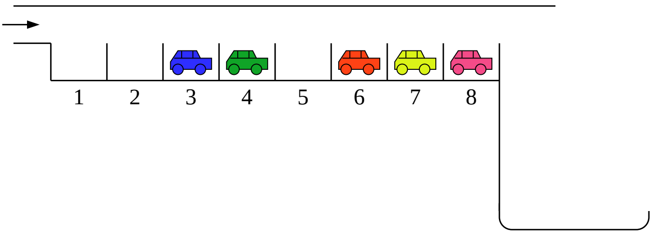

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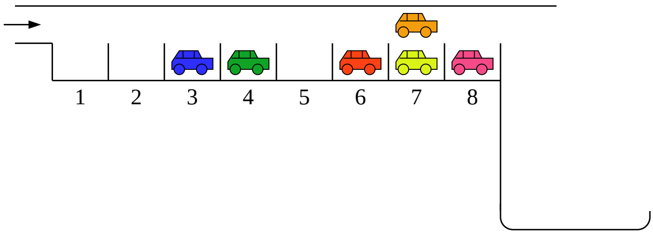

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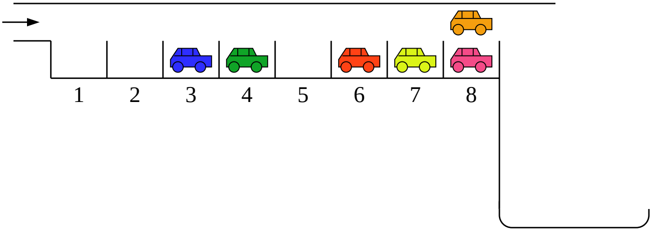

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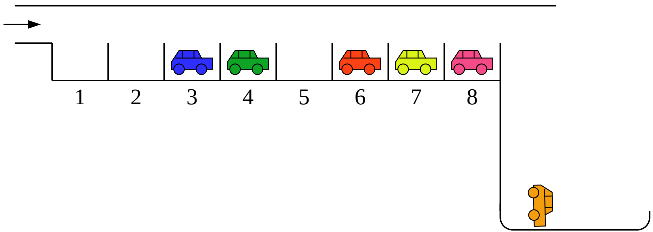

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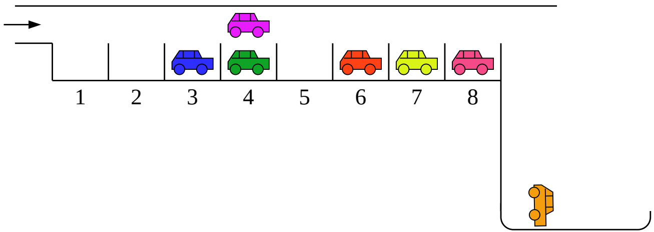

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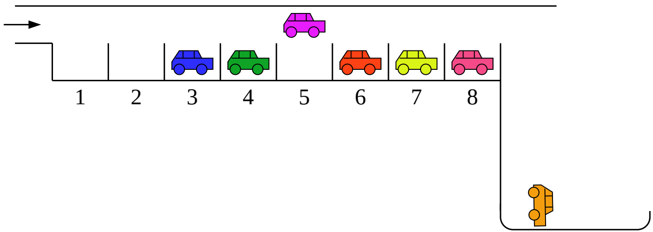

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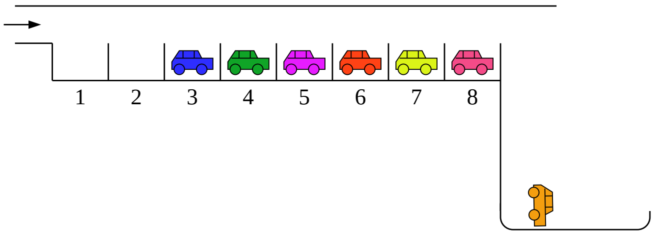

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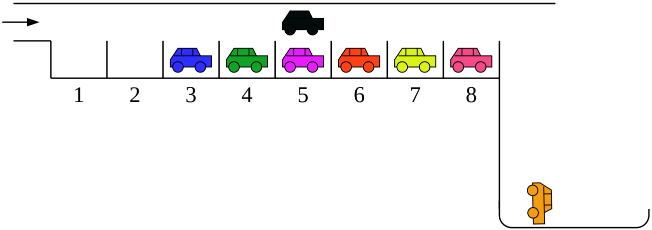

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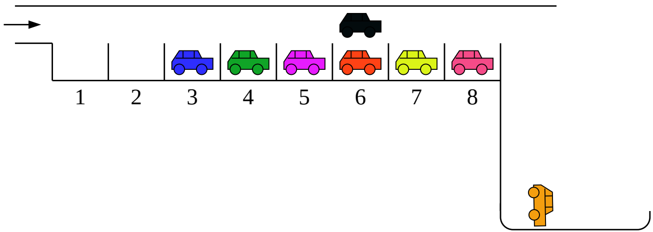

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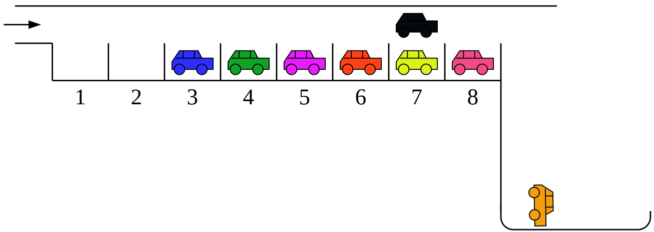

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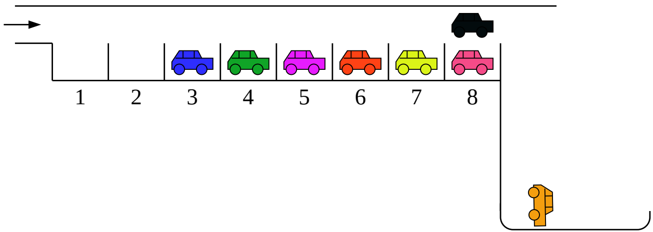

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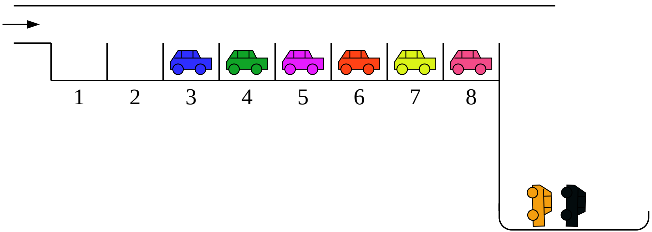

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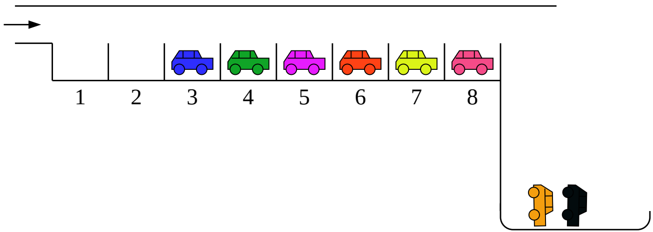

⇒ 2 cars are unsuccessful

Discrete parking problems: Parking functions

Number of unsuccessful cars:

Parking sequence $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$
$\Rightarrow$ k unsuccessful cars $(\max(n - m, 0) \leq k \leq n - 1)$

Parking functions: special instance $k = 0$
$\Rightarrow$ all cars can be parked

Introduced by Konheim and Weiss [1966]:
in analysis of linear probing hashing algorithm

- m places at a round table ($\cong$ memory addresses)
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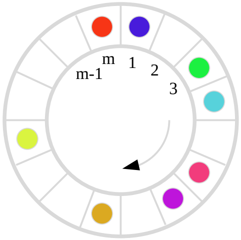

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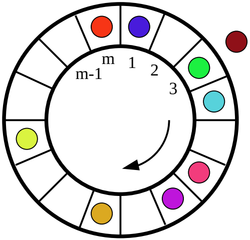

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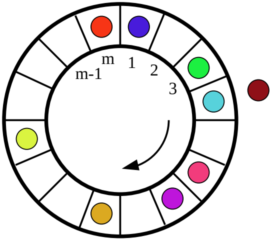

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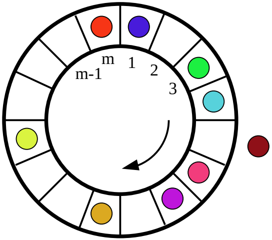

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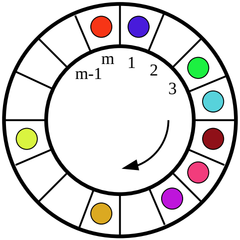

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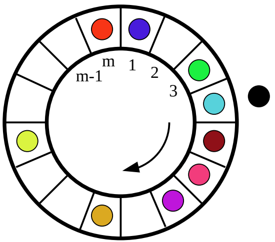

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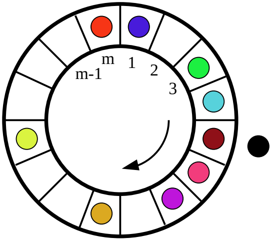

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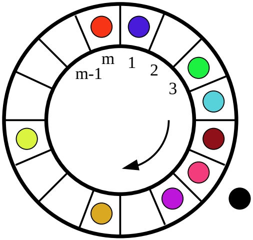

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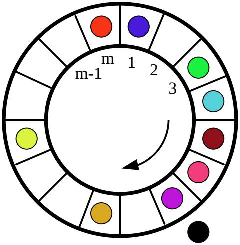

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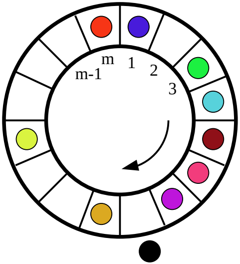

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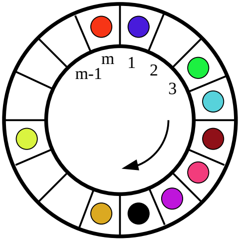

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Discrete parking problems: Enumeration results

Enumeration result for parking sequences:
Konheim and Weiss [1966]

$g(m, n)$: number of parking functions
for m parking spaces and n cars

$$g(m, n) = (m - n + 1)(m + 1)^{n-1}$$

Questions for general parking sequences:
"Combinatorial question":

What is the number $g(m, n, k)$ of parking sequences $a_1, \dots, a_n \in \{1, \dots, m\}^n$ such that exactly k drivers are unsuccessful?

