DYNAMICAL SYSTEMS, TRANSFER OPERATORS

and FUNCTIONAL ANALYSIS

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Séminaire CALIN, LIPN, 5 octobre 2010

Dynamical analysis of a Euclidean Algorithm.

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A Euclidean Algorithm

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Arithmetic properties of the division

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Geometric properties of the branches

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Spectral properties of the transfer operator

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Geometric properties of the branches

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Analytical properties of the Quasi-Inverse of the transfer operator

⇓

Analytical properties of the generating function

⇓

Probabilistic analysis of the Euclidean Algorithm

The (standard) Euclid Algorithm: the grand father of all the algorithms.

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$$\left\{\begin{array}{ccccccc} u_0 & = & m_1 u_1 & + & u_2 & & 0 < u_2 < u_1 \\ u_1 & = & m_2 u_2 & + & u_3 & & 0 < u_3 < u_2 \\ \ldots & = & \ldots & + & & & \\ u_{p-2} & = & m_{p-1} u_{p-1} & + & u_p & & 0 < u_p < u_{p-1} \\ u_{p-1} & = & m_p u_p & + & 0 & & u_{p+1} = 0 \end{array}\right\}$$

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CFE of $\frac{u}{v}$:

$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p}}}},$$

The underlying Euclidean dynamical system (I).

The trace of the execution of the Euclid Algorithm on (u_1, u_0) is:

$$(u_1, u_0) \to (u_2, u_1) \to (u_3, u_2) \to \ldots \to (u_{p-1}, u_p) \to (u_{p+1}, u_p) = (0, u_p)$$

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Replace the integer pair (u_i, u_{i-1}) by the rational $x_i := \dfrac{u_i}{u_{i-1}}$.

The division $u_{i-1} = m_i u_i + u_{i+1}$ is then written as

$$x_{i+1} = \frac{1}{x_i} - \left\lfloor \frac{1}{x_i} \right\rfloor \qquad \text{or} \qquad x_{i+1} = T(x_i), \qquad \text{where}$$

$$T : [0,1] \longrightarrow [0,1], \quad T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for } x \neq 0, \quad T(0) = 0$$

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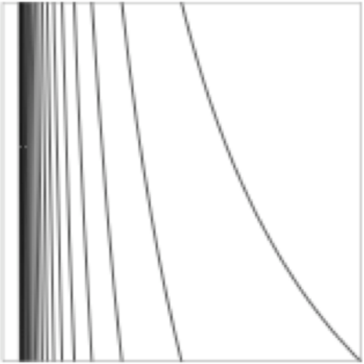
An execution of the Euclidean Algorithm $(x, T(x), T^2(x), \ldots, 0)$

$=$ A rational trajectory of the Dynamical System $([0, 1], T)$

$=$ a trajectory that reaches 0.

The dynamical system is a continuous extension of the algorithm.



$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

$$T_{[m]} : \left]\frac{1}{m+1}, \frac{1}{m}\right[\longrightarrow]0,1[,$$

$$T_{[m]}(x) := \frac{1}{x} - m$$

$$h_{[m]} :]0,1[\longrightarrow \left]\frac{1}{m+1}, \frac{1}{m}\right[$$

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A dynamical system with a denumerable system of branches $(T_{[m]})_{m\geq 1}$,

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The set $\mathcal{H}$ of the inverse branches of T is

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The set $\mathcal{H}$ builds one step of the CF's.

The set $\mathcal{H}^n$ of the inverse branches of T^n builds CF's of depth n.

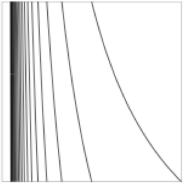
The set $\mathcal{H}^\star := \bigcup \mathcal{H}^n$ builds all the (finite) CF's.

$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p}}}} = h_{[m_1]} \circ h_{[m_2]} \circ \ldots \circ h_{[m_p]}(0)$$

The transfer operator (I).