- Exact formulæ for $g(m, n, k)$?

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> *What is the number $g(m,n,k)$ of parking sequences $a_1,\ldots,a_n \in \{1,\ldots,m\}^n$ such that exactly k drivers are unsuccessful?*

- Exact formulæ for $g(m,n,k)$?

Discrete parking problems: Enumeration results

"Probabilistic question":

> *What is the probability that for a randomly chosen parking sequence $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$ exactly k drivers are unsuccessful ?*

r.v. $X_{m,n}$: counts number of unsuccessful cars for a randomly chosen parking sequence

- Probability distribution of $X_{m,n}$?
- Limiting distribution results (depending on growth of m, n) ?

Discrete parking problems: Enumeration results

Cameron, Johannsen, Prellberg and Schweitzer [2008]; Panholzer [2008]

Number $g(m,n,k)$ of parking sequences for m parking spaces and n drivers such that exactly k drivers are unsuccessful ($n \le m+k$):

$$g(m,n,k) = (m-n+k) \sum_{\ell=0}^{n-k} \binom{n}{\ell} (m-n+k+\ell)^{\ell-1} (n-k-\ell)^{n-\ell}$$
$$-(m-n+k+1) \sum_{\ell=0}^{n-k-1} \binom{n}{\ell} (m-n+k+1+\ell)^{\ell-1} (n-k-1-\ell)^{n-\ell}$$

Discrete parking problems: Enumeration results

Abel's generalization of the binomial theorem:

$$(x+y)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} x(x-\ell z)^{\ell-1}(y+\ell z)^{n-\ell}$$

$\Rightarrow$ **alternative expression for** $g(m,n,k)$ useful for k small

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Abel's generalization of the binomial theorem:

$$(x+y)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} x(x-\ell z)^{\ell-1}(y+\ell z)^{n-\ell}$$

$\Rightarrow$ **alternative expression for** $g(m,n,k)$ useful for k small

Examples for small numbers k of unsuccessful cars:

$$g(m,n,0) = (m-n+1)(m+1)^{n-1}$$

$$g(m,n,1) = (m-n+2)(m+2)^{n-1} + (n^2-n-m^2-2m-1)(m+1)^{n-2}$$

$$\begin{aligned} g(m,n,2) = {} & (m-n+3)(m+3)^{n-1} \\ & + (2n^2-mn-m^2-4n-4m-4)(m+2)^{n-2} \\ & + \frac{1}{2}n(-n^2-mn+2m^2+2n-5m+1)(m+1)^{n-3} \end{aligned}$$

Discrete parking problems: Limiting distribution results

Exact probability distribution of $X_{m,n}$:

$$\mathbb{P}\{X_{m,n} = k\} = \frac{g(m,n,k)}{m^n}$$

Expectation of $X_{m,n}$: Gonnet and Munro [1984]

Studied in analysis of algorithm "linear probing sort"

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Depending on growth of $m, n \Rightarrow$

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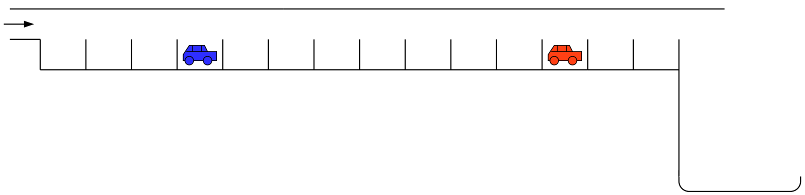
Weak convergence of $X_{m,n}$ (m parking spaces, n cars):

$n \ll m$: $\quad X_{m,n} \xrightarrow{(d)} X$

$$\mathbb{P}\{X = 0\} = 1$$

degenerate limit law



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Weak convergence of $X_{m,n}$ (m parking spaces, n cars):

$n \sim \rho m, \quad 0 < \rho < 1: \qquad X_{m,n} \xrightarrow{(d)} X_\rho$

$$\mathbb{P}\{X_\rho \le k\} = (1-\rho) \sum_{\ell=0}^{k} (-1)^{k-\ell} \frac{(\ell+1)^{k-\ell}}{(k-\ell)!} \rho^{k-\ell} e^{(\ell+1)\rho}$$

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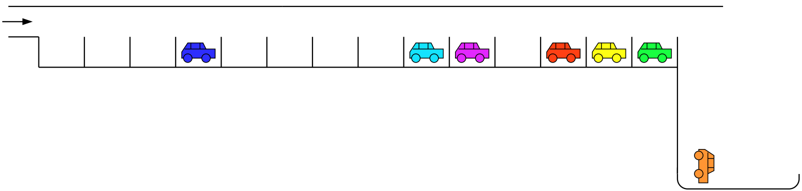

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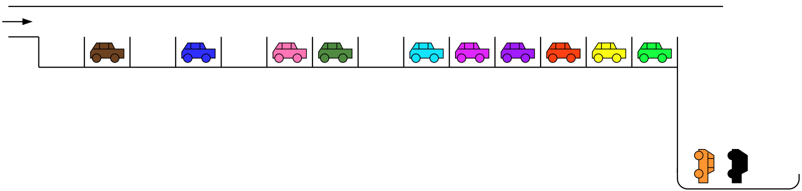
Weak convergence of $X_{m,n}$ (m parking spaces, n cars):

$\sqrt{m} \ll \Delta := m - n \ll m$: $\frac{\Delta}{m} X_{m,n} \xrightarrow{(d)} X \stackrel{(d)}{=} \text{EXP}(2)$

survival function: $\mathbb{P}\{X \geq x\} = e^{-2x}, \quad x \geq 0$

asymptotically exponential distributed



Discrete parking problems: Limiting distribution results

Weak convergence of $X_{m,n}$ (m parking spaces, n cars):

$$\Delta := m - n \sim \rho\sqrt{m}, \quad \rho > 0: \qquad \frac{1}{\sqrt{m}} X_{m,n} \xrightarrow{(d)} X_\rho \stackrel{(d)}{=} \mathrm{LINEXP}(2, \rho)$$

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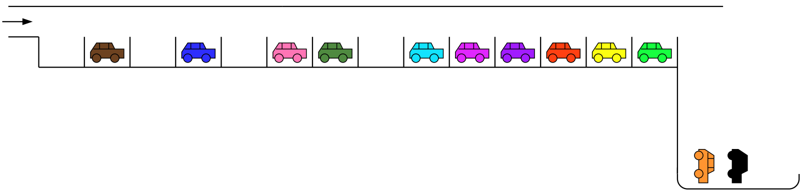

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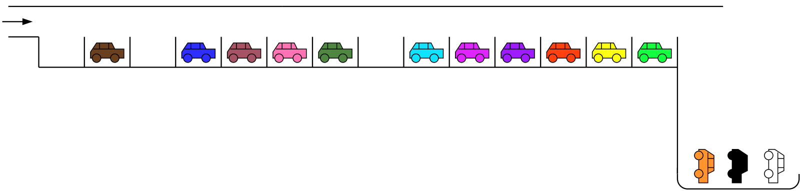

Discrete parking problems: Limiting distribution results

Weak convergence of $X_{m,n}$ (m parking spaces, n cars):

$0 \leq \Delta := n - m \ll \sqrt{n}$: $\quad \frac{X_{m,n+m-n}}{\sqrt{n}} \xrightarrow{(d)} X \stackrel{(d)}{=} \text{RAYLEIGH}(2)$

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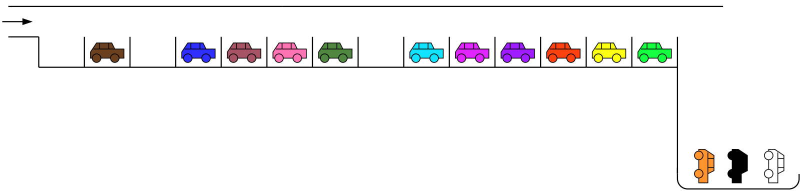

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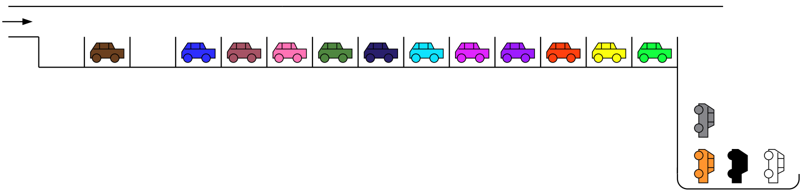

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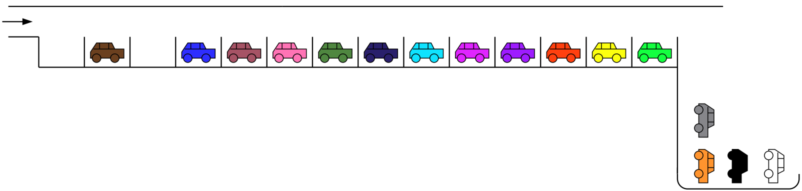

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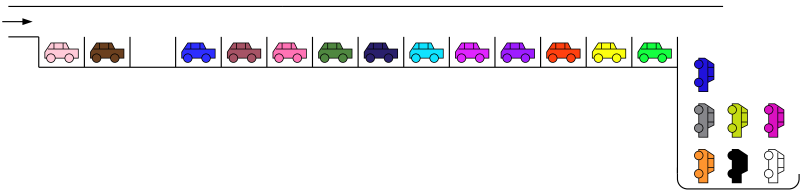
discrete limit law



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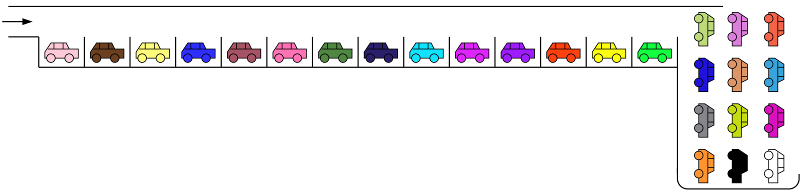
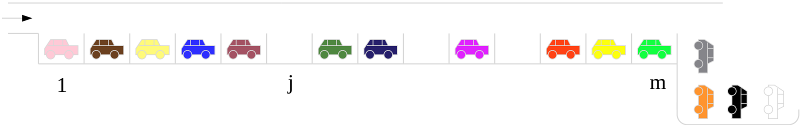


Discrete parking problems: Analysis

Few words on analysis:

Derivation of exact enumeration results:

- Recursive description of parameter via block decomposition
  - * **Case** $n < m + k$: decomposition after first empty space j:



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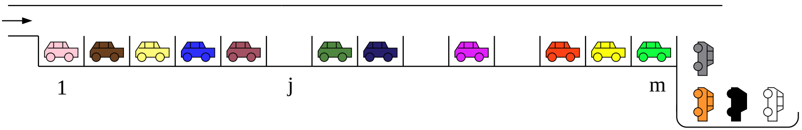

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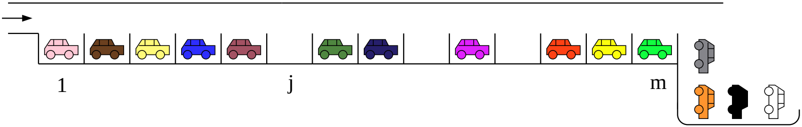

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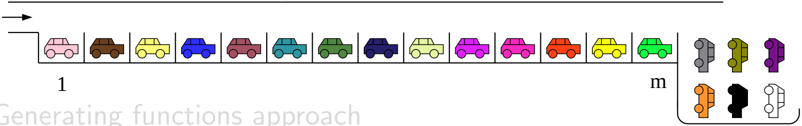

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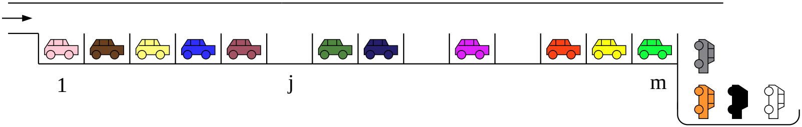

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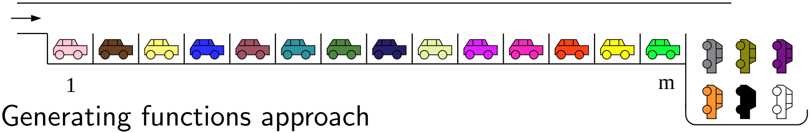

- Generating functions approach

Discrete parking problems: Analysis

- Exact formula for suitable generating function:

$$G(z,u,v) = \frac{1 - \frac{T(zu)}{zv}}{\left(1 - \frac{T(zu)}{z}\right) \cdot \left(1 - \frac{u}{v} e^{zv}\right)}$$

- Special function "tree function" is appearing:

$$T(z) := \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!}$$

$T(z)$: satisfies functional equation $T(z) = ze^{T(z)}$

Discrete parking problems: Analysis

Exact generating function useful for analysing $X_{m,n}$ via analytic combinatorics (applying complex-analytic techniques)

Example: special instance: m (parking spaces) $= n$ (cars)

Contour integral for GF of diagonal: $F(u,v) = \frac{1}{2\pi i} \oint \frac{G(t, \frac{u}{t}, v)}{t} dt$

Applying "method of moments":

- Studying derivatives of $F(u,v)$ evaluated at $v = 1$:
  - local expansion around dominant singularity $u = \frac{1}{e}$
  - Singularity analysis, Flajolet and Odlyzko [1990]
- $\Rightarrow$ r-th moments converge to moments of Rayleigh r.v.

Theorem of Fréchet and Shohat:

$$\frac{X_{m,m}}{\sqrt{m}} \xrightarrow{(d)} \text{RAYLEIGH}(2)$$

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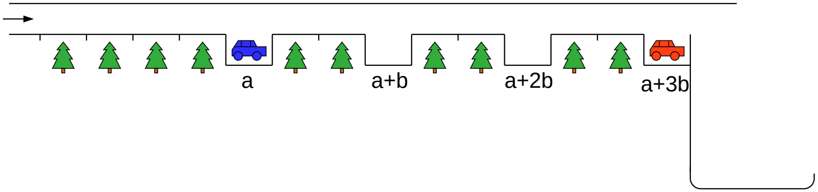
Discrete parking problems: Further research

Generalized parking scheme

Stanley [1996], Yan [1997]: (a, b)-parking scheme

- $a + (m - 1)b$ addresses
- m parking spaces
- parking permitted only at addresses
  $a,\ a + b,\ a + 2b,\ \dots,\ a + (m - 1)b$

Example: $a = 5,\ b = 3$



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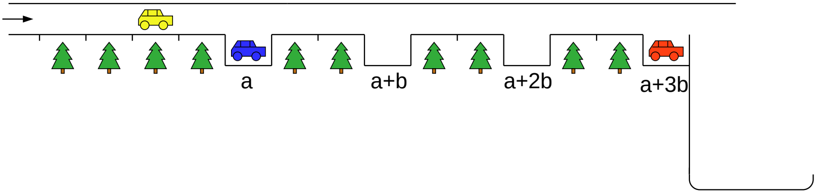

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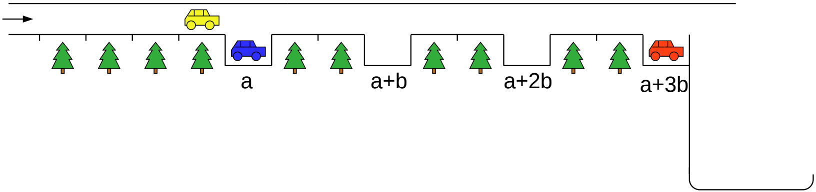

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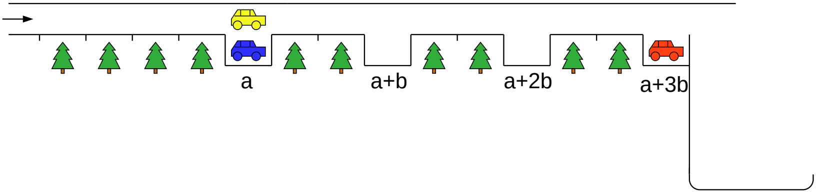

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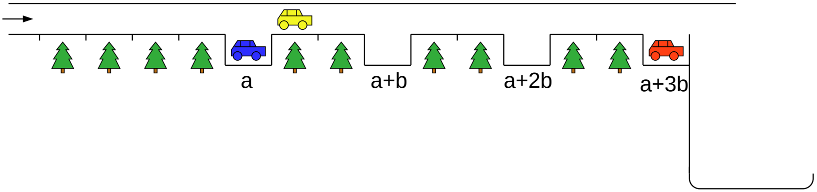