Density Transformer:

For a density f on $[0,1]$, $\mathbf{H}[f]$ is the density on $[0,1]$ after one iteration of the shift



$$\mathbf{H}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)| \, f \circ h(x) = \sum_{m \in \mathbb{N}} \frac{1}{(m+x)^2} f(\frac{1}{m+x}).$$

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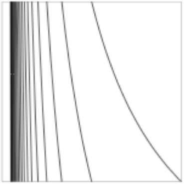

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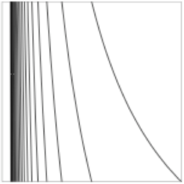

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The k-th iterate satisfies:

$$\mathbf{H}_s^k[f](x) = \sum_{h\in\mathcal{H}^k} |h'(x)|^s\, f\circ h(x)$$

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The density transformer $\mathbf{H}$ expresses the new density f_1 as a function of the old density f_0, as $f_1 = \mathbf{H}[f_0]$. It involves the set $\mathcal{H}$

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With a cost $c : \mathcal{H} \to \mathbf{R}^+$ extended to $\mathcal{H}^\star$ by additivity,

it gives rise to the weighted transfer operator

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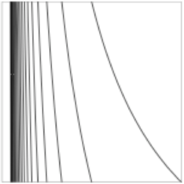
The quasi inverse $(I - \mathbf{H}_{s,w})^{-1} = \sum_{n \geq 0} \mathbf{H}^n_{s,w}$ generates all the finite CFs.

Properties of the dynamical system: the Good Class

$$M_h := \sup\{|h'(x)|, \quad x \in X\}$$



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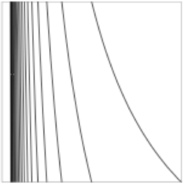

(1) Uniform contraction.

$$\forall h \in \mathcal{H}, \quad M_h \leq 1$$

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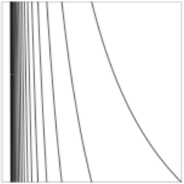

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(2) Bounded distortion.

$$\exists K > 0, \forall h \in \mathcal{H}, \forall x \in X, \quad |h''(x)| \leq K\, |h'(x)|.$$

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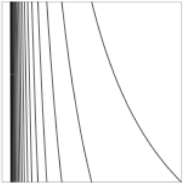

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(3) Convergence on the left of $\Re s = 1$.

$$\exists \sigma_0 < 1, \forall \sigma > \sigma_0, \quad \sum_{h \in \mathcal{H}} M_h^{\sigma} < \infty$$

Properties of the cost

A cost $c : \mathcal{H} \to \mathbf{R}^+$ first defined on $\mathcal{H}$,
then extended to $\mathcal{H}^\star$ by additivity $c(h \circ k) := c(h) + c(k)$.

A cost is of moderate growth if $c(h) = O(|\log M_h|)$

What is needed on the operator $\mathbf{H}_{s,w}$ for the analysis of the algorithm?

For the average case,

only properties on $(I - \mathbf{H}_s)^{-1}$ near $\Re s = 1$

For the distributional analysis,

properties on $(I - \mathbf{H}_{s,w})^{-1}$ on the left of $\Re s = 1$.

Quasi-Compactness

For an operator $\mathbf{L}$,

– the spectrum $\mathrm{Sp}(\mathbf{L}) := \{\lambda \in \mathbb{C}; \quad L - \lambda I \quad \text{non inversible}\}$

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– the essential spectral radius $R_e(\mathbf{L}) =$ the smallest $r > 0$ s.t
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– $\mathbf{L}$ is quasi-compact if the inequality $R_e(\mathbf{L}) < R(\mathbf{L})$ holds.

Then, outside the closed disk of radius $R_e(\mathbf{L})$, the spectrum of the operator consists of isolated eigenvalues of finite multiplicity.

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If $\mathbf{L}$ is a bounded operator on $(\mathcal{F}, ||.||)$ for which there exist two sequences $\{r_n \geq 0\}$ and $\{t_n \geq 0\}$ s.t.