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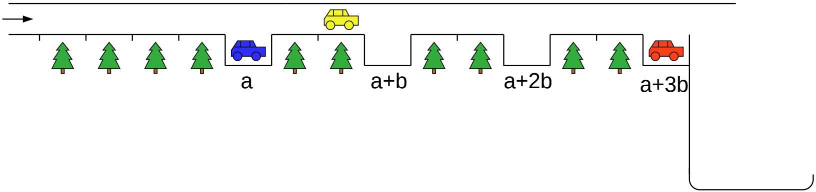

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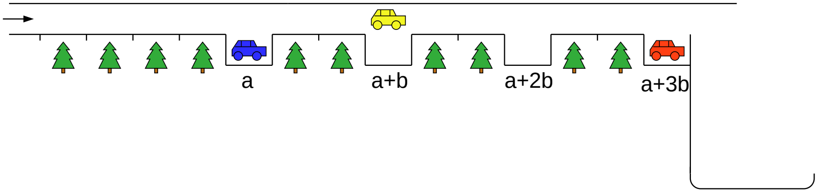

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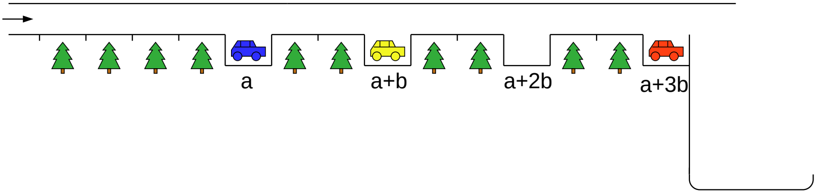

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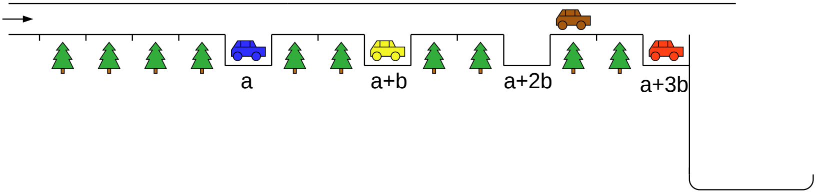

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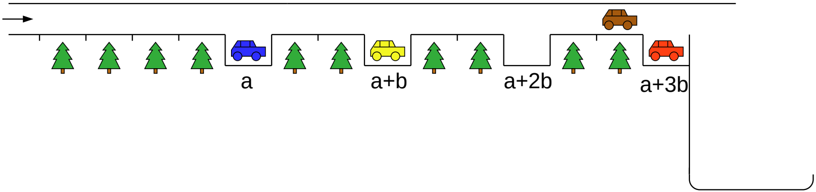

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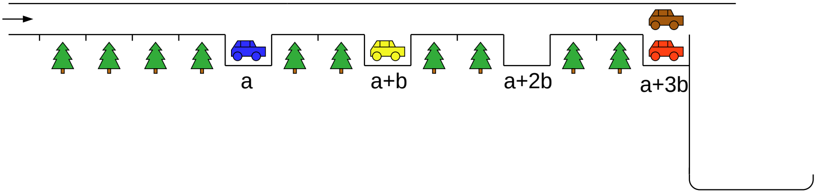

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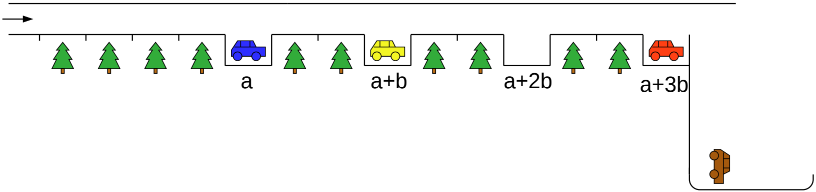

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Question for generalized parking scheme:

What is the number $g^{(a,b)}(m,n,k)$ of parking sequences $a_1, \ldots, a_n \in \{1, \ldots, a+(m-1)b\}^n$ such that exactly k drivers are unsuccessful?

Exact formula for $g^{(a,b)}(m,n,k)$:

$$g^{(a,b)}(m,n,k) = (a+b(m-n+k-1)) \sum_{\ell=0}^{n-k} \binom{n}{\ell} (a+b(m-n+k-1+\ell))^{\ell-1} (b(n-k-\ell))^{n-\ell} - (a+b(m-n+k)) \sum_{\ell=0}^{n-k-1} \binom{n}{\ell} (a+b(m-n+k+\ell))^{\ell-1} (b(n-k-1-\ell))^{n-\ell}$$

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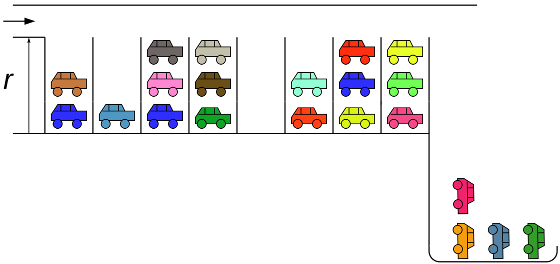
Bucket parking scheme

Blake and Konheim [1976]:

- Each parking space can hold up to r cars
- Related to analysis of bucket hashing algorithms



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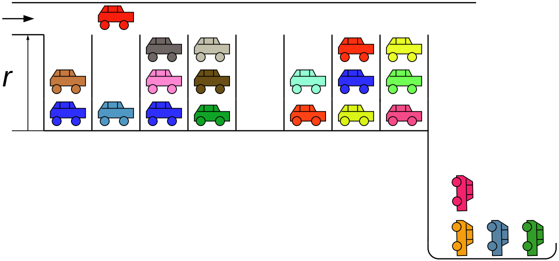

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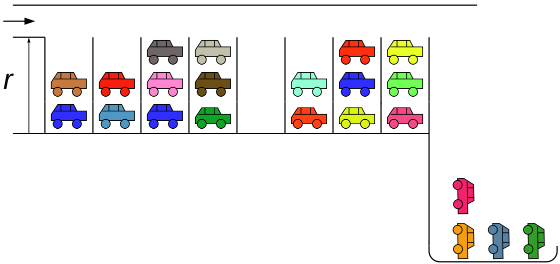

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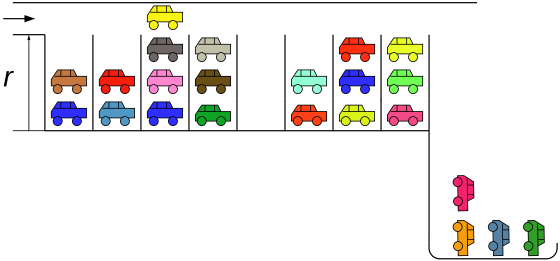

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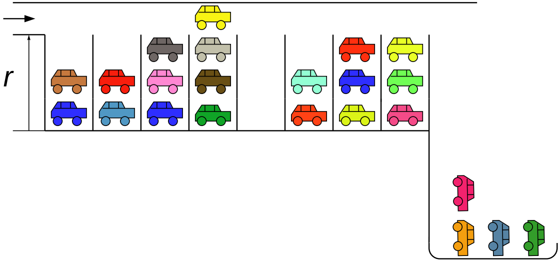

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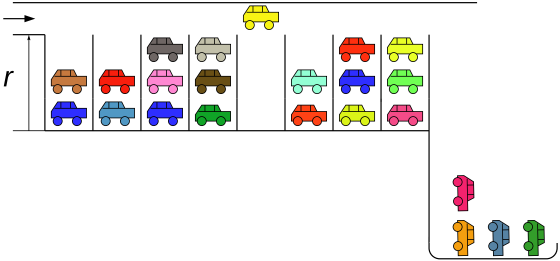

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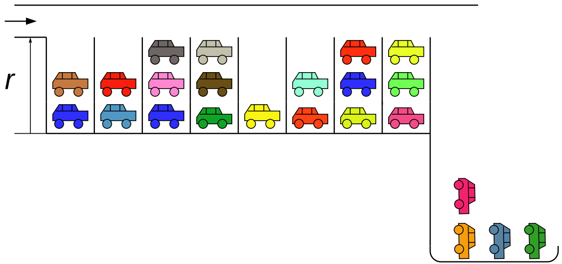

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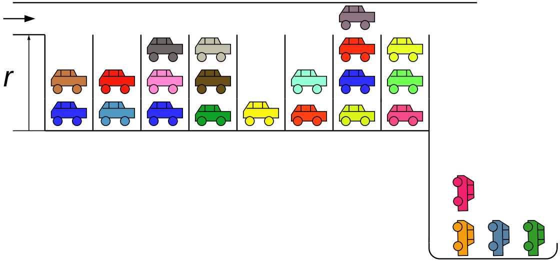

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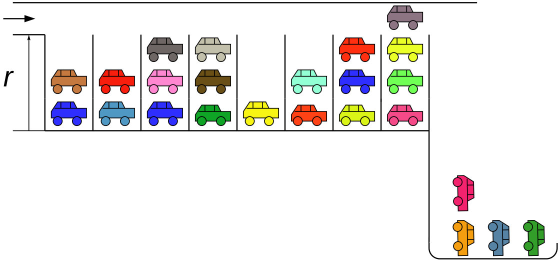

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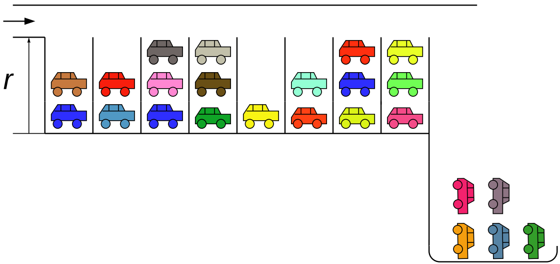

Discrete parking problems: Further research

Question for bucket parking scheme:

What is the number $g^{(r)}(m, n, k)$ of parking sequences $a_1, \ldots, a_n \in \{1, \ldots, m\}^n$ such that exactly k drivers are unsuccessful?

Exact expression for suitable generating function $G_r(z, u, v)$:

$$G_r(z,u,v) = \frac{1}{1 - \frac{u}{v^r} e^{zv}} \frac{\prod_{j=0}^{r-1} \left(1 - \frac{r}{zv} T(\frac{1}{r}\omega_r^j z u^{1/r})\right)}{\prod_{j=0}^{r-1} \left(1 - \frac{r}{z} T(\frac{1}{r}\omega_r^j z u^{1/r})\right)}$$

$\omega_r := e^{\frac{2\pi i}{r}}$: primitive r-th root of unity

Problems for analysis:

- no suitable explicit expression for coefficients available
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Joint study with "terminal block size"

Refinement in analysis:

- k: number of unsuccessful drivers
- ℓ: size of terminal block of occupied parking spaces

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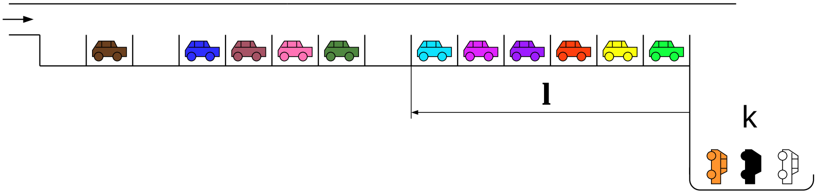

Discrete parking problems: Further research

Exact enumeration result:

Numbers $g(m, n, \ell, k)$ of parking sequences for m parking spaces and n drivers such that exactly k drivers are unsuccessful and the size of the terminal block is ℓ:

$$g(m,n,\ell,k) = \binom{n}{k+\ell}(m-n+k)(m-\ell)^{n-\ell-k-1} \times \left(\ell^{k+\ell} - \sum_{q=0}^{\ell-1}\binom{k+\ell}{q}(q+1)^{q-1}(\ell-1-q)^{k+\ell-q}\right)$$

Pólya-Eggenberger urn models

(together with H.-K. Hwang, Academia Sinica, Taipei
and M. Kuba, TU Wien)

Pólya-Eggenberger urn models: Definition

Pólya-Eggenberger urn models:

- two types of balls: urn contains n white balls and m black balls
- evolution of urn occurs in discrete time steps
- at every step: ball is drawn at random from urn
- color of ball is inspected and then ball is reinserted into urn
- according to observed color of ball, balls are added/removed due to following rules:
- white ball drawn $\Rightarrow$ a white balls and b black balls are added
- black ball drawn $\Rightarrow$ c white balls and d black balls are added

Ball replacement matrix specifies urn model:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}$$

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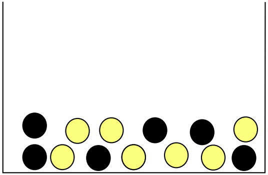
Example:

- ball replacement matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$
- initial configuration:
  $n = 7$ yellow (white) balls and $m = 6$ black balls



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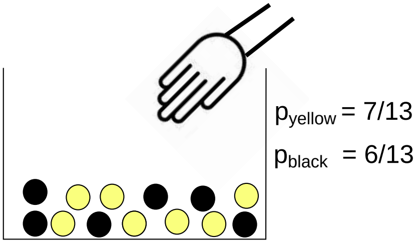

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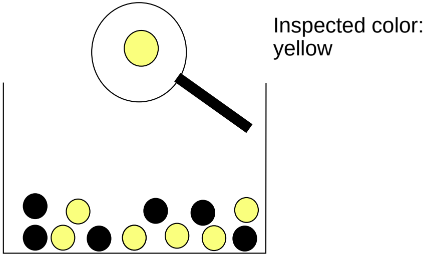

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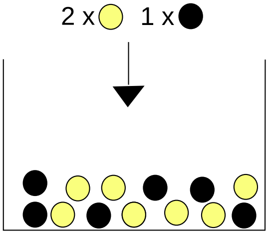

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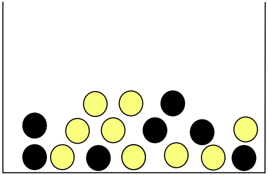

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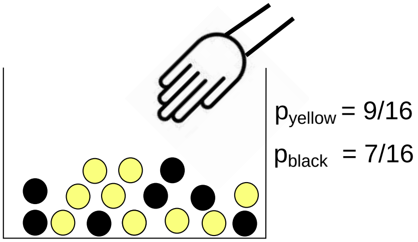

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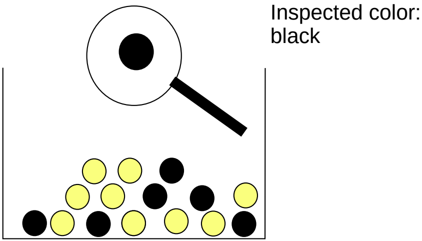

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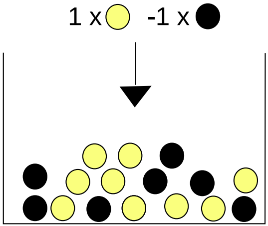

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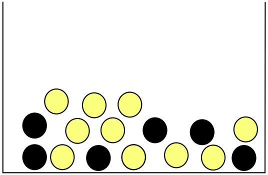

Pólya-Eggenberger urn models: Diminishing urns

Diminishing urn models:

- Pólya-Eggenberger urn model with ball replacement matrix M
- in addition: set of absorbing states $\mathcal{A} \subset \mathbb{N}_0 \times \mathbb{N}_0$.
  state $= (i, j) \quad \hat{=} \quad$ urn contains i black balls, j white balls
- urn evolves according to matrix M until absorbing state $(i, j) \in \mathcal{A}$ is reached
- consider only well defined urns:
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Why should we study such urn models?

Motivation:

- such models appear in various contexts
- often have different nature compared to "usually" studied urns
- different question arising:

  What is the terminal configuration of urn when starting with m black and n white balls?

Examples of urns arising in applications:

- Pill's problem urn and generalizations
- Cannibal urn problem
- OK Corral urn problem

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OK Corral urn: introduced as model in theory of warfare

- two groups A and B of gunmen are fighting
- one gunmen is selected uniformly at random and shoots (kills) then a member of the opposing group
- fight ends if all members of one group are killed

Main questions:

- Which group will survive?
- How many survivors, say of group A, are there when the fight is over?

Historical remark: 1881 Wyatt Earp, Morgan Earp, Virgil Earp, and Doc Holliday were fighting against Frank McLaury, Tom McLaury, Ike Clanton, Billy Clanton, Billy Claiborne, and Wes Fuller the OK Corral ranch.

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OK Corral urn: described via diminishing urn model

- ball replacement matrix $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
- absorbing states $\mathcal{A} = \{(0,n) | n \in \mathbb{N}_0\} \cup \{(m,0) | m \in \mathbb{N}_0\}$

Mathematical description:

- $X_{m,n}$: r.v. counting number of white balls (survivors) when all black balls have been drawn
- probability gen. function: $h_{m,n}(v) = \sum_{k \geq 0} \mathbb{P}\{X_{m,n} = k\} v^k$

Recurrence for $h_{m,n}(v)$:

$$h_{m,n}(v) = \frac{n}{n+m} h_{m-1,n}(v) + \frac{m}{n+m} h_{m,n-1}(v), \quad n \geq 1, m \geq 1$$

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Generalized OK Corral urn:

Arms of group A have power $\alpha \in \mathbb{N}$

Arms of group B have power $\beta \in \mathbb{N}$

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Cannibal urn: model for behavior of cannibals in biological population

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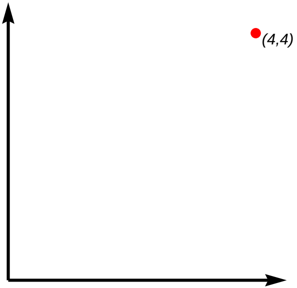
Pólya-Eggenberger urn models: Examples

Evolution of urn: can be described via **weighted lattice paths**


Example: OK Corral urn



Pólya-Eggenberger urn models: Examples

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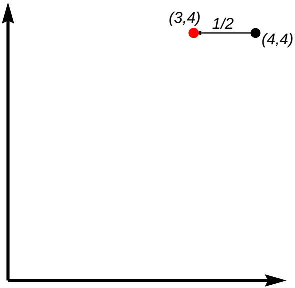

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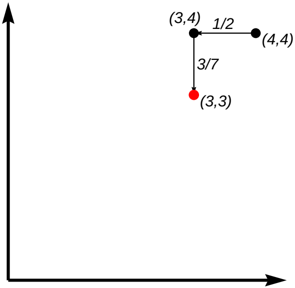



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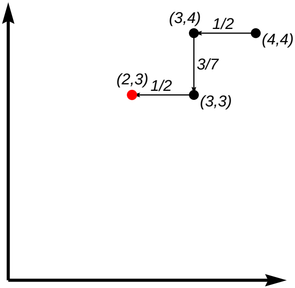

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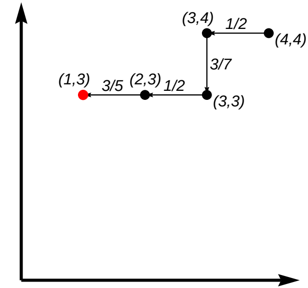

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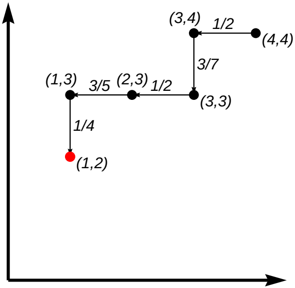

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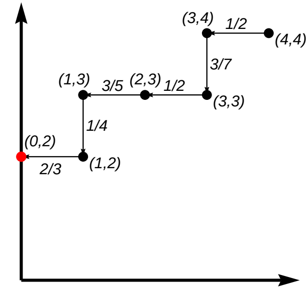

State $(0,2)$ is reached!