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For systems of the Good Class, $\mathcal{F} := \mathcal{C}^1(X)$,

– the weak norm is the sup-norm $||f||_0 := \sup|f(t)|$,

– the strong norm is the norm $||f||_1 := \sup|f(t)| + \sup|f'(t)|$.

– the density transformer satisfies the hypotheses of Hennion's Theorem.

Main Analytical Properties of $\mathbf{H}_{s,w}$ for a dynamical system of the Good Class and a digit-cost c of moderate growth.

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$\mathbf{H}_{s,w}$ acts on $\mathcal{C}^1(\mathcal{I})$ for $\Re s > \sigma_0$ and $\Re w$ small enough

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A spectral decomposition $\mathbf{H}_{s,w} = \lambda(s,w) \cdot \mathbf{P}_{s,w} + \mathbf{N}_{s,w}$.

$\mathbf{P}_{s,w}$ is the projector on the dominant eigensubspace.
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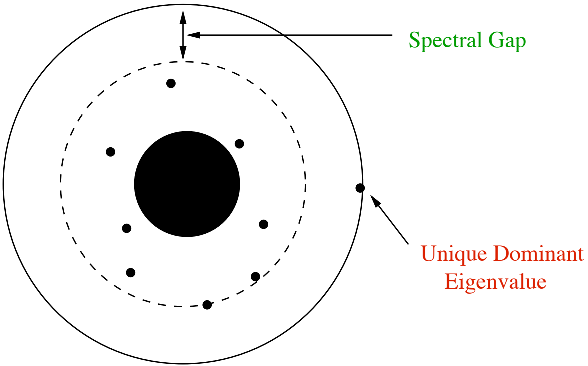
.....which extends to all $n \ge 1$, $\quad \mathbf{H}^n_{s,w} = \lambda^n(s,w) \cdot \mathbf{P}_{s,w} + \mathbf{N}^n_{s,w}$.


Spectral Gap
Unique Dominant
Eigenvalue

Then, a Quasi–Power Property

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"Dominant" (polar) singularities of $(I - \mathbf{H}_{s,w})^{-1}$ near the point $(1,0)$: along a curve $s = \sigma(w)$ on which the dominant eigenvalue satisfies

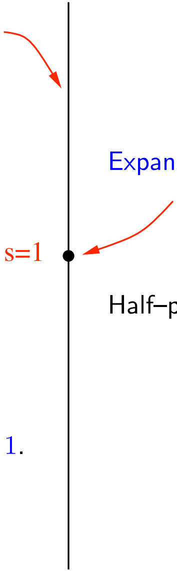
$$\lambda(\sigma(w), w) = 1$$

Another important condition: the Aperiodicity condition:

On the line $\Re s = 1$, $1 \notin \mathrm{Sp}\mathbf{H}_s$.

The triple $UDE + SG + \mathrm{Aperiodicity}$ entails good properties for $(I - \mathbf{H}_s)^{-1}$, sufficient for applying Tauberian Theorems

$s = 1$ is the only pole
on the line $\Re s = 1$



Expansion near the pole $s = 1$

$$(I - \mathbf{H}_s)^{-1} \sim \frac{a}{s-1}$$

Half–plane of convergence $\Re s > 1$

No hypothesis needed
on the half–plane $\Re s < 1$.

Property $US(s, w)$: Uniformity on Vertical Strips

There exist $\alpha > 0, \beta > 0$ such that,

on the vertical strip $\mathcal{S} := \{s; |\Re(s) - 1| < \alpha\}$,

and uniformly when $w \in \mathcal{W} := \{w; |\Re w] < \beta\}$,

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(i) [Strong aperiodicity] $s \mapsto (I - \mathbf{H}_{s,w})^{-1}$ has a unique pole inside $\mathcal{S}$;
it is located at $s = \sigma(w)$ defined by $\lambda(\sigma(w), w) = 1$.

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(ii) [Uniform polynomial estimates] For any $\gamma > 0$, there exists $\xi > 0$ s.t,

$$(I - \mathbf{H}_{s,w})^{-1}[1] = O(|\Im s|^{\xi}) \qquad \forall s \in \mathcal{S}, \quad |t| > \gamma, \quad w \in \mathcal{W}$$

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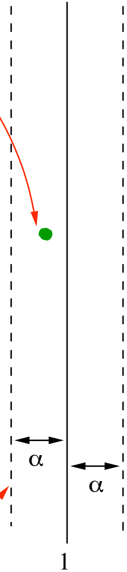
With the Property *US*,

it is easy to deform the contour of the Perron Formula

and use Cauchy's Theorem ...

Near $w = 0$, the function σ is defined by $\lambda(\sigma(w), w) = 1$

$s = \sigma(w)$ is the only pole
on the strip $|\Re s - 1| \le \alpha$



Expansion

near the pole $s = \sigma(w)$

$$(I - \mathbf{H}_{s,w})^{-1} \sim \frac{a}{s - \sigma(w)}$$

Half–plane of
convergence $\Re s > \sigma(w)$

Uniform polynomial estimates

needed on the left domain

$1 - \alpha \le \Re s \le 1, |\Im s| \ge \gamma$.

Property $US(s)$ is not always true

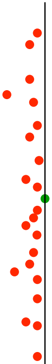
Item (i) is always false for Dynamical Systems with affine branches.

Example: Location of poles of $(I - \mathbf{H}_s)^{-1}$ near $\Re s = 1$
in the case of affine branches of slopes $1/p$ and $1/q$ with $p + q = 1$.

Two main cases

If $\dfrac{\log p}{\log q} \in \mathbf{Q}$

Regularly spaced poles
on $\Re s = 1$



If $\dfrac{\log p}{\log q} \notin \mathbf{Q}$

Only one pole at $s = 1$
on $\Re s = 1$
but accumulation of poles
on the left of $\Re s = 1$

Three main facts.

(a) There exist various conditions, (introduced by Dolgopyat),

the Conditions *UNI* that express that

"the dynamical system is quite different from a system

with piecewise affine branches"

(b) For a good Dynamical system

[complete, strongly expansive, with bounded distortion],

Conditions *UNI* imply the Uniform Property $US(s,w)$.

(c) Conditions *UNI* are true in the Euclid context.

Dolgopyat (98) proves the Item (b) but

- only for Dynamical Systems with a finite number of branches
- He considers only the $US(s)$ Property

Baladi-Vallée adapt his arguments to generalize this result:

For a Dynamical System
with a denumerable number of branches (possibly infinite),
Conditions UNI [Strong or Weak] imply $US(s, w)$.

Precisions about the UNI Conditions

Distance Δ. $\Delta(h,k) := \inf_{x\in\mathcal{I}} \Psi'_{h,k}(x)$, with $\Psi_{h,k}(x) := \log \frac{|h'(x)|}{|k'(x)|}$

Contraction ratio ρ. $\rho := \limsup \left(\{\max |h'(x)|; h \in \mathcal{H}^n, x \in \mathcal{I}\}\right)^{1/n}$.

Probability $\Pr_n$ on $\mathcal{H}^n \times \mathcal{H}^n$. $\Pr_n(h,k) := |h(\mathcal{I})| \cdot |k(\mathcal{I})|$

For a system $\mathcal{C}^2$–conjugated with a piecewise-affine system :
For any $\hat\rho$ with $\rho < \hat\rho < 1$, for any n, $\Pr_n[\Delta < \hat\rho^n] = 1$

Strong Condition UNI.

For any $\hat\rho$ with $\rho < \hat\rho < 1$, for any n, $\Pr_n[\Delta < \hat\rho^n] \ll \hat\rho^n$

Weak Condition UNI.

$\exists D > 0, \exists n_0 \geq 1$, $\forall n \geq n_0$, $\Pr_n[\Delta \leq D] < 1$.