Pólya-Eggenberger urn models: Analysis

Outline of analytic approach:

- generating function approach
- recurrences for prob. gen. fct. $h_{n,m}(v)$ translated into first order linear partial differential equations
- applying method of characteristics

Problems where all boundary behaviors are known:

- use ordinary generating function $H(z,w) = H(z,w,v)$:

$$H(z,w) := \sum_{n\geq 1} \sum_{m \geq 1} h_{n,m}(v) z^n w^m$$

Problems with unknown boundary values:

- use suitably modified generating functions to get rid of the unknown boundary values $h_{m,0}(v)$
- E.g., for cannibal urn we use

$$H(z,w) := \sum_{n\geq 0} \sum_{m\geq 1} \frac{1}{m} \binom{n+m-1}{m-1} h_{n,m}(v) z^n w^m.$$

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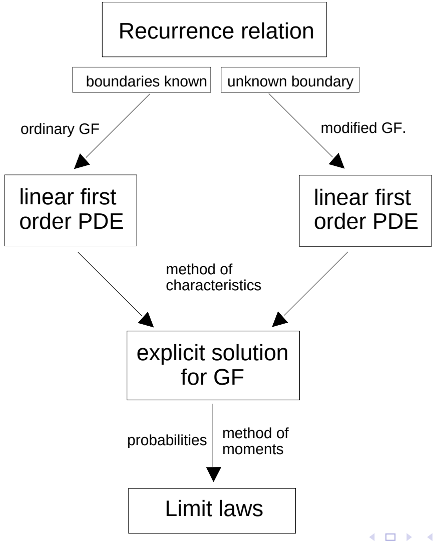
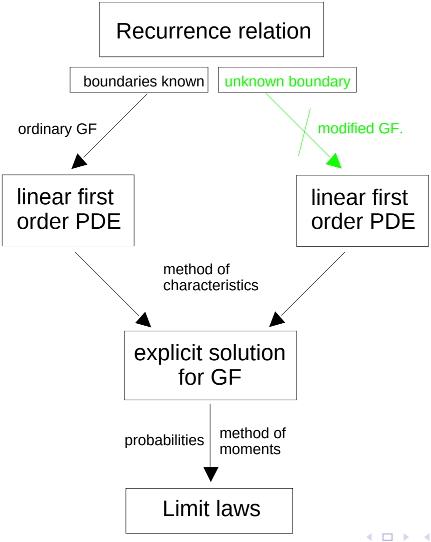
Pólya-Eggenberger urn models: Analysis

In short we proceed as follows:



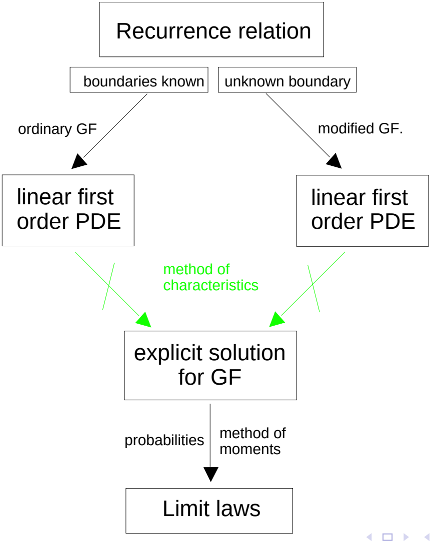
Pólya-Eggenberger urn models: Analysis

Can we manage to find a suitably modified GF?



Pólya-Eggenberger urn models: Analysis

Can we find a “handy” first integral?



Pólya-Eggenberger urn models: Analysis

For many interesting urn models we obtain explicit solutions!

Example: generalized OK Corral urn

Linear first-order PDE with $H(z,0) = H(0,w) = 0$:

$$\beta z(1-w)H_z(z,w) + \alpha w(1-z)H_w(z,w) = \frac{\beta wzv^{\beta}}{(1-v^{\beta}z)^2} + \frac{\alpha wz}{(1-w)^2}$$

System of characteristic differential equations:

$$\dot{z} = \beta z(1-w), \quad \dot{w} = \alpha w(1-z)$$

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Pólya-Eggenberger urn models: Analysis

⇒ **first integral:**

$$\xi(z,w) := \frac{z^{\alpha/\beta}}{w} e^{w - z\alpha/\beta} = \text{const.}$$

Using transformation:

$$\xi = \frac{z^{\alpha/\beta}}{w} e^{w - z\alpha/\beta} \quad \text{and} \quad \eta = z$$

⇒ **explicit GF solution** involving tree function $T(z)$:

$$H(z,w) = z\int_0^1 \frac{v^\beta T(wq^{\alpha/\beta} e^{\beta z(1-q)/\alpha - w}) dq}{(1 - v^\beta zq)^2 (1 - T(wq^{\alpha/\beta} e^{\beta z(1-q)/\alpha - w}))} + z\int_0^1 \frac{\alpha T(wq^{\alpha/\beta} e^{\beta z(1-q)/\alpha - w}) dq}{\beta (1 - T(wq^{\alpha/\beta} e^{\beta z(1-q)/\alpha - w}))^3}.$$

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Pólya-Eggenberger urn models: Results

As a consequence:

- *For many interesting urn models we obtain explicit formulæ for probabilities, probability generating functions, moments, etc.*
- *Explicit formulæ useful for describing limiting behaviour of random variables.*

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- *Explicit formulæ useful for describing limiting behaviour of random variables.*

Pólya-Eggenberger urn models: Results

Example: Generalized OK Corral urn

Theorem

Starting with βn white balls and αm black balls.

$p_{\alpha m,\beta n}$: *probability that all black balls are removed (group of white balls "survive"):*

$$p_{\alpha m,\beta n} = \frac{1}{m!n!} \frac{\beta^m}{\alpha^m} \sum_{\ell=1}^{n} (-1)^{n-\ell} \frac{\binom{n}{\ell}}{\binom{m+\frac{\beta}{\alpha}\ell}{m}} \ell^{n+m}$$

$\mathbb{P}\{X_{\alpha m,\beta n} = \beta k\}$: *probability that exactly βk white balls "survive":*

$$\mathbb{P}\{X_{\alpha m,\beta n} = \beta k\} = \frac{k}{(n-k)!m!} \frac{\beta^m}{\alpha^m} \sum_{\ell=0}^{n} (-1)^{n-\ell} \frac{\binom{n-k}{\ell-k}}{\binom{m+\frac{\beta}{\alpha}\ell}{m}} \ell^{m+n-1-k}$$

Pólya-Eggenberger urn models: Results

Limiting distribution results:

- model very sensitive to relative sizes of initial groups
- influence of "power of arms": according to the square roots of powers
- If $\sqrt{\alpha}m \sim \sqrt{\beta}n$ does not hold then fight is unfair!
- results dependend on behaviour of quantities

$$A_1(n,m) = \beta\frac{n(n+1)}{2} - \alpha\frac{m(m+1)}{2}$$

and

$$A_2(n,m) = \beta^2\frac{n(n+1)(2n+1)}{6} + \alpha^2\frac{m(m+1)(2m+1)}{6}$$

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Pólya-Eggenberger urn models: Results

Theorem

Which group will survive?

- *Region "Black balls survive":* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to -\infty$:

  $p_{\alpha m,\beta n} \to 0$
- *"Fair" region:* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to \theta \in \mathbb{R}$:

  $p_{\alpha m,\beta n} \to F(\theta)$, *function $F(\theta)$ can be described explicitly.*
- *Region "White balls survive":* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to \infty$:

  $p_{\alpha m,\beta n} \to 1$

Pólya-Eggenberger urn models: Results

Theorem

How many survivors in group of white balls?

- *Region "No survivors":* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to -\infty$:

  $X_{\alpha m, \beta n} \xrightarrow{(d)} X$ *with* $\mathbb{P}\{X = 0\} = 1$

- *"Fair" region:* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to \theta \in \mathbb{R}$:

  $\frac{X_{\alpha m, \beta n}}{\sqrt{A_2(n,m)}} \xrightarrow{(d)} X$, *with* $\mathbb{P}\{X \le x\} = \Phi\big(\frac{\beta x^2}{2} - \theta\big)$, $x \ge 0$

  $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$: *standard normal distribution function*

- *Region "White group of balls survive":* $\frac{A_1(n,m)}{\sqrt{A_2(n,m)}} \to \infty$:

  various subregions with different behaviour

Pólya-Eggenberger urn models: Higher dimensions

Higher dimensional urn models: approach applicable to several urns

Example: r**-dimensional Pills problem urn:**

- ball replacement matrix:

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

- absorbing states: hyperplane $\mathcal{A} = \{(n_1, \dots, n_{r-1}, 0) | n_1, \dots, n_{r-1} \in \mathbb{N}_0\}$

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- ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$
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- start with 6 large pills and one small pill

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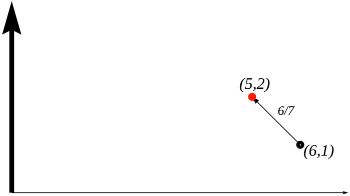

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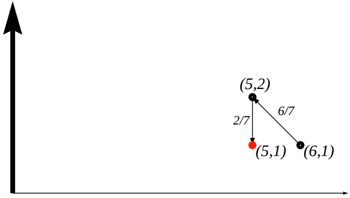

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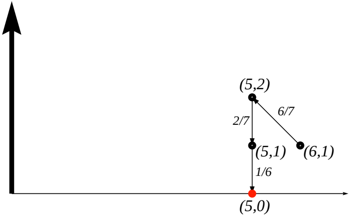

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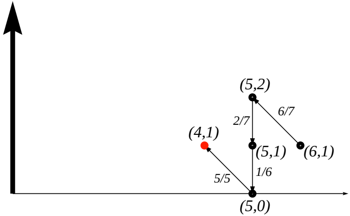

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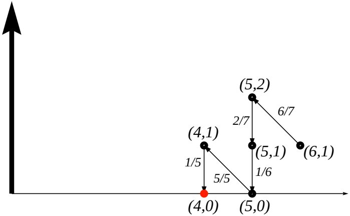

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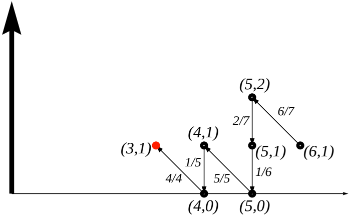

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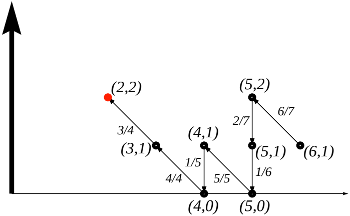

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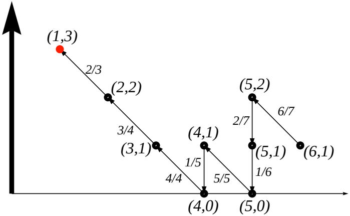

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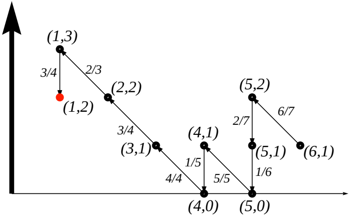

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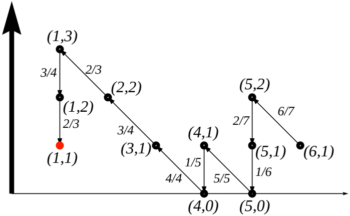

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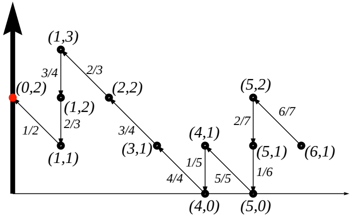

$\Rightarrow$ the state $(0, 2) \in \mathcal{A}$ is reached

Pólya-Eggenberger urn models: Higher dimensions

First order linear PDE:

$$\sum_{j=1}^{r-1}(z_j - z_1 z_j - z_{j+1})H_{z_j}(\mathbf{z}) + (z_r - z_1 z_r)H_{z_r}(\mathbf{z}) - z_1 H(\mathbf{z})$$
$$= \frac{v_{r-1} z_r}{(1 - v_1 z_1 - v_2 z_2 - \cdots - v_{r-1} z_{r-1})^2}.$$

Chracteristic system of DEs:

$$\dot{z}_1 = z_1 - z_1^2 - z_2, \quad \dot{z}_2 = z_2 - z_1 z_2 - z_3, \quad \ldots,$$
$$\dot{z}_{r-1} = z_{r-1} - z_1 z_{r-1} - z_r, \quad \dot{z}_r = z_r - z_1 z_r.$$

Pólya-Eggenberger urn models: Higher dimensions

First order linear PDE:

$$\sum_{j=1}^{r-1}(z_j - z_1 z_j - z_{j+1})H_{z_j}(\mathbf{z}) + (z_r - z_1 z_r)H_{z_r}(\mathbf{z}) - z_1 H(\mathbf{z})$$

$$= \frac{v_{r-1}z_r}{(1 - v_1 z_1 - v_2 z_2 - \cdots - v_{r-1}z_{r-1})^2}.$$

Chracteristic system of DEs:

$$\dot{z}_1 = z_1 - z_1^2 - z_2, \quad \dot{z}_2 = z_2 - z_1 z_2 - z_3, \quad \ldots,$$
$$\dot{z}_{r-1} = z_{r-1} - z_1 z_{r-1} - z_r, \quad \dot{z}_r = z_r - z_1 z_r.$$

Pólya-Eggenberger urn models: Higher dimensions

Independent first integrals $\xi_1, \dots, \xi_{r-2}$: characterized as solution of system of linear equations

$$\begin{aligned}
\frac{z_{r-2}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^2}{2!} + \xi_{r-2},\\
\frac{z_{r-3}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^3}{3!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_{r-3},\\
\frac{z_{r-4}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^4}{4!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)^2}{2!} + \xi_{r-3}\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_{r-4},\\
\vdots &= \vdots\\
\frac{z_1}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-1}}{(r-1)!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-3}}{(r-3)!} + \xi_{r-3}\frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-4}}{(r-4)!} + \cdots + \xi_2\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_1.
\end{aligned}$$

$(r-1)$-th independent first integral:

$$\xi_{r-1} = \frac{z_r}{1 - z_1 - \cdots - z_r} e^{\frac{z_{r-1}}{z_r}}.$$

Pólya-Eggenberger urn models: Higher dimensions

Theorem

Explicit generating functions solution:

$$H(\mathbf{z}) = v_{r-1} z_r \int_0^1 \frac{dq}{\left(f(\mathbf{z}, \mathbf{v}, q)\right)^2},$$

with

$$f(\mathbf{z}, \mathbf{v}, q) = 1 - \sum_{\ell=1}^{r-1} z_\ell \Big(1 - q \sum_{k=1}^{\ell} \frac{(1-v_k)(-1)^{\ell-k} \log^{\ell-k} q}{(\ell-k)!}\Big)$$
$$- z_r \Big(1 - q - q \sum_{k=1}^{r-1} \frac{(1-v_k)(-1)^{r-k} \log^{r-k} q}{(r-k)!}\Big).$$

Exact and asymptotic results follow from that!

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Exact and asymptotic results follow from that!

Network models

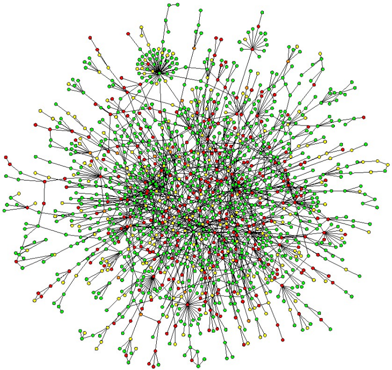
(partially together with M. Kuba, TU Wien
partially together with M. Drmota and B. Gittenberger, TU Wien
partially together with G. Seitz, TU Wien)




Network models: Introduction

Experimental study of real networks:
(e.g., Watts and Strogatz [1998])

- neural networks
- collaboration graphs
- power grid of US



Network models: Introduction

Occuring phenomena:

- **"small-world"-phenomen:**
  diameters are smaller than regularly constructed graphs
- **degree-distribution follows "power-law":**
  probability p_k that node has degree k satisfies

$$p_k \sim k^{-\gamma}, \quad \gamma \in \mathbb{R}^+$$

  $\Rightarrow$ **Scale-free networks**
  (e.g., protein networks, citation networks, some social networks)

$\Rightarrow$ different behaviour than "classical" graph models
(e.g., $G(n,p)$: Erdős-Rényi-graphs)

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Network models: Introduction

Of interest:

- Modelling scale-free networks by random graphs defined by simple rules
- Precise mathematical analysis of models

Famous model: Barabasi-Albert model [1999]:

- Start with small number of vertices
- At each time step:
  add new vertex and connect it to m different existing vertices
- Special rule "**Preferential attachement**":
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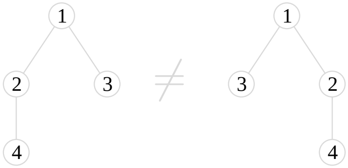
Network models: PORTs

Special case: $m = 1 \Rightarrow$ family of random trees:

Plane-oriented recursive trees (PORTs)
(introduced by Prodinger and Urbanek [1983]; Szymansky [1985])

The order of the subtrees is important!



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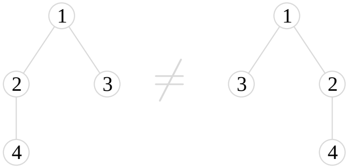

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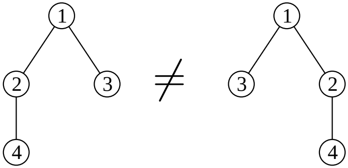

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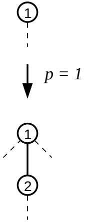

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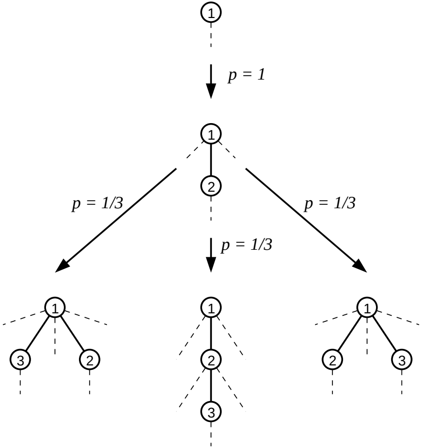

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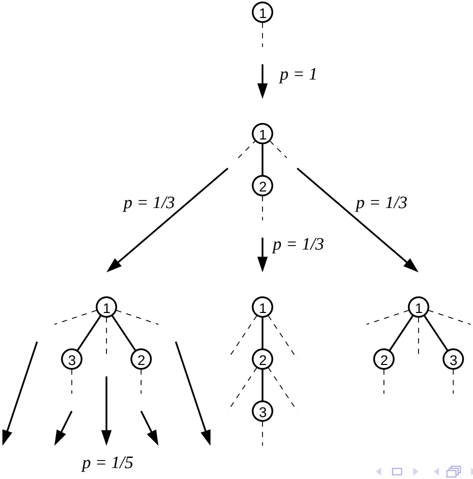

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Kuba and Panholzer [2006, 2007]:

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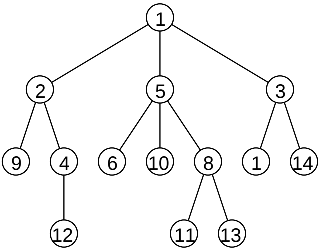

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- Depth of specified nodes
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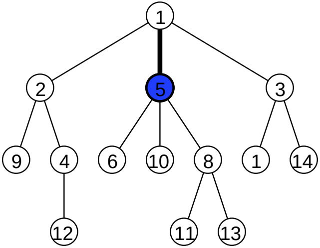

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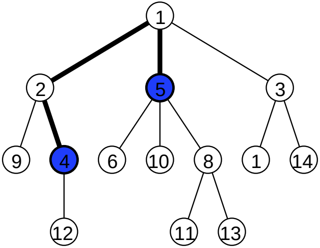

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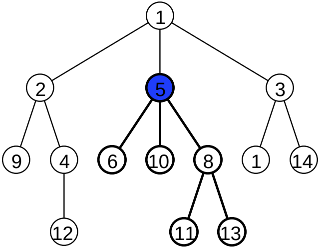

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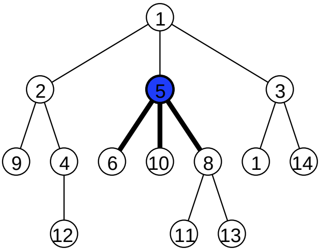

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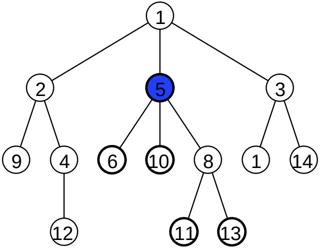

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Network models: Thickened trees

But after all: PORTs are trees!

Richer structures:

"Thickened trees": Drmota, Gittenberger and Panholzer [2008], Drmota, Gittenberger and Kutzelnigg [2009]

- **Substitution process:**
  start with PORTs, replace nodes by certain graphs

- **inspired from some real networks:**
  local structure: clusters, global structure: tree-like

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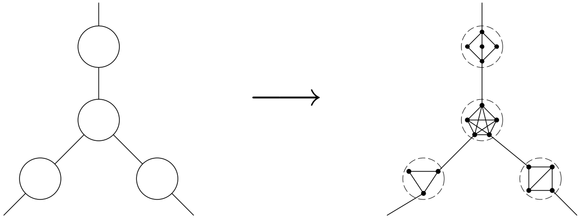

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Processes generating other graph families:

Panholzer and Seitz [2009+]

$\Rightarrow$ **"Ordered k-trees"** by attaching nodes to existing k-cliques

Example of a rooted 2-tree:

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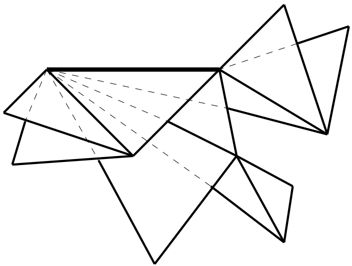

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Ordered k-trees:

- Start with k-clique
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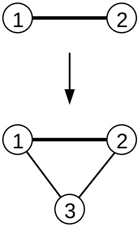

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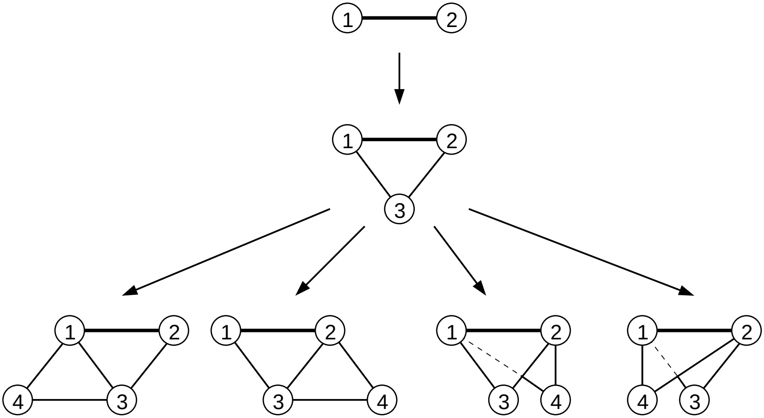

Network models: k-trees

Analysis of parameters in k-trees:

Two descriptions:

- bottom-up: insertion process
- top-down: decomposition according to root k-clique

Exact and asymptotic results for analysed parameters:

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- Degree of nodes (specified nodes, random nodes)
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Theorem (Panholzer and Seitz, 2009)

D_n: **Distance between node 1 and node** *n in ordered k-tree*

Expectation and Variance of D_n:

$$\mathbb{E}(D_n) = \frac{1}{(k+1)H_k} \log n + \mathcal{O}(1),$$

$$\mathbb{V}(D_n) = \frac{H_k^{(2)}}{(k+1)H_k^3} \log n + \mathcal{O}(1).$$

Normalized random variable asympotically Gaussian distributed:

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{D_n - \mathbb{E}(D_n)}{\sqrt{\mathbb{V}(D_n)}} \le x \right\} - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{\sqrt{\log n}} \right).$$

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Top-down approach: $\Rightarrow$ system of ordinary DE for generating functions $S_1(z, v), \ldots, S_k(z, v)$:

$$\begin{aligned}
\frac{\partial}{\partial z} S_1(z, v) &= \frac{k-1}{1-(k+1)z}\big(S_1(z, v) + S_2(z, v)\big), \\
\frac{\partial}{\partial z} S_2(z, v) &= \frac{k-2}{1-(k+1)z}\big(S_2(z, v) + S_3(z, v)\big), \\
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\vdots &= \vdots, \\
\frac{\partial}{\partial z} S_{k-1}(z, v) &= \frac{1}{1-(k+1)z}\big(S_{k-1}(z, v) + S_k(z, v)\big), \\
\frac{\partial}{\partial z} S_k(z, v) &= \frac{kv}{1-(k+1)z} S_1(z, v).
\end{aligned}$$

Network models: k-trees

System of DEs can be solved explicitly:

$$S_\ell(z,v) = \sum_{j=1}^{k} \frac{A_j^{(\ell)}(v)}{(1-(k+1)z)^{\alpha_j(v)}}, \quad 1 \le \ell \le k.$$

$A_j^{(\ell)}(v)$: certain functions analytic in v

$\alpha_j(v), 1 \le j \le k$: different solutions of equation

$$\alpha \cdot (\alpha - \frac{1}{k+1}) \cdot (\alpha - \frac{2}{k+1}) \cdots (\alpha - \frac{k-1}{k+1}) = \frac{k!}{(k+1)^k} v.$$

Results follow immediately by applying methods from analytic combinatorics!

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